Control Systems - January 10, 2022
with updated solution

1) Consider the plant

$$P(s) = \frac{2000}{s\,(0.001s + 1)}$$

Design a feedback control system such that

1. at steady state, when a reference $r(t) = t\delta_{-1}(t)$ is applied, the error is $|e_{ss}(t)| \leq 0.001$
2. a constant disturbance acting at the plant's input does not affect the output at steady state
3. we have a crossover frequency $\omega_c^* = 10$ rad/s and a phase margin $PM \geq 45°$

2) Using the Nyquist criterion, discuss in terms of K the closed loop stability of the unit feedback system having the following loop function

$$L(s) = \frac{K(s+1)^2}{s^2(s+0.01)}$$

Check your result with the Routh criterion.

3) Consider the plant shown in Fig. 1

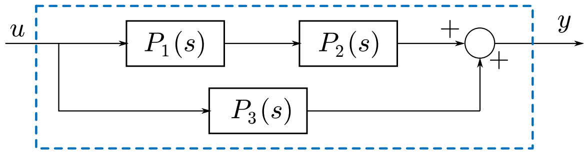

Figure 1: Plant for Exercise 3.

with

$$P_1(s) = \frac{1}{s-1}, \quad P_2(s) = \frac{s+1}{s+2}, \qquad P_3(s) = \frac{-3}{s+2}$$

1. Build, if possible, an asymptotic observer.
2. Is the system stabilizable using the separation principle? Explain.
3. Can we stabilize the system with an output feedback and a static controller $C(s) = K_c$? If yes, find the interval for the stabilizing values of K_c. Check your result with the root locus plot.
4. Find, if possible, an output feedback controller which assigns the poles in $\{-1, -1, -2\}$. Any particular remark about the resulting controller?

4) Consider again the plant given in Exercise 3 and determine:

1. the plant's natural modes,
2. its impulse response,
3. the set, if any, of initial conditions that lead to a free output response (ZIR) which does not diverge.

1 - Sol. The second specification which requires making the closed loop system astatic w.r.t. a constant disturbance acting at the plant's input (point 2) requires the presence of a pole in $s = 0$ before the entry point of the disturbance and therefore this pole will necessarily be in the controller. This pole makes the closed loop system of type 2 due to the presence of another pole in $s = 0$ in the plant, and therefore (when the closed loop system will be made asymptotically stable) the error at steady state w.r.t. a ramp input (order 1) will be zero. This implies that there is no requirement on the controller gain (except being positive in order to be able to apply the Bode stability theorem). From the modified plant Bode diagrams we see that we need

- a phase increase by at least 45° at 10 rad/s
- an attenuation of the magnitude by -26 dB (we need to evaluate the magnitude of the modified plant at 10 rad/s: we have twice the contribution of the monomial at denominator, that is -40 dB, and the contribution of the gain $2000|_{dB} = (2 \times 1000)|_{dB} = 2|_{dB} + 60\,\mathrm{dB} = -66\,\mathrm{dB}$ so a total of 66-40 = 26 dB).

We can therefore choose the remaining part of the controller in such a way to achieve a loop phase at 10 rad/s greater equal than $-135°$ (so to guarantee the required phase margin of at least 45°) and then assign the required crossover frequency with a gain. Two possible alternatives are:

1. choose a lead function that guarantees the required increase of 45° (e.g., for example $m_a = 6$ and normalized frequency roughly equal to 1.7, so that $\omega_a = 1.7/10$). The required attenuation is therefore of -26 dB (modified plant) plus -5 dB (to compensate for the amplification introduced by the lead function), that is $K_c = 0.1 * 0.5 * 10^{-5/20}$ (corresponding respectively to -20 dB, -6 dB and -5 dB).
2. Since the controller already has a pole, we can obtain the required increase of phase with a simple zero in -10 that is with the binomial $(1 + 0.1s)$ so not to alter the gain. This zero with cut-off frequency 10 rad/s also introduces an amplification of 3 dB and therefore the gain needs to be chosen as $K_c = 0.1 * 0.5 * 1/\sqrt{2}$.

The resulting two loop functions are shown in Figure 2.

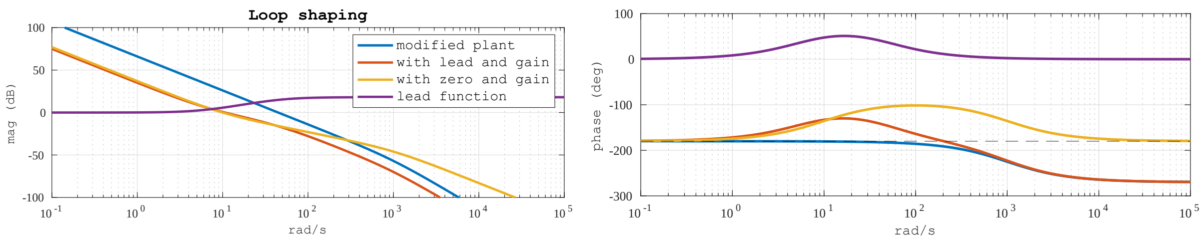

Figure 2: Bode plots Exercise 1.

The closed loop system is asymptotically stable since the loop function has no poles with positive real part, positive gain, a unique crossover frequency and a positive phase margin (Bode's stability theorem).

Typical errors:

- getting the wrong cut off frequencies of the binomial terms (in particular of $(1 + 0.001\,s)$.
- Not noticing that the disturbance acts at the plant's input and not output and therefore the plant's pole in $s = 0$ is not making the closed loop astatic w.r.t. the constant disturbance at the plant's input.
- Assigning a value to the controller gain (typically $K_c = 1/2$ since it satisfies the reference specification) from the first requirement, even if the introduction of the pole in $s = 0$ in the controller from the astatism specification will make the requirement on the gain useless. Some, after having chosen K_c, used a lead (for the phase increase) and a lag function to obtain the required attenuation which guarantees the desired crossover frequency; although this final controller meets the requirements, this is not a stated choice made by the designer but rather a consequence.
- BTW This problem is, in practice, identical to Problem 4 in `PublishedExamsB.pdf`.

2 - Sol. The Bode canonical form of $L(s)$ is

$$L(s) = \frac{K(s+1)^2}{s^2(s+0.01)} = 100K\frac{(1+s)^2}{s^2(1+100s)}$$

while the approximate Bode plots (for $K = 1$) are shown on the left of Figure 3. In this situation, the phase plot crosses $-\pi$ at exactly 1 rad/s where also the magnitude is 1 (that is 0 dB); therefore in this first approximation the Nyquist diagram would pass exactly through the point $(-1, 0)$. However we know that at the cut-off frequency of the double zero in $s = -1$, the true magnitude differs by 6 dB (that is approximately 2) w.r.t. the segment approximation and therefore the point P (corresponding to the crossing at $-\pi$) is to the left of the point $(-1, 0)$ since the magnitude is greater than 1. Clearly if the point $(-1, 0)$ lies in the area A, the Nyquist criteria is met (no encirclements and no open loop poles with positive real part) and the closed loop system is asymptotically stable while if it lies in the area B the closed loop system is unstable. For negative values of K there is an additional contribution of $-\pi$ to the phase plot, the whole Nyquist diagram rotates by $-\pi$ and encircles clockwise the point $(-1, 0)$ independently from the absolute value of K (closure at infinity) so the closed loop is unstable.

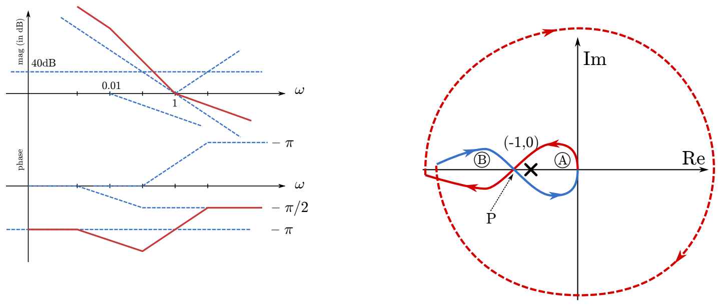

Figure 3: Approximate Bode plots (left) and Nyquist diagram (right) for Exercise 2.

There is clearly a critical value for the gain K (for sure smaller than 1 from the previous discussion) and it can be found using the Routh criterion on the closed loop poles polynomial $p_{CL}(s)$

$$p_{CL}(s) = s^2(s + 0.01) + K(s + 1)^2 = s^3 + (K + 0.01)s^2 + 2K\, s + K$$

so that the Routh table is (assuming the necessary conditions $K > 0$ and $K > -0.01$ are satisfied) becomes

$$\left| \begin{array}{cc} 1 & 2K \\ K + 0.01 & K \\ (*) & \\ K & \end{array} \right.$$

where $(*)$ is given by

$$(*) = 2(K)^2 + 0.02K - K = 2(K - 0.49)K$$

which is positive provided $K > 0.49$. The closed loop system is asymptotically stable for $K > K_{\text{crit}} = 0.49$. As a (numerical) check, for $K = K_{\text{crit}} = 0.49$, the closed loop poles are in - 0.5 and $\pm 0.9899j$.

As an exercise one could trace the root locus.

Typical errors:

- numerical issues: for example transforming $s + 0.01$ in $0.01(1 + 100s)$, or in the Routh table
- Wrong Bode diagrams, in particular for the phase or getting the wrong cut-off frequency. Some still try to compute the frequency response explicitly by expanding all the terms. We have seen all the separate contributions to avoid this computation.
- Wrong Nyquist, in particular either the total absence of the closure at infinity or doing the closure counterclockwise instead of clockwise.
- Not discussing or commenting the Nyquist diagram: just drawing a correct Nyquist diagram does not fully answer the question.
- BTW We have seen a very similar example during the course.

3 - Sol. The interconnection is made of a series (between $P_1(s)$ and $P_2(s)$) and then a parallel with $P_3(s)$. While in the series interconnection there is no pole cancellation, we clearly see a cancellation of the common pole in $s = -2$ in the parallel interconnection. The transfer function of the interconnected system is computed as the series of $P_1(s)$ and $P_2(s)$ which is then in parallel with $P_3(s)$

$$P(s) = P_1(s)P_2(s) + P_3(s) = \frac{s + 1}{(s - 1)(s + 2)} - \frac{3}{s + 2} = \frac{-2s + 4}{(s - 1)(s + 2)}$$

The final number of poles is 2 while we know the interconnected system has 3 eigenvalues so there is a one-dimensional hidden dynamics.

We know from the theory (see slides on parallel interconnection) that two systems in parallel which share a common pole leads to both a loss of controllability and observability. However, since the common eigenvalue (hidden dynamics) is characterized by the eigenvalue $\lambda_2 = -2$ (negative real part), the unobservable dynamics is asymptotically stable and therefore the plant is detectable. We can therefore build an asymptotic observer. Note that this is the only hidden dynamics generated by the interconnection.

1) As a check, we can derive the plant's state space representation by interconnecting the 3 single transfer functions representations (no known hidden dynamics in any of the three subsystems since we are given the transfer functions)

$$\begin{aligned}
P_1(s) &= \frac{1}{s-1} &\Rightarrow\quad& \dot{x}_1 = x_1 + u_1, & y_1 &= x_1\\
P_2(s) &= \frac{s+1}{s+2} &\Rightarrow\quad& \dot{x}_2 = -2x_2 + u_2, & y_2 &= -x_2 + u_2\\
P_3(s) &= \frac{-3}{s+2} &\Rightarrow\quad& \dot{x}_3 = -2x_3 + u_3, & y_3 &= -3x_3
\end{aligned}$$

with the interconnection equations $u = u_1 = u_3$, $u_2 = y_1$ and $y = y_2 + y_3$. The resulting plant's state space representation, choosing $x^T = (x_1\ x_2\ x_3)^T$, is

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & -1 & -3 \end{pmatrix} x$$

The associated observability matrix is

$$O = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 2 & 6 \\ 2 & -4 & -12 \end{pmatrix}, \qquad \det(O) = 0, \qquad \mathrm{Ker}(O) = \mathrm{gen}\left\{ \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right\}$$

We can perform a Kalman decomposition w.r.t. observability with the following change of coordinates

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} = T \qquad \text{(surprinsingly)}$$

In the new coordinates $z = Tx$ we obtain the following decomposition

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \mathbf{0} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$$

Note the asymptotically stable unobservable subsystem characterized by the eigenvalue $\lambda_2 = -2$ as predicted. Moreover $(\tilde{A}_{11}, \tilde{C}_1)$ is observable and has the corresponding observability matrix

$$\mathrm{obs}(\tilde{A}_{11}, \tilde{C}_1) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \star & 0.5 \\ \star & 0.5 \end{pmatrix} = \begin{pmatrix} \Diamond & \bar{\gamma}^T \end{pmatrix}$$

We can choose $\tilde{K}_1$ so to assign the eigenvalues to $\tilde{A}_{11} - \tilde{K}_1\tilde{C}_1$. For example, in order to assign the eigenvalues $(\lambda_1^*, \lambda_2^*) = (-10, -20)$ solutions of $p^*(\lambda) = (\lambda + 10)(\lambda + 20) = \lambda^2 + 30\lambda + 200$, we need to choose $\tilde{K}_1$ as

$$\tilde{K}_1 = p^*(\tilde{A}_{11})\bar{\gamma}^T = \begin{pmatrix} 231 & 0 \\ 29 & 144 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 115.5 \\ 86.5 \end{pmatrix}$$

or in the original coordinates

$$K = T^{-1} \begin{pmatrix} \tilde{K}_1 \\ 0 \end{pmatrix}$$

The final observer is

$$\dot{\xi} = (A - KC)\xi + Bu + Ky$$

1 bis) A possible alternative consists in computing first the series interconnection $P_{12}(s) = P_2(s)P_1(s)$, noticing that there are no cancellations and taking a state space representation (typically in the controller canonical form)

$$P_{12}(s) = \frac{s+1}{(s-1)(s+2)} = \frac{s+1}{s^2 + s - 2} \quad \Rightarrow \quad A_{12} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and then making the parallel with $\mathcal{S}_3$, obtaining (being $A_3 = -2$, $B_3 = 1$ and $C_3 = -3$),

$$A = \begin{pmatrix} A_{12} & \mathbf{0} \\ \mathbf{0} & A_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} B_{12} \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} C_{12} & C_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \end{pmatrix}.$$

The eigenvalues, as a check, are clearly $\lambda_3 = -2$ and the eigenvalues of A_{12}, that is $\lambda_1 = -1$ and $\lambda_2 = -2$. The resulting observability matrix and possible change of coordinates are

$$O = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 0 & 2 & -12 \end{pmatrix}, \qquad \det(O) = 0, \qquad \mathrm{Ker}(O) = \mathrm{gen}\left\{ \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix} \right\}$$

and

$$T^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -1 \end{pmatrix} = T \qquad \text{(again?!)}$$

The subsequent steps are similar (just different numerical values).

2) We need also to verify if the system is stabilizable via state feedback (necessary condition for output stabilization). Computing the controllability matrix using the first state space representation,

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -2 & 4 \end{pmatrix}, \qquad \det(P) = 0$$

we clearly see that P has rank 2 (a minor of dimension 2 is evident). We know from the theory that the eigenvalue $\lambda_2 = -2$ is uncontrollable (otherwise, from the previous step, having found the eigenvalues, we can check the controllability of the unstable eigenvalues with the PBH test) and now we know that the uncontrollable subsystem has dimension one so it is characterized by the asymptotically stable eigenvalue λ_2. The system is therefore stabilizable with state feedback.

Since the system has an asymptotically stable uncontrollable subsystem and admits an asymptotic observer (or equivalently has asymptotically stable unobservable dynamics), it can be stabilized via output feedback using the separation principle (or directly since the system is detectable and stabilizable).

3) The system transfer function (we can consider the transfer function since the hidden dynamics is asymptotically stable and we are only interested in the closed loop stability) is

$$P(s) = \frac{-2(s-2)}{(s-1)(s+2)}$$

The transfer function can alternatively be determined from the previous Kalman decomposition after we compute also the new input matrix $\tilde{B} = TB$

$$\tilde{B} = TB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} \tilde{B}_1 \\ -1 \end{pmatrix}$$

Since

$$\begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}^{-1} = \begin{pmatrix} M_{11}^{-1} & 0 \\ \# & M_{22}^{-1} \end{pmatrix}$$

when we compute the transfer function (independent from the particular coordinates) we have

$$P(s) = C(sI-A)^{-1}B = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B} = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix} \begin{pmatrix} sI - \tilde{A}_{11} & \mathbf{0} \\ -\tilde{A}_{21} & sI - \tilde{A}_{22} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{B}_1 \\ -1 \end{pmatrix} = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1$$

that is

$$P(s) = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s-1 & 0 \\ -1 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{(s-1)(s+2)} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 1 & s-1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{-2(s-2)}{(s-1)(s+2)}$$

When we use a pure static controller $C(s) = K_c$, the closed loop pole polynomial becomes

$$p(s, K_c) = s^2 + s - 2 - 2K_c s + 4K_c = s^2 + s(1 - 2K_c) - 2(1 - 2K_c)$$

which clearly cannot have $1-2K_c$ simultaneously positive and negative (here it is a necessary and sufficient condition). Therefore the closed loop system is not asymptotically stable. To plot the root locus, we define $K' = -2K_c$ so that

$$L(s) = C(s)P(s) = \frac{-2K_c(s-2)}{(s-1)(s+2)} = \frac{K'(s-2)}{(s-1)(s+2)}$$

is in the correct form. The closed loop pole polynomial becomes

$$p(s,K') = s^2 + s(1+K') - 2(1+K')$$

The resulting possible root locus is shown in Fig. 4 where the position of the singular point s^* is still unknown (left) while the correct locus is shown on the right. If we are in the first situation (left plot), there exists an interval of negative values for K' such that all the closed loop poles lie simultaneously in the left half plane.

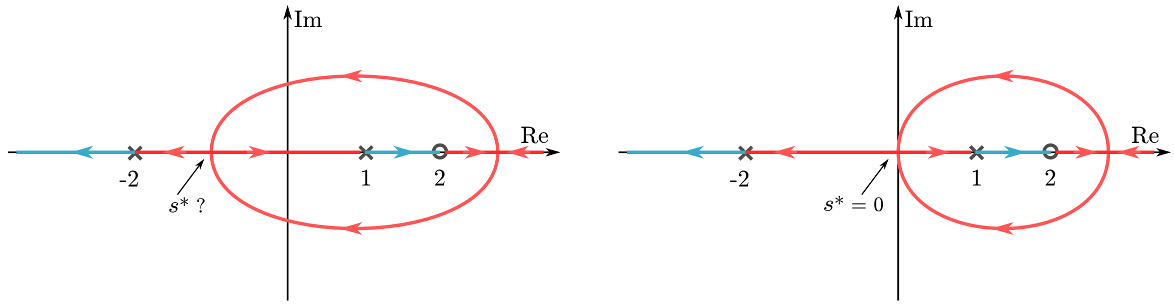

Figure 4: Root locus: singular point has not been determined (left) and with $s^* = 0$ (right) for Exercise 3.

It is important to determine, in this case, the position of the singular point s^*. We can go through the definition and solve the equation for the candidate singular points

$$(s-2)\frac{d}{ds}(s^2+s-2) - (s^2+s-2)\frac{d}{ds}(s-2) = (s-2)(2s+1) - s^2 - s + 2 = s(s-4) = 0$$

or similarly solve (no big difference since there are no multiple open loop zeros or poles)

$$\frac{1}{s-1} + \frac{1}{s+2} - \frac{1}{s-2} = \frac{s^2-4+s^2-3s+2-s^2-s+2}{*} = \frac{s^2-4s}{*} = 0.$$

Clearly, being real, both solutions $s_1^* = 0$ and $s_2^* = 4$ are singular points. We are interested in s_1^*. Note that it is also evident from the pole polynomial that for $K' = -1$ we have

$$p(s, K' = -1) = s^2$$

which clearly shows the presence of the singular point with multiplicity 2 at the origin. We therefore see that there is no interval of values for K' (and therefore of K) for which the closed loop is asymptotically stable.

4) Again, being the hidden dynamics asymptotically stable, we can also stabilize the plant from the output using the pole placement technique. Since the plant has dimension $n = 2$, the controller is going to be of dimension $r = n - 1 = 1$

$$P(s) = \frac{-2s+4}{s^2+s-2} \qquad \Rightarrow \qquad C(s) = \frac{as+b}{s+c}$$

The closed loop polynomial is then

$$\begin{aligned} p_{\text{CL}}(s) &= (s^2+s-2)(s+c) + (-2s+4)(as+b) \\ &= s^3 + (1+c-2a)s^2 + (4a-2b+c-2)s + 4b - 2c \end{aligned}$$

The desired polynomial is

$$p^*_{\text{CL}}(s) = (s+2)(s+1)^2 = s^3 + 4s^2 + 5s + 2$$

We therefore have to solve the following set of equations (equating the coefficients of the terms with the same power)

$$\begin{aligned} -2a + c &= 3 \\ 4a - 2b + c &= 7 \\ 2b - c &= 1 \end{aligned}$$

which can be (simply) solved, giving

$$a = 2, \quad b = 4, \quad c = 7 \qquad \Rightarrow \qquad C(s) = \frac{2(s+2)}{s+7}$$

We immediately notice that the controller cancels the plant's pole in $s = -2$ and this results from having required a closed loop pole coincident with a plant's pole.

Typical errors:

- Numerical issues: copying $(s-2)$ instead of $(s+2)$, for example, clearly changes everything; errors in matrix multiplication, computing a rank, computing an inverse.
- Inconsistent results: for example noticing the eigenvalue coincidence in the parallel interconnection and knowing that there is a loss of observability and controllability but obtaining a non-singular observability (or controllability) matrix.
- Apparently some have misunderstood how to detect the presence of hidden dynamics when interconnecting, for example, in parallel. Take for example two systems in parallel described respectively by $F_1(s)$ and $F_2(s)$ with

  $$F_1(s) = \frac{s+1}{(s-1)(s+2)} \qquad F_2(s) = \frac{-3}{(s+2)}$$

  System 1 has 2 poles and therefore 2 eigenvalues, while System 2 has 1 pole (eigenvalue). The parallel has dimension 3 (the sum of each subsystem dimension) but when we compute the sum of the two transfer functions

  $$F(s) = F_1(s) + F_2(s) = \frac{s+1}{(s-1)(s+2)} + \frac{-3}{(s+2)} = \frac{s+1-3(s-1)}{(s-1)(s+2)} = \frac{-2(s-2)}{(s-1)(s+2)}$$

  apparently there was no simplification between the numerator and denominator (this is clearly due to the fact that the two denominators have a common factor). However we know the interconnected system has 3 eigenvalues so there is a one dimensional hidden dynamics.
- Computing the interconnected system transfer function from the state space realization (3 dimensional) is a huge loss of time.
- Comment: one could say that since we know there is an unobservable hidden dynamics (from the parallel interconnection) the resulting observable part will coincide (modulo a possible change of coordinates) with the realization of the transfer function of the interconnected system

  $$P(s) = \frac{-2s+4}{(s-1)(s+2)} \qquad \Rightarrow \qquad A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 4 & -2 \end{pmatrix}$$

  However, doing this will result in a 2-dimensional observer which will reconstruct asymptotically only the state of the observable sub-system while we are interested in reconstructing the whole 3-dimensional system state.

4 - Sol. From the plant's realization (a different one but we could have used the previous one), we have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$$

and the eigenvalues, due to the lower triangular structure of the A matrix, are easily seen to be $\lambda_1 = 1$ and $\lambda_2 = -2$ with algebraic multiplicity equal to 2. This was evident from the poles/eigenvalues of the single subsystems and remembering that both series and parallel interconnections do not alter the eigenvalues. To determine the natural mode corresponding to λ_2, we need to determine its geometric multiplicity and therefore look at the dimension of the nullspace of $A - \lambda_2 I$

$$\text{Ker}(A - \lambda_2 I) = \text{Ker} \begin{pmatrix} 3 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{gen} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad m_g = \dim\left[\text{Ker}(A - \lambda_2 I)\right] = 2$$

The natural modes are then

$$e^t, \quad e^{-2t}$$

There is no natural mode te^{-2t}. We know that the (output) impulse response is the inverse Laplace transform of the transfer function which can be expanded in simple fractions as

$$P(s) = \frac{R_1}{s-1} + \frac{R_2}{s+2} \qquad \text{with} \quad R_1 = \left[(s-1)P(s)\right]_{s=1} = \frac{2}{3} \qquad R_2 = \left[(s21)P(s)\right]_{s=-2} = -\frac{8}{3}$$

and therefore

$$p(t) = \mathcal{L}^{-1}\left(P(s)\right) = R_1 e^t + R_2 e^{-2t}$$

Finally, since the geometric multiplicity of $\lambda_2 = -2$ is 2, there are two independent associated eigenvectors (shown previously) and therefore any initial condition belonging to the subspace spanned by these two eigenvectors will lead to a converging to zero state trajectory.