

# Planning Desired Center of Mass and Zero Moment Point Trajectories for Bipedal Locomotion

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**Abstract**—In this paper, we consider the problem of simultaneously synthesizing desired trajectories for both the center of mass (CoM) and zero moment point (ZMP) of a bipedal robot. The approach extends our past work, in which we derived an explicit constraint that ensures boundedness of CoM trajectories for a given reference ZMP trajectory. We parameterize the desired ZMP trajectory by a sequence of basis functions; these design parameters are optimized by an algorithm, in which the CoM boundedness constraint as well as ZMP and other CoM specifications are simultaneously satisfied. When these specifications do not fully constrain the solution, we apply a null-space projection approach that allows the incorporation of additional design criteria. We present simulation results for both the linear inverted pendulum model, and for a model that includes swing leg dynamics.

## I. INTRODUCTION

In this paper, we address the problem of simultaneous planning of desired center of mass (CoM) and zero moment point (ZMP) trajectories, a problem that has been of interest in the humanoid robotics community for several years [1]–[4]. The method that we present in this paper continues to build on our past results [5], [6]. In [5], we derived an explicit constraint that guarantees boundedness of the CoM trajectories for a given ZMP trajectory. We derived an analytical solution for any input ZMP trajectory and applied this result to the problems of planning individual footsteps and disturbance rejection. In [6], we showed how this problem can be expressed as a stable inversion problem for non-minimum phase systems, and showed connections to other well-known methods [2], [7], [8]. Here, we show how this approach can be extended to simultaneous CoM and ZMP trajectory planning, first for a simple linear inverted pendulum (LIP) model, and subsequently for a model that includes swing leg dynamics.

To achieve simultaneous CoM/ZMP planning, we consider a class of parameterized basis functions that can be sequenced together to construct a full ZMP trajectory. Values for the parameters are chosen to satisfy the boundedness constraint on the CoM trajectory in the spirit of [5] as well as performance criteria for the ZMP or CoM trajectory. This is similar to approaches that restrict the class of admissible desired ZMP trajectories to be the family of polynomial functions (e.g., [2], [3], [9], [10]) in order to plan both CoM and ZMP trajectories. To solve this problem, past approaches

have relaxed constraints on the desired ZMP trajectory, for example, modifying the coefficients of the first segment of a spline so that CoM continuity conditions are satisfied (e.g., [2]), however there is no more strict control on the ZMP. Another interesting line of research is presented in [11], which considers the walking motion generation as a CoM state transfer. In [4], an MPC-based approach is used to generate a gait based on the desired locomotion direction.

For the problem that we address, in [5], we looked for a reference CoM trajectory  $x_c^d(t)$  which guarantees that the resulting ZMP given by the LIP second order differential equation follows a desired evolution  $x_{zmp}^d(t)$ . This tool can be extended and used for compensating known terms such as the effect of a swinging leg on the ZMP (see [6]) in the same spirit as [12]–[14]. This  $x_c^d(t)$  can then be the reference for control algorithms, e.g., based on inverse kinematics.

The remainder of this paper is organized as follows. In Section II we review the basic problem formulation and recap previous results from [5], [6]. In Section III we introduce the design approach for simultaneous desired CoM and ZMP trajectory generation, and apply it to a set of increasingly difficult problems. In Section IV we present simulation results to demonstrate the method.

## II. PRELIMINARIES

For simplicity we consider the LIP dynamics only in the sagittal plane, given by the linear differential equation

$$\ddot{x}_c(t) - \omega_o^2 x_c(t) + \omega_o^2 x_{zmp}(t) = 0 \quad (1)$$

in which  $x_c$  denotes the position of the center of mass (CoM), and  $x_{zmp}$  the zero moment point (ZMP) [8], [15], [16]. As is well known, the LIP has stable and unstable modes which can be highlighted by the change of coordinates  $(x_s, x_u)$

$$x_s = x_c - \frac{1}{\omega_o} \dot{x}_c, \quad x_u = x_c + \frac{1}{\omega_o} \dot{x}_c \quad (2)$$

This change of coordinates has been introduced in [17], and used also in [18], where the two motions were respectively defined as convergent and divergent components of motion. Note that the unstable state variable  $x_u$  coincides with the definition of the Instantaneous Capture Point of [16], also named as Extrapolated Center of Mass of [19], where the terminologies were chosen to highlight salient aspects of the system.

In previous work [5], [6], we have shown that it is possible to determine bounded solutions  $x_c^d(t)$  of the CoM trajectory

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for a given ZMP trajectory  $x_{\text{zmp}}^{\text{d}}(t)$  i.e.,

$$\ddot{x}_c^{\text{d}}(t) - \omega_o^2 x_c^{\text{d}}(t) = -\omega_o^2 x_{\text{zmp}}^{\text{d}}(t) \quad (3)$$

$$\text{s.t. } x_c^{\text{d}}(0) + \frac{1}{\omega_o} \dot{x}_c^{\text{d}}(0) = x_u^*(0; x_{\text{zmp}}^{\text{d}}(t)) \quad (4)$$

$$\text{where } x_u^*(0; x_{\text{zmp}}^{\text{d}}(t)) = \omega_o \int_0^\infty e^{-\omega_o \tau} x_{\text{zmp}}^{\text{d}}(\tau) d\tau \quad (5)$$

in which the superscript d is used to denote a desired, or reference trajectory.

As shown in [5], [6] the condition (4) guarantees boundedness of the solution  $x_c^{\text{d}}(t)$  with respect to  $x_{\text{zmp}}^{\text{d}}(t)$ , and therefore we refer to this condition as the ‘‘Boundedness Constraint’’. This condition is a generalization of the Instantaneous Capture Point, as will be seen in Section III. Furthermore, this approach can easily be adapted for rejecting known force disturbances (e.g., change of ground slope in the future landing foot, or known effects as the swinging foot dynamics). Finally, note that, since the system is linear,  $x_{\text{zmp}}^{\text{d}}(t)$  together with the initial conditions on CoM position and velocity completely determine the CoM trajectory.

Viewed in terms of finding a reference CoM trajectory, to obtain a unique solution of (3), we need to define two boundary conditions (e.g., any two of  $x_c^{\text{d}}(0)$ ,  $\dot{x}_c^{\text{d}}(0)$ ,  $x_s^{\text{d}}(0)$ ,  $x_u^{\text{d}}(0)$ , or, conditions on the corresponding final values). The Boundedness Constraint (4) provides one such constraint, which can concisely be written as

$$x_u^{\text{d}}(0) = x_u^*(0; x_{\text{zmp}}^{\text{d}}(t)) \quad (6)$$

This leaves one remaining degree of freedom (boundary condition) to assign. If we express this constraint in terms of  $x_s$  (using, e.g., the change of coordinates given in (2)) we have several alternative options, including setting  $x_c^{\text{d}}(0)$  as the actual initial CoM position  $x_c(0)$ , setting  $\dot{x}_c^{\text{d}}(0)$  as the actual initial CoM velocity  $\dot{x}_c(0)$ , or setting  $x_c^{\text{d}}(T)$  as the desired final CoM position  $x_c(T)$ .

### III. DESIGN

A first important aspect of the proposed approach is that it gives an analytical bounded CoM trajectory for any given ZMP trajectory. However, if the actual initial CoM position and velocity do not satisfy (4) (unmatched case), the actual CoM, solution of (1), will diverge exponentially. To start from the actual initial state, a solution is to relax the reference ZMP trajectory so that the integral (5) evaluates to a value that satisfies (4). This approach has been introduced in [5] for the point foot case.

To accomplish this, we construct the desired ZMP trajectory as a sequence of parameterized segments. The  $i^{\text{th}}$  ZMP segment  $\alpha_i u_b^i(t)$  for the  $i^{\text{th}}$  step occurring during the interval  $[t_{i-1}, t_i]$  is a scaled version of the unit-length basic step  $u_b^i(t)$ , monotonic in  $t$  and zero for  $t < t_{i-1}$ . Choices for  $u_b^i(t)$  could include, for example, the Heaviside unit-step function  $u_{\text{step}}(t)$  or a Constant/Cubic/Constant polynomial pattern as illustrated in Section III-B.

#### A. Design: starting from a generic initial CoM state

In this section, we develop our method through a sequence of increasingly complex examples, all starting from the actual initial CoM state  $(x_c(0), \dot{x}_c(0))$ , ranging from considering a single step to planning a sequence of steps.

1) *One variable step:* We consider a single step of length  $\alpha$ , occurring at time  $t_1$ , i.e.  $x_{\text{zmp}}^{\text{d}}(t) = \alpha u_b(t)$  with, to simplify notation,  $u_b^i = u_b$  and  $u_b(t) = 0$  for all  $t < t_1$ . The ensuing three constraints, boundedness, initial actual position and velocity, being  $x_u^*(0; \alpha u_b(t)) = \alpha x_u^*(0; u_b(t))$ , are

$$x_u^{\text{d}}(0) = x_u^*(0; x_{\text{zmp}}^{\text{d}}(t)) = \alpha x_u^*(0; u_b(t)) \quad (7)$$

$$x_c^{\text{d}}(0) = x_c(0) = \frac{1}{2}(x_u^{\text{d}}(0) + x_s^{\text{d}}(0)) \quad (8)$$

$$\dot{x}_c^{\text{d}}(0) = \dot{x}_c(0) = \frac{\omega_o}{2}(x_u^{\text{d}}(0) - x_s^{\text{d}}(0)) \quad (9)$$

Combining the last two equations in the first one leads to

$$\alpha x_u^*(0; u_b(t)) = x_c(0) + \frac{\dot{x}_c(0)}{\omega_o} \quad (10)$$

For a fixed basis function  $u_b$ , the  $x_u^*(0; u_b(t))$  takes a known value which can be computed offline a priori from (5) and therefore we can solve (10) in terms of  $\alpha$ , the step length.

For example, if we consider the special point feet situation, i.e. with  $u_b(t) = u_{\text{step}}(t - t_1)$ ,

$$x_u^*(0; u_{\text{step}}(t - t_1)) = e^{-\omega_o t_1} \quad (11)$$

we can choose as design parameter either the step length

$$\alpha = e^{\omega_o t_1} \left[ x_c(0) + \frac{\dot{x}_c(0)}{\omega_o} \right] \quad (12)$$

or the step duration

$$t_1 = -\frac{1}{\omega_o} \ln \frac{1}{\alpha} \left[ x_c(0) + \frac{\dot{x}_c(0)}{\omega_o} \right] \quad (13)$$

Some interesting remarks are in order.

- To be able to start from any initial CoM velocity we should take at time  $t_1$  a step with length given by (12). In the special case of  $t_1 = 0$ , the length becomes

$$\alpha = x_c(0) + \frac{\dot{x}_c(0)}{\omega_o} \quad (14)$$

which is the original definition of the *instantaneous capture point* [16], i.e. where the LIP should make an instantaneous step in order to recover balance after an impulsive perturbation has brought the generic state in  $(x_c(0), \dot{x}_c(0))$ .

- Keeping the step length fixed, we can vary its duration. Since  $t_1 \geq 0$  and  $\ln$  should be defined, this requires that

$$0 < \left| \frac{1}{\alpha} \left( x_c(0) + \frac{\dot{x}_c(0)}{\omega_o} \right) \right| \leq 1 \quad (15)$$

which leads to a similar analysis as in [20], i.e.

- If the CoM initial state is  $(0, 0)$ , there is no need to make a step at any time.

- To make a step of length  $\alpha = x_c(0) + \dot{x}_c(0)/\omega_o$  and maintain balance, it is sufficient to make the step instantaneously, i.e.  $t_1 = 0$ , which is a different interpretation of the instantaneous capture point.
- If the actual state is s.t.  $x_c(0) + \dot{x}_c(0)/\omega_o < 0$ , then  $\alpha < 0$ , i.e. the LIP needs to make a backwards step.
- If  $|\alpha(x_c(0) + \dot{x}_c(0)/\omega_o)| > 1$  no balance is possible with a fixed step length  $\alpha$ , no matter what its duration is (i.e., no 1-step capturability with fixed size footstep).

We can therefore see that the expression (10) generalises the concept of capture point at any instant and for any chosen basic footstep function  $u_b(\cdot)$ . In [21], the author describes the capture point as “a point about which the system is theoretically open-loop stable”, i.e. a degenerate case of the boundedness concept.

2) *Multiple steps - only first step variable*: We could also envisage a strategy where only the first step, of a sequence, is adapted in order to cope with the initial CoM conditions. The only change occurs in the Boundedness Constraint

$$\alpha_1 x_u^*(0; u_b^1(t)) + \sum_{i=2}^n \alpha_i x_u^*(0; u_b^i(t)) = x_c(0) + \dot{x}_c(0)/\omega_o$$

in which the  $t_i$  and  $\alpha_i$  are fixed for  $i = 2, \dots, N$ . For point feet, we can choose as design parameter either the step length  $\alpha_1$  or its duration  $t_1$

$$\alpha_1 = e^{\omega_o t_1} \left[ x_c(0) + \dot{x}_c(0)/\omega_o - \sum_{i=2}^n \alpha_i e^{-\omega_o t_i} \right]$$

$$t_1 = -\frac{1}{\omega_o} \ln \left[ 1/\alpha_1 (x_c(0) + \dot{x}_c(0)/\omega_o - \sum_{i=2}^n \alpha_i e^{-\omega_o t_i}) \right]$$

An illustrative simulation is reported in Section IV-A.

From the examples above, it is possible to highlight several important characteristics of our approach.

First, how the solution is obtained depends upon how the ZMP design parameters enter. If they appear nonlinearly, e.g. timing of steps  $t_i$ , there is no a-priori guarantee that a closed-form solution can be easily found. If they enter linearly, a set of linear equations in the unknown coefficients is obtained and these can be solved very efficiently. We will therefore, unless for some specific situations, prefer constraints where the unknown design parameters enter linearly.

The basic principle of our approach is to choose a sufficiently rich family of functions  $u_b^i$ , such that there is a sufficient number of free design parameters to allow us to comply with the required constraints. This general concept has previously been used in [2] and [9] or in [22] by placing an extra ZMP control point in the middle of the first double support phase (the double support phase being less critical, due to higher stability margins).

### B. Simultaneous CoM/ZMP design

As shown in Section III-A, considering variable steps  $\alpha_i u_b^i(t)$  leads to linear equations in the design parameters

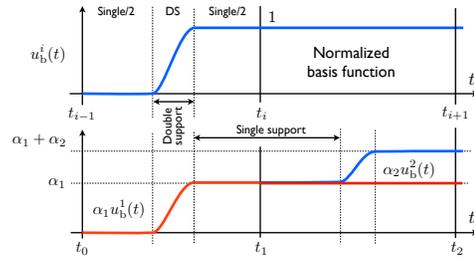


Fig. 1. Design: basis function and its use (2 steps example)

$\alpha_i$ . The general  $N$ -steps ZMP, yet to be determined, is

$$x_{zmp}(t) = \sum_{i=1}^N \alpha_i u_b^i(t) \quad (16)$$

in which  $u_b^i(t) = 0$  for  $t < t_{i-1}$ .

We generally assume that  $u_b^i$  has normalized step length, such that  $u_b^i(t) = 1$  for  $t \geq t_i$ . Changing the step length is accomplished by our selection of  $\alpha_i$ . For example, the ZMP trajectory in the interval  $[t_{i-1}, t_i]$  might be defined as a spline  $B(t)$  with boundary conditions  $B(t_{i-1}) = 0$ ,  $B(t_i) = 1$  such that  $B(t) = 0$  for  $t < t_{i-1}$ , and  $t > t_i$ . We would then have  $u_b^i(t) = B(t) + u_{step}(t - t_i)$  for  $t \geq t_{i-1}$ .

We have considered the following possibilities for  $u_b^i$ .

- Perhaps the simplest choice is  $u_b^i(t) = u_{step}(t - t_{i-1})$ , which corresponds to an instantaneous step at time  $t_{i-1}$ . This is typical for point-foot gaits (see [16]) or Divergent-Component of Motion based ones [23].
- A second choice is to use splines, e.g., unit Constant/Cubic/Constant polynomials, which could represent Single/Double/Single support phases as in [24]. Our choice, shown in Fig. 1, differs slightly as the true Single Support is split among two steps and there is no need of considering any continuity constraint at the intermediate instants  $t_i$ ,  $i = 2, \dots, N - 1$  when the different length steps are put together to make the gait. Considering instead a Linear/Cubic/Linear basis function would require some continuity condition at those instants and thus additional degrees of freedom.

For any chosen basis function  $u_b^i(t)$ , the Boundedness Constraint for the ZMP (16), is<sup>1</sup> linear w.r.t. the unknown  $\alpha_i$ 's

$$x_c(0) + \frac{\dot{x}_c(0)}{\omega_o} = \sum_{i=1}^N \alpha_i x_u^*(0; u_b^i(t)) \quad (17)$$

and each term  $x_u^*(0; u_b^i(t))$  can be precomputed. Adding extra constraints also linear in the unknown parameters, as illustrated in [5] for the point foot case, leads to a simultaneous CoM/ZMP design in the same spirit of [2] and [3] but with fixed shaped basis ZMP functions and with total control on the nominal ZMP.

As an example, using the more general basis function of Fig. 1, we consider the following constraints: we want to

<sup>1</sup>We directly assumed to start from the actual initial CoM state and solved the corresponding two constraints as previously done to get (10).

start from the actual CoM initial position and velocity, set a final velocity  $\dot{x}_c^d(T)$  in  $t = T$  and walk a total distance  $L$ . With a total number of constraints given by 5, (BC) plus 4 constraints, we need at least 3 steps

$$x_{\text{zmp}}(t) = \sum_{i=1}^3 \alpha_i u_b^i(t) \quad (18)$$

and the constraints become

$$x_u^d(0) = \sum_{i=1}^3 \alpha_i x_u^*(0; u_b^i(t)), \quad \sum_{i=1}^3 \alpha_i = L$$

$$\sum_{i=1}^3 \alpha_i \cosh(\omega_o t_i) = \frac{1}{2}(x_u^d(0) + x_s^d(0)) + \frac{\dot{x}_c^d(T)e^{\omega_o T}}{\omega_o}$$

plus the two initial conditions constraints (8) and (9) which can be solved independently. In matrix form  $C\alpha = b$  with  $\alpha$ ,  $b$  and the coefficient matrix  $C$  given by

$$C = \begin{bmatrix} x_u^*(0; u_b^1(t)) & x_u^*(0; u_b^2(t)) & x_u^*(0; u_b^3(t)) \\ \cosh(\omega_o t_1) & \cosh(\omega_o t_2) & \cosh(\omega_o t_3) \\ 1 & 1 & 1 \end{bmatrix}$$

$$\alpha^T = [\alpha_1 \quad \alpha_2 \quad \alpha_3]$$

$$b^T = \left[ x_c^d(0) + \frac{\dot{x}_c^d(0)}{\omega_o} \quad x_c^d(0) + \frac{\dot{x}_c^d(T)e^{\omega_o T}}{\omega_o} \quad L \right]$$

As shown in [6], if the humanoid is initially at rest (CoM starts from  $(0, 0)$ ), the design algorithm autonomously generates a first step which is necessarily negative (backward step) in order to destabilize the humanoid and allow the motion. A more natural way to achieve this is to consider the effect of the swinging foot motion on the overall ZMP.

### C. Design: the underdetermined case

In the previous sections we have implicitly shown that adding a step corresponds to gaining an extra degree of freedom (dof) which can be used to make the solution verify an additional constraint. In order to have a bounded CoM desired trajectory (one constraint (4)) and to comply with an additional  $k$  constraints, being (1) a second order differential equation, we need at least  $k - 1$  variable length steps.

If we plan to make more steps, then we have more design parameters (e.g., the  $\alpha_i$ 's in (16)) than equations; the set of equations representing the  $k$  constraints (including (4))

$$C\alpha = b, \quad C : p \times q, \quad \text{with } q > p \quad (19)$$

is underdetermined and the general solution can be written:

$$\alpha = C^+ b + [I - C^+ C] v \quad (20)$$

with  $v$  an arbitrary vector. The extra unused dofs can therefore be used by choosing  $v$  so to minimise a performance index. This opportunity will be explored in future research while a short example is reported in Section IV-B.

### D. Design with swinging foot compensation

In order to illustrate the potential of our design approach we briefly also show how the effect of the swinging foot can be exactly compensated in the generation of the CoM desired trajectory. The original compensation idea was initially introduced in [12] and used in [2], [13], [17] to cite a few.

We have two masses  $M$  and  $m$  with, respectively, coordinates  $(x_c, z_c)$  and  $(x_{\text{swg}}, z_{\text{swg}})$  representing the body (upper-body) and a swinging foot/leg. Each of the masses will contribute to the total ZMP  $x_{\text{zmp}}^{\text{tot}}$  which is defined through the moment balance, under constant  $z_c$ ,

$$-Mz_c \ddot{x}_c = mz_{\text{swg}} \ddot{x}_{\text{swg}} - Mg(x_c - x_{\text{zmp}}^{\text{tot}}) - mg(x_{\text{swg}} - x_{\text{zmp}}^{\text{tot}}) + m(x_{\text{zmp}}^{\text{tot}} - x_{\text{swg}}) \ddot{z}_{\text{swg}} \quad (21)$$

which we rewrite, setting  $\omega_o^2 = g/z_c$ , as

$$\ddot{x}_c(t) - \omega_o^2 x_c(t) = -\omega_o^2 \frac{m+M}{M} x_{\text{zmp}}^{\text{tot}}(t) - \omega_o^2 \frac{m}{M} \left( \ddot{x}_{\text{swg}}(t) \frac{z_{\text{swg}}(t)}{g} - x_{\text{swg}}(t) \right) - \omega_o^2 \frac{m}{M} (x_{\text{zmp}}^{\text{tot}}(t) - x_{\text{swg}}(t)) \frac{\ddot{z}_{\text{swg}}(t)}{g} \quad (22)$$

The quantities in the previous moment balance are relative (referred to the stance foot) and therefore  $x_{\text{zmp}}^{\text{tot}}$  here indicates the relative total ZMP which is the quantity we focus on to avoid tilting of the stance foot. If  $(x_{\text{swg}}, z_{\text{swg}})$  is the assigned swinging foot trajectory and if  $x_{\text{zmp}}^{\text{tot}}$  is a desired  $x_{\text{zmp}}^{\text{tot,des}}$ , all the terms in the right hand-side are known functions. We can therefore see (22) as the equation of a LIP governing the body CoM  $x_c$  driven by the *equivalent* ZMP  $x_{\text{zmp}}^{\text{eq}}$

$$x_{\text{zmp}}^{\text{eq}}(t) = \frac{m+M}{M} x_{\text{zmp}}^{\text{tot,des}}(t) + \frac{m}{M} \left( \ddot{x}_{\text{swg}}(t) \frac{z_{\text{swg}}(t)}{g} + \ddot{z}_{\text{swg}}(t) \frac{x_{\text{zmp}}^{\text{tot,des}}(t) - x_{\text{swg}}(t)}{g} - x_{\text{swg}}(t) \right) \quad (23)$$

The basic idea is synthetized in Fig. 2.

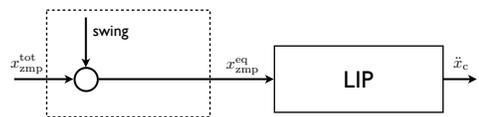


Fig. 2. Swing leg compensation conceptual scheme

From a design perspective, we need to relate the swinging foot motion to the, still to be found, step lengths  $\alpha_i$ . Moreover, to keep the constraints, and therefore  $x_{\text{zmp}}^{\text{eq}}$ , linear in the design parameters  $\alpha_i$ , we choose the vertical motion  $z_{\text{swg}}$  of the swinging leg to be independent from the step length, for example assuming all the steps to be of equal height. For the horizontal motion  $x_{\text{swg}}$ , with  $u_{b,\text{swg}}^{x,i}$  a basis function for the swinging foot  $x$ -motion, we choose

$$x_{\text{swg}}(t) = \sum_{i=1}^N \alpha_i u_{b,\text{swg}}^{x,i}(t) \quad (24)$$

as previously done in (16). With these choices, the equivalent ZMP  $x_{\text{zmp}}^{\text{eq}}$  becomes linear in the  $\alpha_i$  as desired. An illustrative example is reported in Section IV-C.

#### IV. DISCUSSION AND SIMULATIONS

We illustrate and discuss the theoretical results of Section III through some simulations.

##### A. Starting from a generic initial CoM state

Consider the simple situation of Section III-A, i.e. we want the bounded CoM reference trajectory to start from the actual initial state  $(x_c(0), \dot{x}_c(0)) = (0.05, 0.1)$ . The initial desired ZMP trajectory (fixed first step  $\alpha_1 = 0.3$  m,  $t_1 = 0.4$  s) is reported in the first plot of Fig. 3. Starting from  $x_c(0) = 0.05$ , this ZMP requires the initial velocity to be  $\dot{x}_c(0) = 0.28$  (mismatched initial velocity). This may also represent a push recovery maneuver: an impulsive push transfers instantaneously the state in  $(x_c(0), \dot{x}_c(0))$  and the humanoid makes a step in order to recover balance. As shown in Section III-A.2, a properly chosen variable first step allows to start from the actual CoM state. The two design alternatives - variable step length or duration - are reported in Fig. 3. Note that the method may either vary the total path length to achieve desired objectives or maintain a constant, predetermined total path length by allowing the variation of the first step duration.

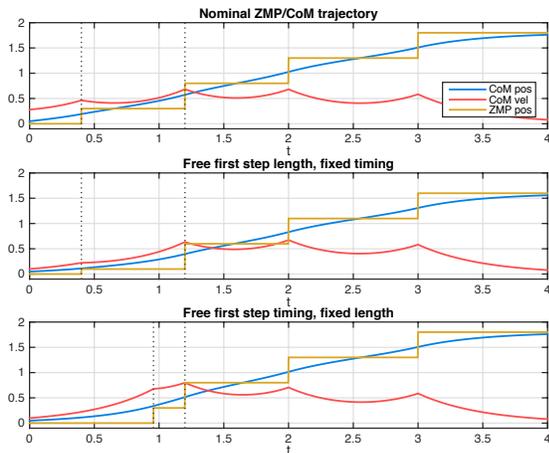


Fig. 3. Design: free first step. Top: nominal trajectory for a given ZMP, initial CoM should be  $(0.05, 0.28)$ . Starting from actual CoM  $(0.05, 0.1)$  with variable step length (Middle) or variable duration (Bottom)

##### B. A simple underdetermined case

Although only one variable step is needed in the previous situation, let's see what can be achieved with two variable steps. We choose, with the notation of Section III-B,

$$x_{\text{zmp}}^{\text{d}}(t) = \alpha_1 u_{\text{b}}^1(t) + \alpha_2 u_{\text{b}}^2(t)$$

with  $\alpha_1$  and  $\alpha_2$  to be determined in order to fulfil the required constraints: actual initial CoM position and velocity the

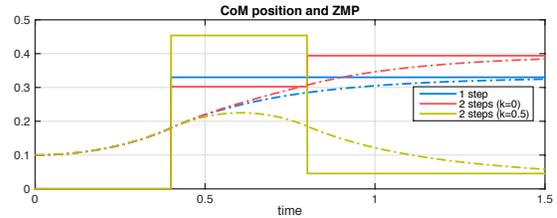


Fig. 4. Design: 1 step vs underdetermined case (2 steps). ZMP (continuous line) and CoM position (dash-dot line). Undetermined solution: pure pseudoinverse solution ( $k = 0$ ) or with a kernel contribution ( $k = 0.5$ )

Boundedness Constraint which becomes<sup>2</sup>

$$\left[ \alpha_1 + \alpha_2 e^{-\omega_o(t_2-t_1)} \right] x_{\text{u}}^*(0; u_{\text{b}}^1(t)) = x_c(0) + \frac{1}{\omega_o} \dot{x}_c(0)$$

In matrix form  $C\alpha = b$ , we have

$$\begin{bmatrix} 1 & e^{-\omega_o(t_2-t_1)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = b$$

$$\text{with } b = \frac{1}{x_{\text{u}}^*(0; u_{\text{b}}^1(t))} \left( x_c(0) + \frac{1}{\omega_o} \dot{x}_c(0) \right)$$

Note that  $b$  corresponds to the one-step solution of (10). The general solution (20) becomes, with  $\Delta_{21} = t_2 - t_1$ ,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{1 + e^{-2\omega_o\Delta_{21}}} \left\{ b \begin{bmatrix} 1 \\ e^{-\omega_o\Delta_{21}} \end{bmatrix} + k \begin{bmatrix} e^{-\omega_o\Delta_{21}} \\ -1 \end{bmatrix} \right\}$$

with  $k \in \mathbf{R}$ . For pure illustration purposes, we report a simulation for the point-foot case in order to highlight that, having no extra constraints, a solution included in this scheme allows the humanoid to recover from an impulsive push even by taking a first step forward and a second backward as shown in Fig. 4 (case  $k = 0.5$ ).

##### C. Design example with swinging foot compensation

We briefly illustrate how the design concept can be applied even when considering the swinging foot influence as explained in Section III-D. For simplicity we choose to deal with a point-foot contact with no double support so that the  $x_{\text{zmp}}^{\text{tot}}$  will just be a sequence of scaled Heaviside functions contributions. For the swinging foot, we choose the horizontal and vertical motion to be expressed in terms of the basis functions illustrated in Fig. 5 which we chose to be a fifth order polynomial for  $u_{\text{b,swg}}^{x,i}$  (which all just differ by a time shifting) and linear for the  $z$  basis function  $u_{\text{b,swg}}^{z,i}$ . Since the  $z$ -basis is chosen to be linear, its acceleration is zero and so will be  $\ddot{z}_{\text{swg}}(t)$ ; this corresponds to neglecting the last contribution in (22).

We simulate a simple 2-steps gait where the swinging foot starts still at its maximum height right above the stance foot (single support); the gait ends similarly with the swinging foot again above the new stance foot (single support). In Fig. 6 it is possible to analyse the effect of each term by comparing its contribution w.r.t. the no swinging foot case

<sup>2</sup>Being  $u_{\text{b}}^2(t) = u_{\text{b}}^1(t - (t_2 - t_1))$ , we use the easy to prove shifting property  $x_{\text{u}}^*(0; g(t - T)) = e^{-\omega_o T} x_{\text{u}}^*(0; g(t))$

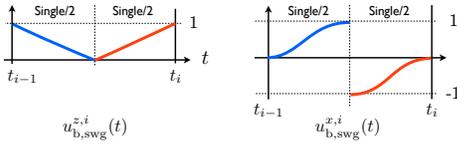


Fig. 5. Design: possible choice of the swinging foot basis functions, linear vertical  $u_{b,swg}^{z,i}(t)$  and polynomial horizontal  $u_{b,swg}^{x,i}(t)$  motions

(first plot). In particular the second plot shows how the contribution of  $x_{swg}$  affects the equivalent ZMP by introducing a destabilizing effect due to the forward movement of the swinging foot. This can be seen as an “almost static” contribution of the swinging foot. The combined effect with the acceleration term  $\ddot{x}_{swg}$  and the vertical motion  $z_{swg}$  is reported in the third plot.

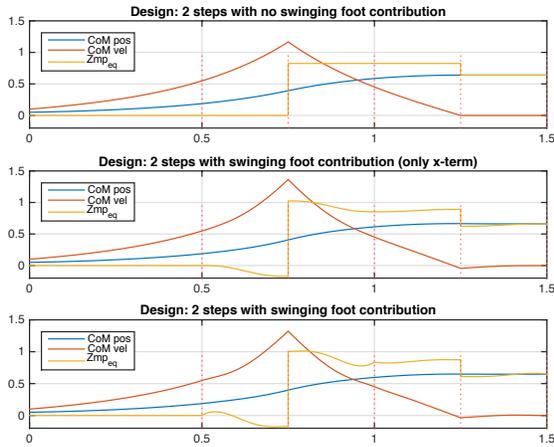


Fig. 6. Design comparison: no swinging foot compensation (top), with only the  $x_{swg}$  contribution (middle) and with also the  $\ddot{x}_{swg}z_{swg}$  term (bottom)

## V. CONCLUSIONS

We presented an approach for simultaneously synthesizing desired trajectories for both the CoM and ZMP of a bipedal robot. We applied our method to a variety of problems, ranging from simple LIP models to more complicated models that incorporate swing dynamics. We evaluated our approach in simulation, which showed the method to be effective. Future work will focus on a real-world humanoid robot implementation and the integration of the actual ZMP knowledge in the correction of the generated CoM desired reference.

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