Control Systems

Control basics II

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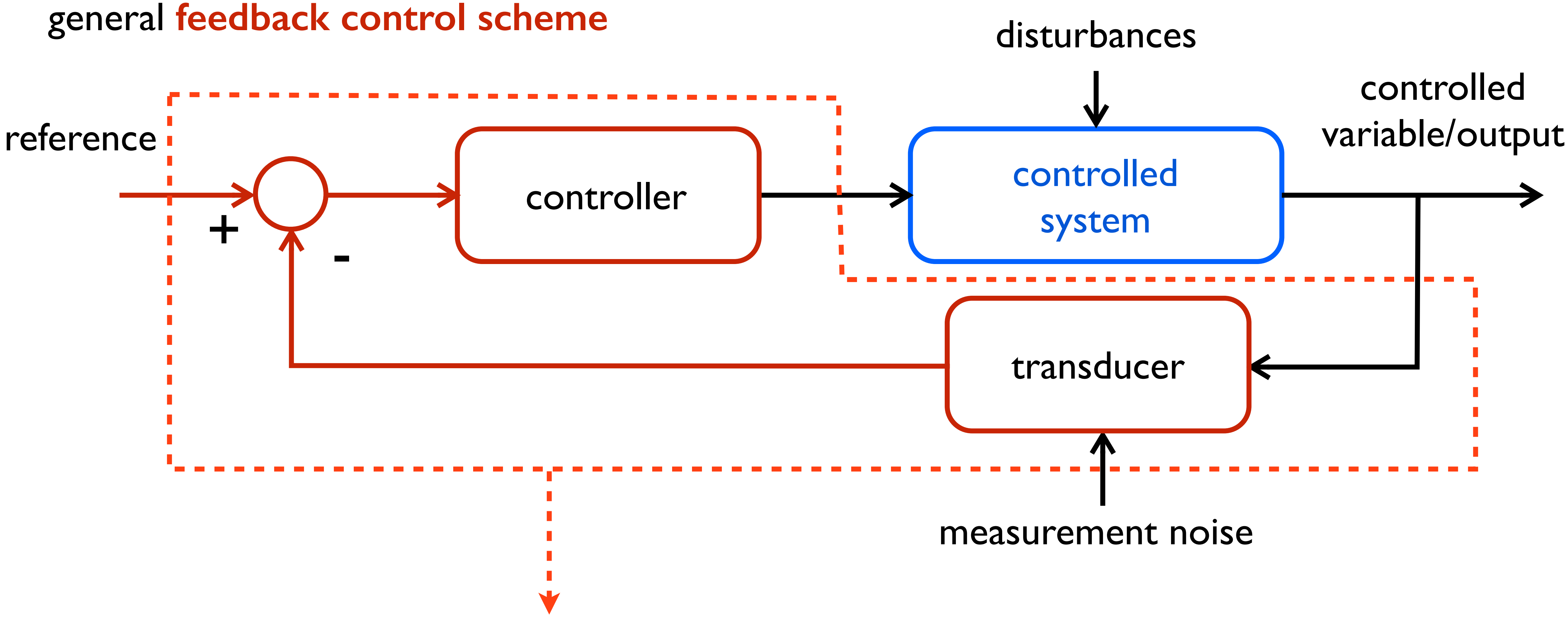
Outline

- a general feedback control scheme
- typical specifications
- the 3 sensitivity functions
- constraints in the specification definitions
- steady-state requirements w.r.t. references
- system type
- steady-state requirements w.r.t. disturbances
- effects of the introduction of integrators
- transient characterization in the frequency domain
- closed-loop to open-loop transient specifications

general **feedback control scheme**



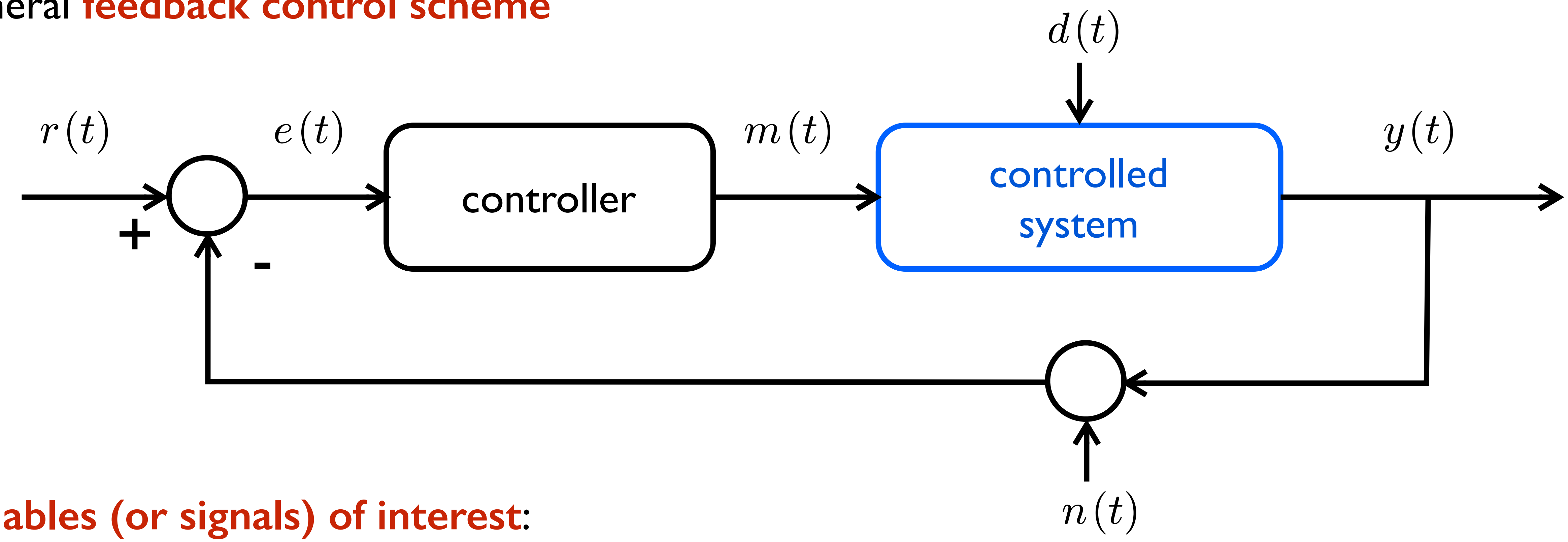
all this is “built” on top of the system to be controlled

→ this choice will influence therefore the type of measurement noise present

→ we focus on the design of the controller rather than on the measurement related issues

all this is “built” on top of the system to be controlled

general **feedback control scheme**

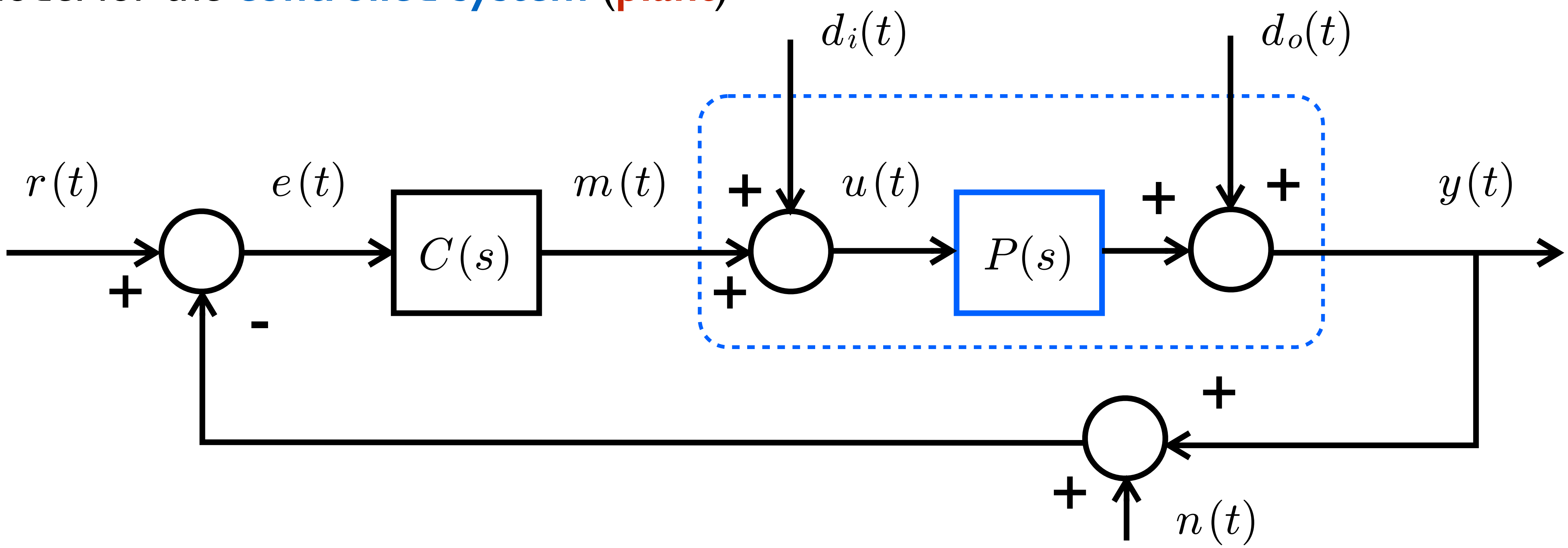


variables (or signals) of interest:

- $r(t)$ reference signal: represents the desired behavior of the controlled output
- $e(t)$ error: this is the error only when $n(t) = 0$
- $m(t)$ control input: output of the controller
- $d(t)$ disturbance: exogenous input affecting the control system
- $y(t)$ controlled output
- $n(t)$ measurement noise: noise introduced from the measurement devices

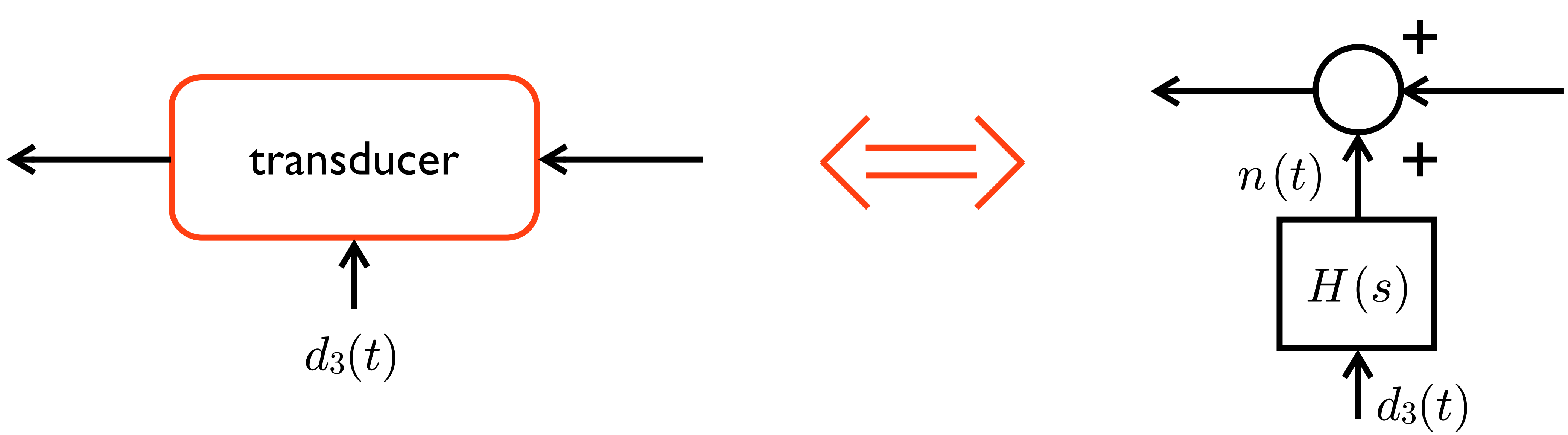
controller: a dynamical (or static) system represented either by its transfer function $C(s)$ or, equivalently, by its state space representation (A_c, B_c, C_c, D_c). It will depend upon the control technique used to solve the control problem

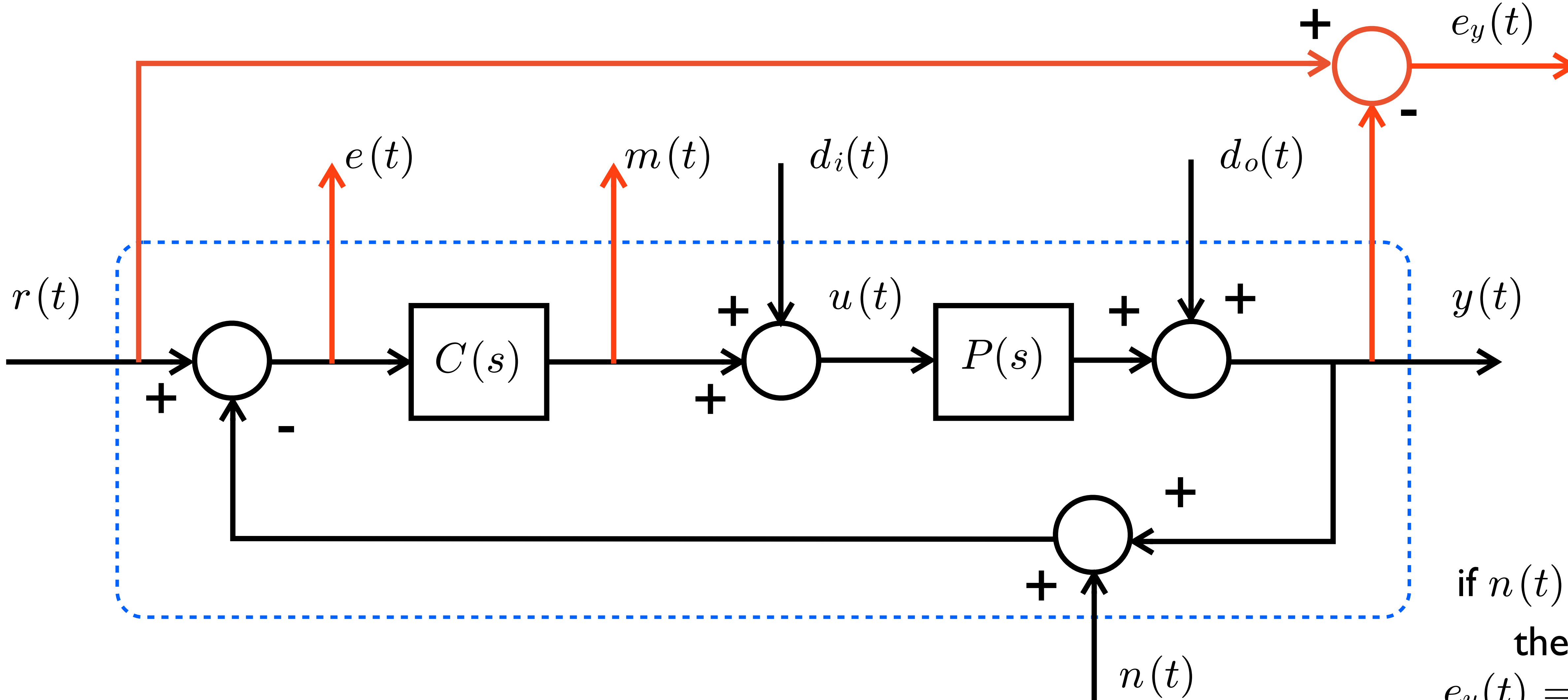
+ model for the controlled system (plant)



distinguish weather the disturbance acts at the input $d_i(t)$ or at the output $d_o(t)$ of the plant

signals $d_i(t), d_o(t)$ and $n(t)$ may be the output of some other system too





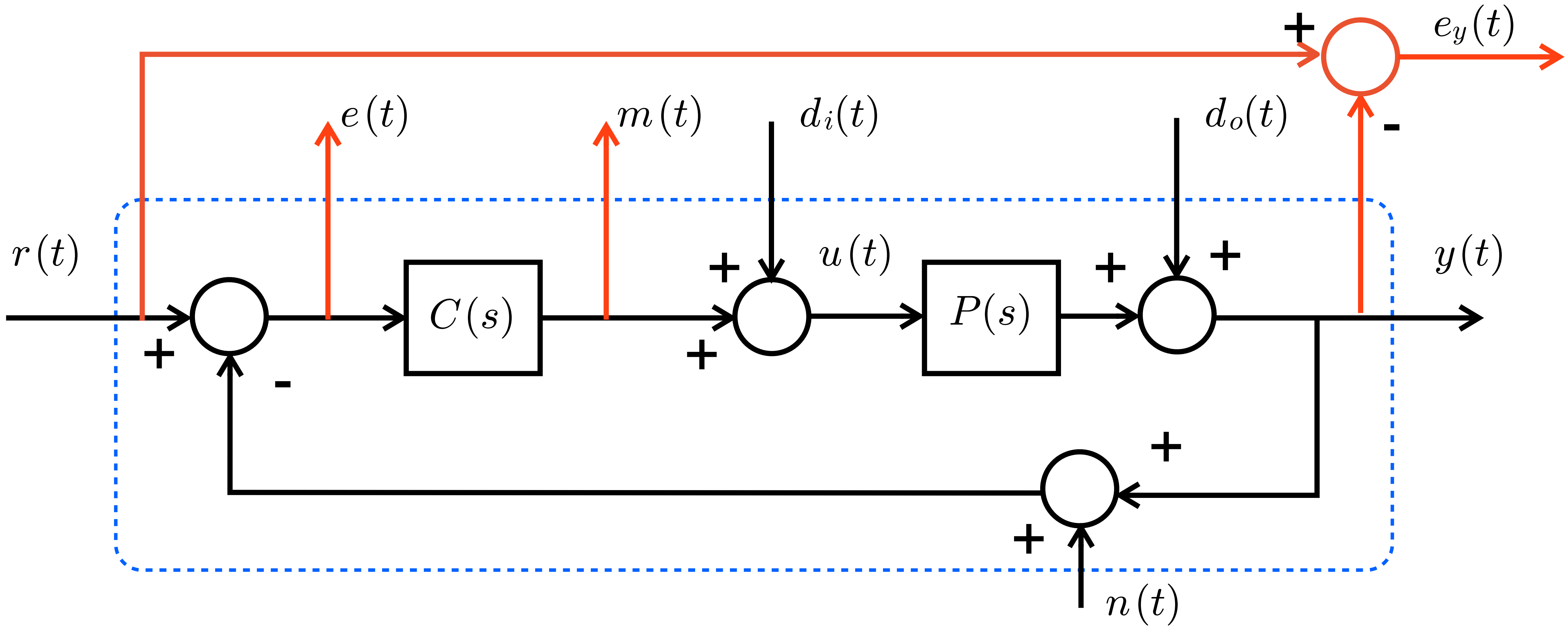
if $n(t) = 0$ then $e_y(t) = e(t)$

a more meaningful error signal is the **tracking error** $e_y(t) = r(t) - y(t)$ since it gives the real measure of how effective is the control action

$$
\begin{aligned}
e_y(s) = r(s) - y(s) &= r(s) - T(s)r(s) - P(s)S(s)d_i(s) - S(s)d_o(s) + T(s)n(s) \\
&= S(s)r(s) - P(s)S(s)d_i(s) - S(s)d_o(s) + T(s)n(s)
\end{aligned}
$$

since $S(s) + T(s) = 1$

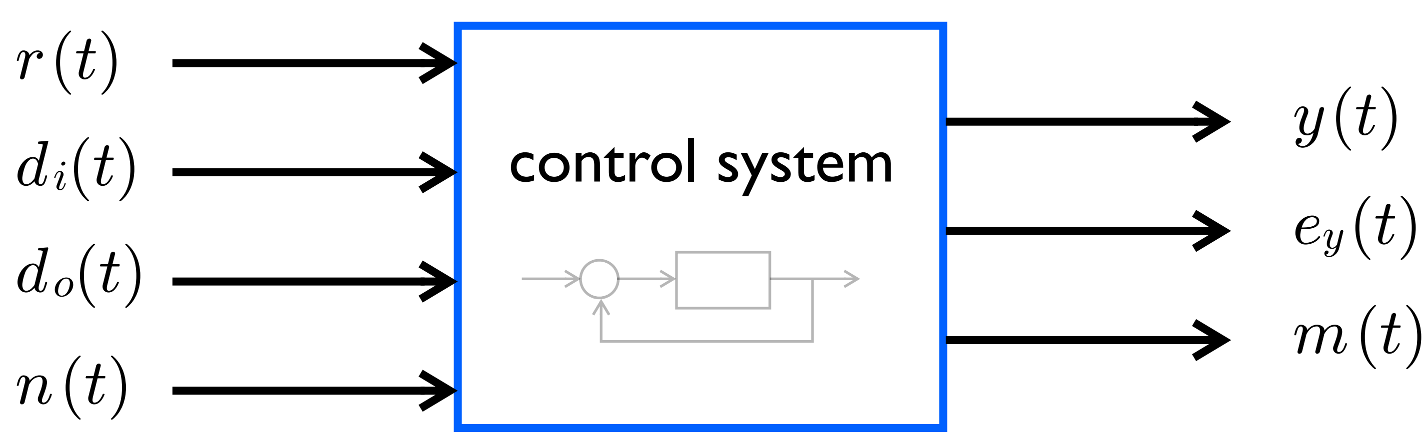
thus if we obtain good tracking ($S(s)$ small) we also reintroduce the measurement noise ($T(s)$ large)

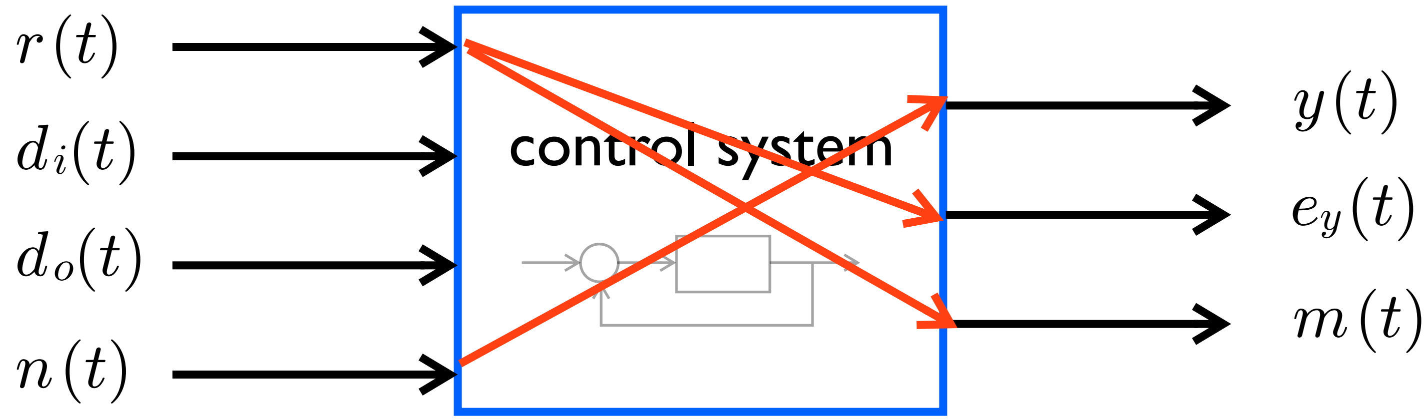


We define the **control system** as the overall interconnected system (plant & controller)

- several inputs act simultaneously on the control system
- we may be interested in several (output) variables

therefore the control system is a dynamical system with multiple inputs and outputs





we may need to give specifications on different Input/Output pairs

the effect of each input on any output is determined using the **superposition principle** that is by considering each input at a time (for example if we want to determine the effect of the input $r(t)$ on $m(t)$ we set $d_i(t) = d_o(t) = n(t) = 0$ and compute the single input - single output transfer function from $r(t)$ to $m(t)$). For the controlled output we obtain

$$y(s) = \underbrace{\frac{C(s)P(s)}{1+C(s)P(s)} r(s)}_{\substack{\text{output when} \\ d_i(t)=0 \\ d_o(t)=0 \\ n(t)=0}} + \underbrace{\frac{P(s)}{1+C(s)P(s)} d_i(s)}_{\substack{\text{output when} \\ r(t)=0 \\ d_o(t)=0 \\ n(t)=0}} + \underbrace{\frac{1}{1+C(s)P(s)} d_o(s)}_{\substack{\text{output when} \\ r(t)=0 \\ d_i(t)=0 \\ n(t)=0}} - \underbrace{\frac{C(s)P(s)}{1+C(s)P(s)} n(s)}_{\substack{\text{output when} \\ r(t)=0 \\ d_i(t)=0 \\ d_o(t)=0}}$$

define the **loop function** $L(s) = C(s)P(s)$ and the following transfer functions

$$S(s) = \frac{1}{1 + C(s)P(s)} = \frac{1}{1 + L(s)}$$ **sensitivity function**

$$T(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{L(s)}{1 + L(s)}$$ **complementary sensitivity function**
since $S(s) + T(s) = 1$

$$S_u(s) = \frac{C(s)}{1 + C(s)P(s)} = \frac{C(s)}{1 + L(s)}$$ **control sensitivity function**

check as an exercise that, using the superposition principle, we can rewrite all the outputs of interest as

$$\begin{aligned} y(s) &= T(s)r(s) + P(s)S(s)d_i(s) + S(s)d_o(s) - T(s)n(s) \\ e(s) &= S(s)r(s) - P(s)S(s)d_i(s) - S(s)d_o(s) - S(s)n(s) \\ m(s) &= S_u(s)r(s) - S_u(s)P(s)d_i(s) - S_u(s)d_o(s) - S_u(s)n(s) \\ e_y(s) &= S(s)r(s) - P(s)S(s)d_i(s) - S(s)d_o(s) + T(s)n(s) \end{aligned}$$

note that all 12 possible transfer functions (between each input and output) are expressed in terms of the three sensitivity functions $S(s), T(s)$ and $S_u(s)$.

Model uncertainties (plant)

recall that the plant may be subject to model uncertainties. We can design the controller based on a **nominal plant** but in general we would like the same controller to be still effective under models which do not differ too much from the nominal one

- **Parametric uncertainties**:

  the real (**perturbed**) parameters of the controlled system (plant) are different from the ones (**nominal**) used to design the controller. Possible causes are:

  - slowly time-varying parameters
  - wear & tear (damage caused by use)
  - difficulty to determine true values
  - change of operating conditions (linearization), ...

- **Unmodeled dynamics**:

  typically high-frequency

  - dynamics deliberately neglected for design simplification,
  - difficulty in modeling

Parametric uncertainties

(MSD example)

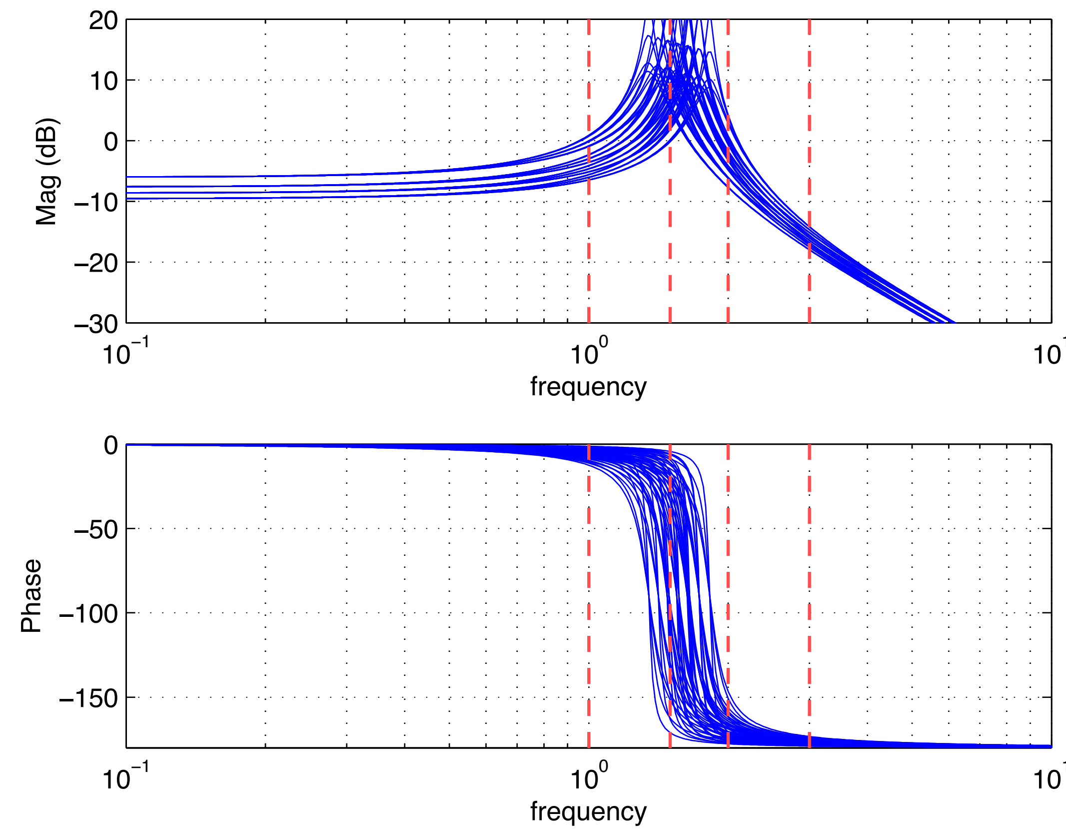
$$P(s) = \frac{1}{ms^2 + \mu s + k}$$

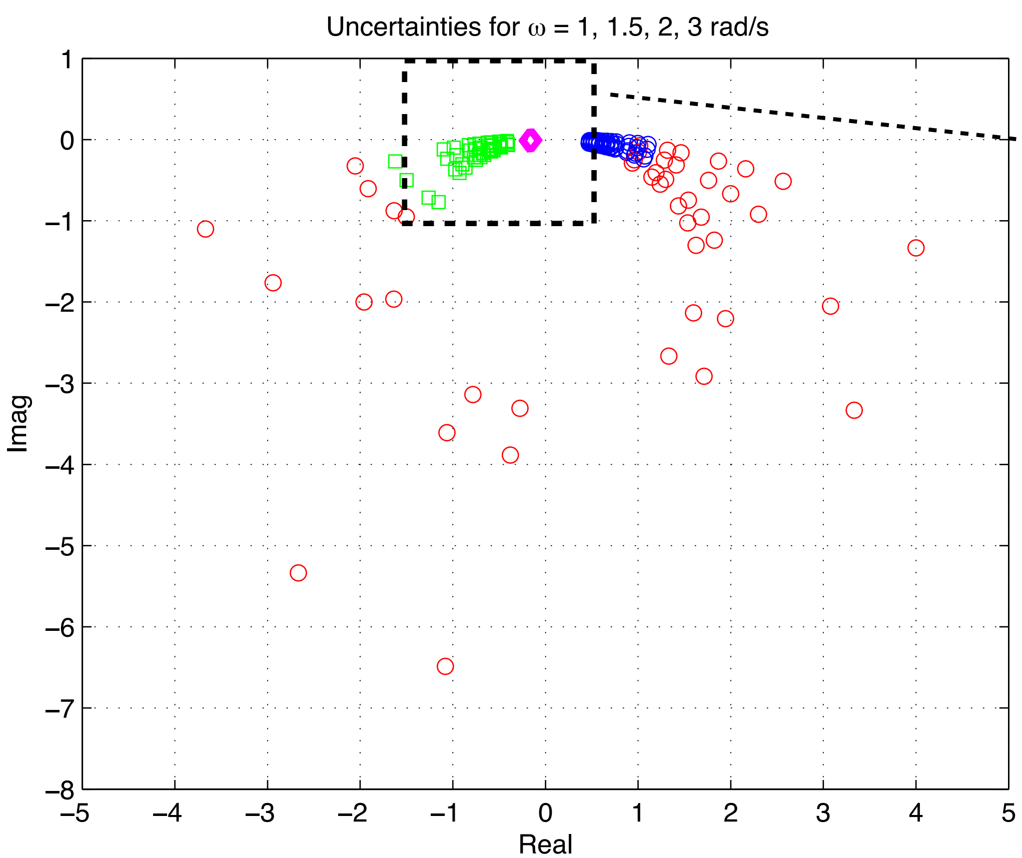
assume the real value of the 3 system parameters can vary as

m in $[0.9,\ 1.1]$

μ in $[0.05,\ 2]$

k in $[2,\ 3]$





zoom

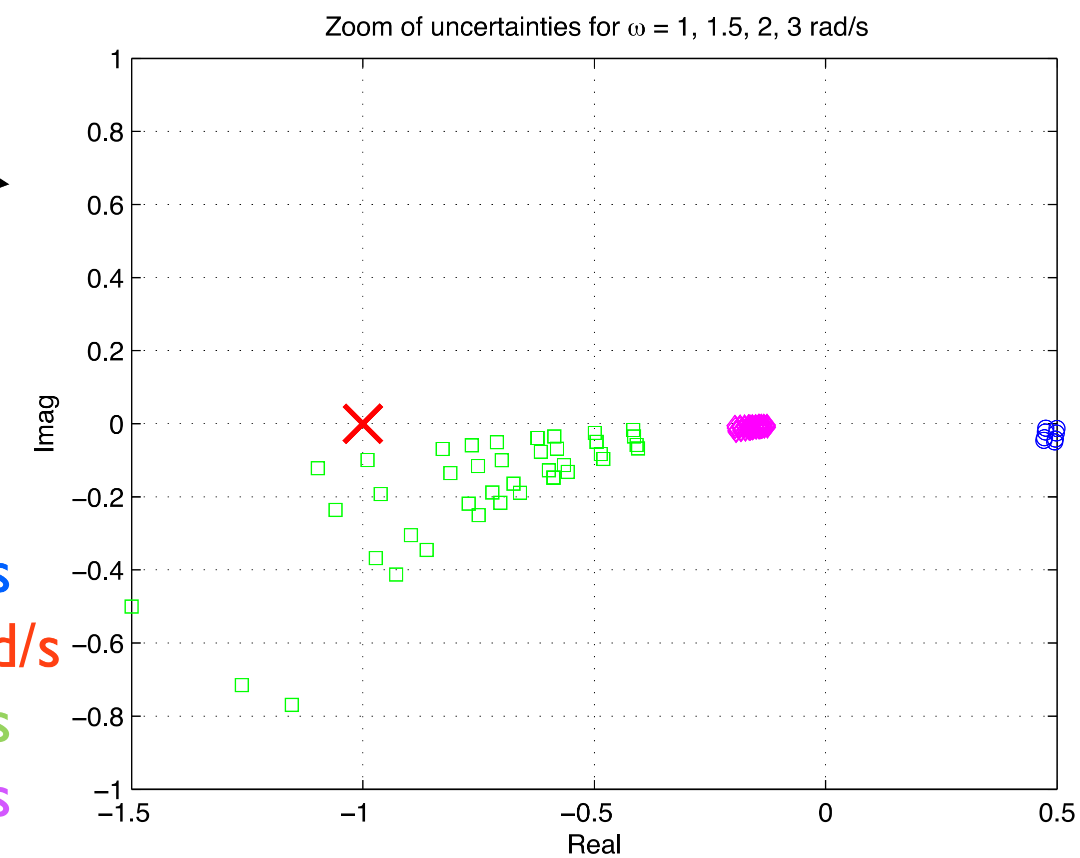
Nyquist plot at different frequencies

$\omega_1 = 1$ rad/s

$\omega_2 = 1.5$ rad/s

$\omega_3 = 2$ rad/s

$\omega_4 = 3$ rad/s



Neglected/Unmodeled dynamics

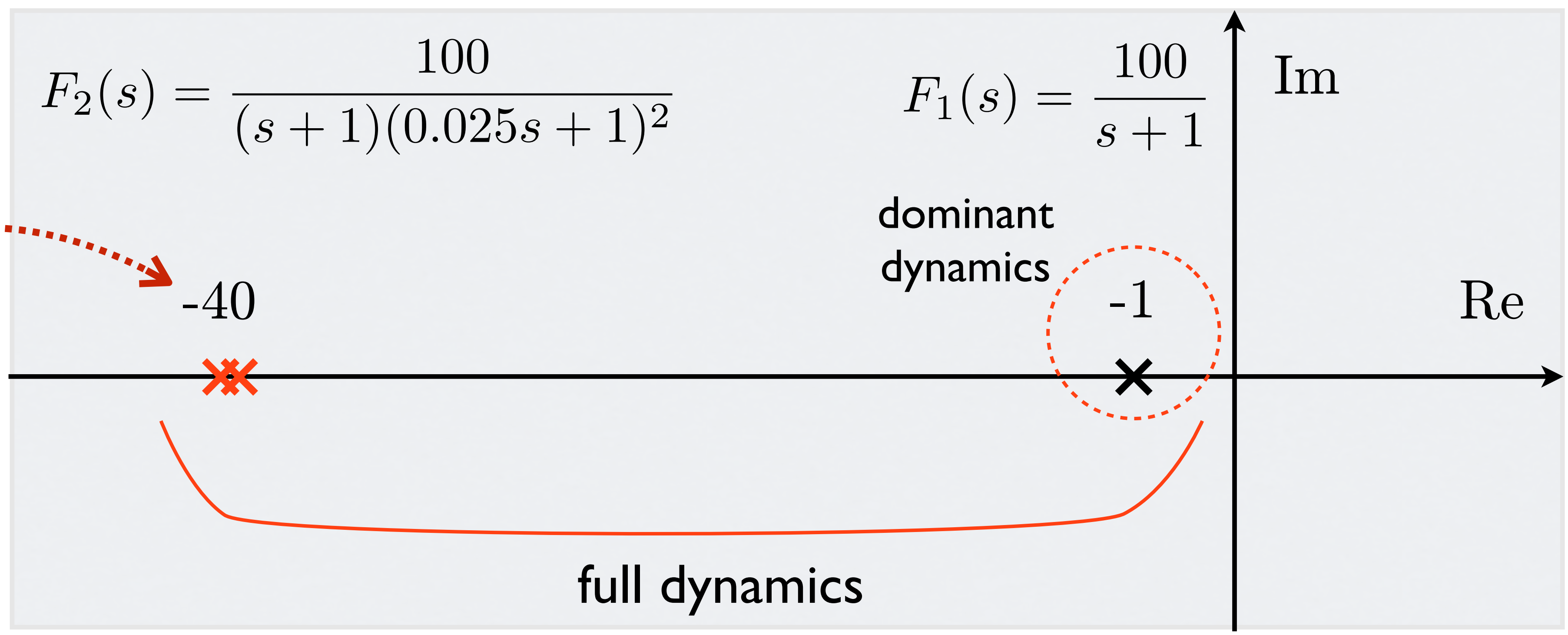
assume the real system is $F_2(s) = \dfrac{100}{(s+1)(0.025s+1)^2}$

- we may either think to approximate the real system with its dominant dynamics (same gain as $F_2(s)$ and only the "low frequency" poles are considered) (**neglected dynamics**)
- or $F_1(s)$ may be directly the outcome of a simplified model derivation which did not consider high frequency dynamics (**unmodeled dynamics**)

the result is

$$F_1(s) = \frac{100}{s+1}$$

high frequency dynamics with gain $= 1$ $\dfrac{1}{(1+0.025s)^2}$



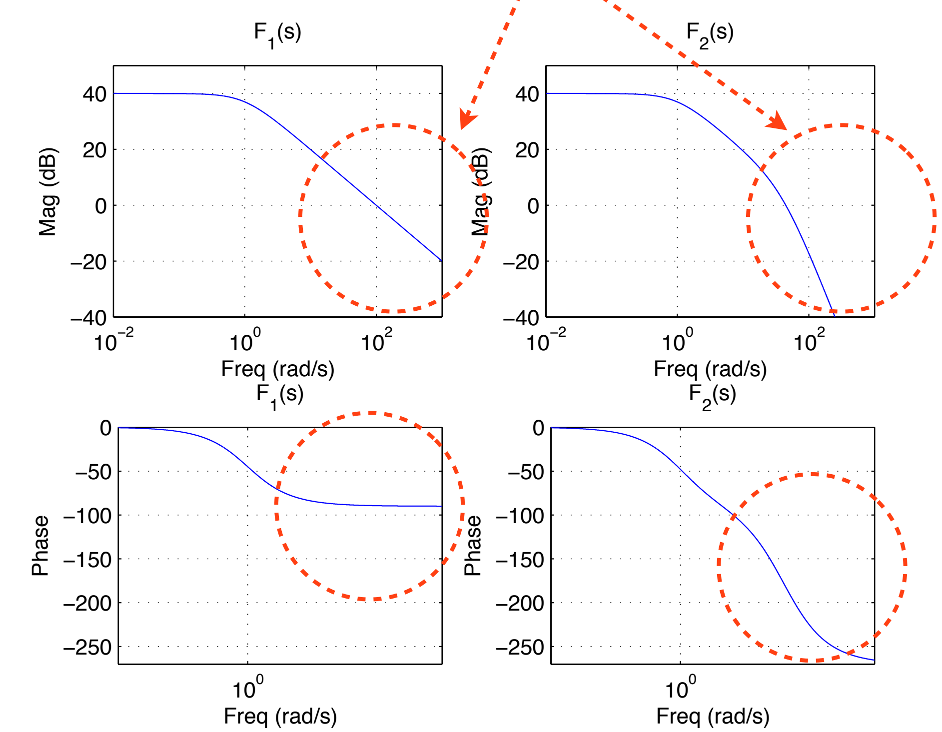
This neglected (or unmodeled dynamics) may have an impact on closed loop stability (of the real system) and performance.

open-loop similar

$$F_2(s) = \frac{100}{(s+1)(0.025s+1)^2}$$ true dynamics

they differ in high frequency content

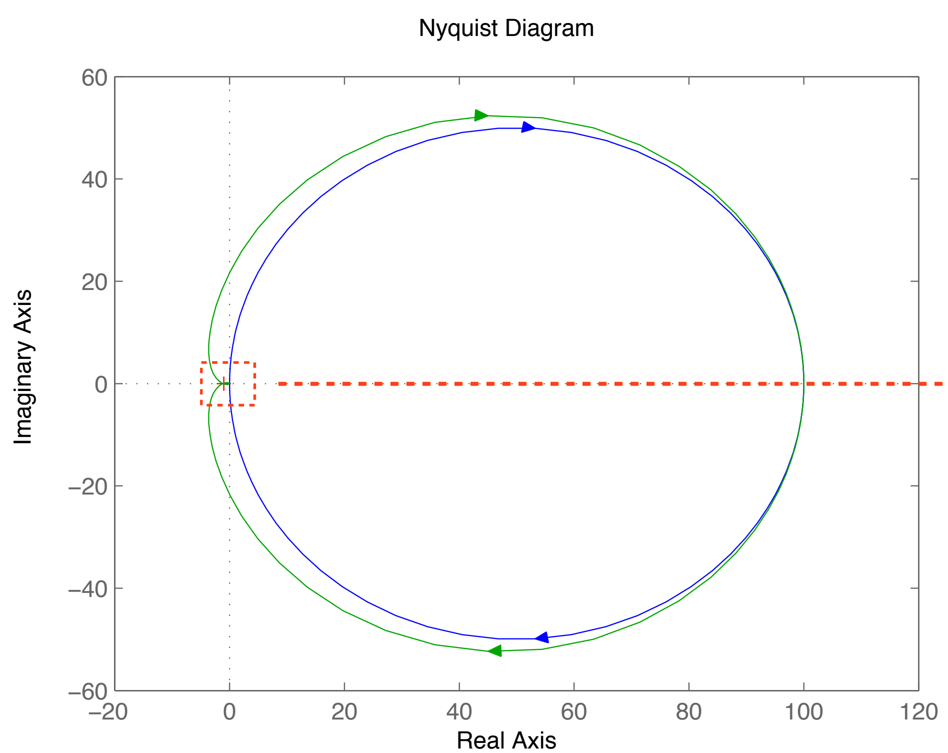
simplified model with only the dominant dynamics $F_1(s) = \dfrac{100}{s+1}$



if we close the loop with the (static) controller $K = 1$ we obtain an asymptotically stable closed loop system if we base the design (but **closed-loop different**)

$$W_1(s) = \frac{100}{s+101}$$ asymptotically stable

$$W_2(s) = \frac{160000}{(s+83.9254)(s^2 - 2.9254s + 1925.5)}$$ unstable closed loop (Nyquist criterion)



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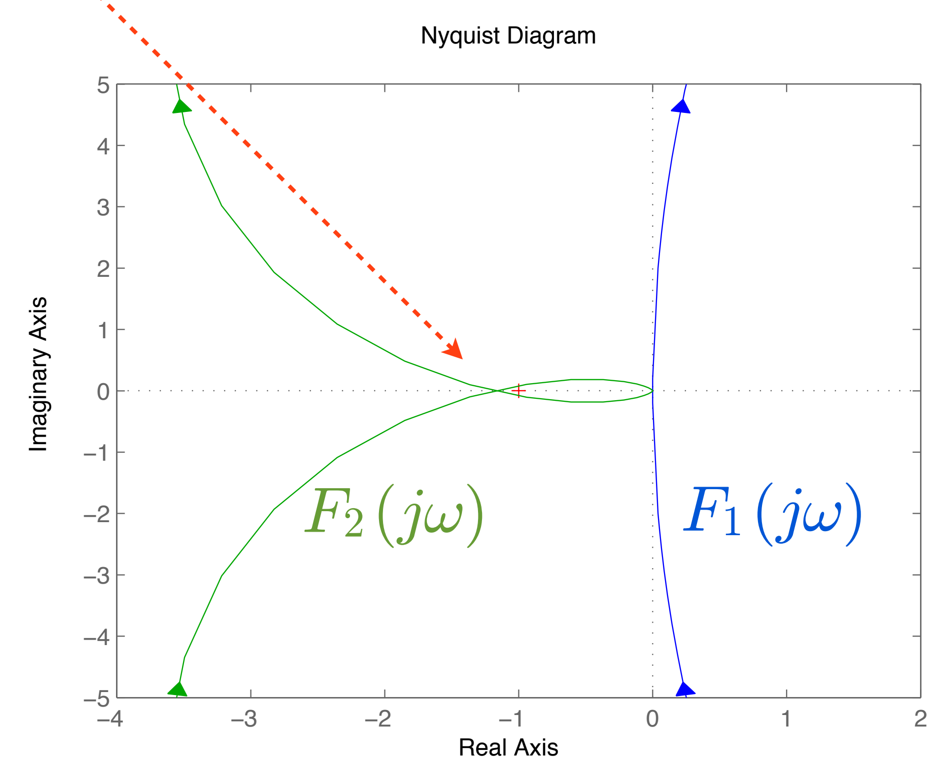


Specifications

Stability of the control system (closed-loop system)

- **nominal stability** (can be checked with Routh, Nyquist, root locus ...)
- **robust stability** guarantees that, even in the presence of parameter uncertainty and/or unmodeled dynamics, stability of the closed-loop system is guaranteed. We have seen two useful indicators (gain and phase margins) others are possible (based on the Nyquist stability criterion or on a surprising result known as the Kharitonov theorem).

Performance

- **nominal performance**
  - **at steady state** (or static) on the desired behavior between the different input/output pairs of interest for the nominal plant
  - **dynamic** on the dynamic behavior during transient
- **robust performance**: we ask that the performance obtained in nominal conditions is also guaranteed, to some extent, under perturbations (parameter variations, unmodeled dynamics).

Specifications

We have obtained

$$\begin{aligned} y(s) &= T(s)r(s) + P(s)S(s)d_i(s) + S(s)d_o(s) - T(s)n(s) \\ e(s) &= S(s)r(s) - P(s)S(s)d_i(s) - S(s)d_o(s) - S(s)n(s) \\ m(s) &= S_u(s)r(s) - S_u(s)P(s)d_i(s) - S_u(s)d_o(s) - S_u(s)n(s) \\ e_y(s) &= S(s)r(s) - P(s)S(s)d_i(s) - S(s)d_o(s) + T(s)n(s) \end{aligned}$$

we used
$T(s) = P(s)S_u(s)$

ideally we would like to have

- the output should accurately reproduce instantaneously the reference:
  we ask the complementary sensitivity $T(s)$ to be as close as possible to 1
- the disturbances $d_i(t)$ and $d_o(t)$ should not affect the output
  the sensitivity $S(s)$ should be as close as possible to 0
- the measurement noise $n(t)$ should not affect the output:
  the complementary sensitivity $T(s)$ should be as close as possible to 0
  (or equivalently the sensitivity $S(s)$ close to 1 being $S(s) + T(s) = 1$)

we are requiring $T(s) = 1$ and $T(s) = 0$ simultaneously: conflicting requirement w.r.t. r and n!

requirements need to be carefully chosen (compromise)

closed-loop system stability

define $$C(s) = \frac{N_C(s)}{D_C(s)} \quad P(s) = \frac{N_P(s)}{D_P(s)}$$

$$S(s) = \frac{1}{1 + \frac{N_C N_P}{D_C D_P}} = \frac{D_C D_P}{D_C D_P + N_C N_P}$$

$$T(s) = \frac{\frac{N_C N_P}{D_C D_P}}{1 + \frac{N_C N_P}{D_C D_P}} = \frac{N_C N_P}{D_C D_P + N_C N_P}$$

$$S_u(s) = \frac{\frac{N_C}{D_C}}{1 + \frac{N_C N_P}{D_C D_P}} = \frac{N_C D_P}{D_C D_P + N_C N_P}$$

$$P(s)S(s) = \frac{N_P}{D_P}\frac{1}{1 + \frac{N_C N_P}{D_C D_P}} = \frac{N_P}{D_P}\frac{D_C D_P}{D_C D_P + N_C N_P} = \frac{N_P D_C}{D_C D_P + N_C N_P}$$

recall that **stability** is a system property, independent from the particular input/output choice

all share the same denominator (**closed loop poles**) provided no hidden dynamics were created in the series controller/plant interconnection

Specifications

an example of possible specifications

- static (at steady-state) reference/output behavior w.r.t. standard signals (sinusoidal or polynomial)
- static disturbance/output behavior for some standard signal (sinusoidal or constant)
- dynamic (transient) reference/output behavior
  - by setting limits to the step response parameters like overshoot or rise time
  - by setting some equivalent bounds on the frequency response (bandwidth, resonance peak defined soon)

+ closed-loop stability ⟵ most important requirement always present even if not explicitly stated

note how we relaxed some requirements on the performance w.r.t. the reference (based on the tracking error e_y) and disturbances d_i and d_o by asking the specification to be **satisfied only at steady-state** that is

$$\lim_{t \to \infty} (r(t) - y(t)) = 0 \qquad \text{instead of} \qquad y(t) = r(t), \quad \forall t$$

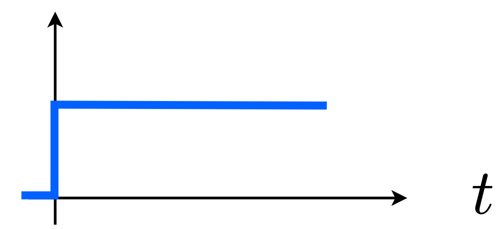
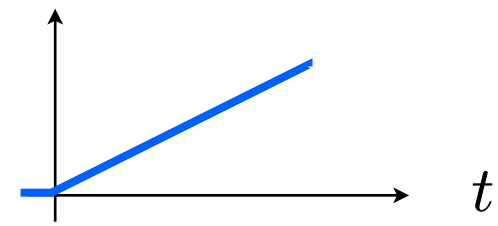
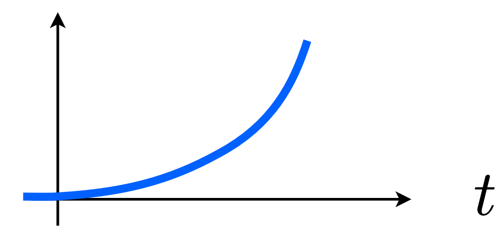
Steady-state specifications - reference

Hyp: closed-loop system will be asymptotically stable (at the end of the control design)

Let the **canonical signal of order k** be defined as

$$\frac{t^k}{k!}\delta_{-1}(t)$$

this representation highlights the different polynomial components of the input signal

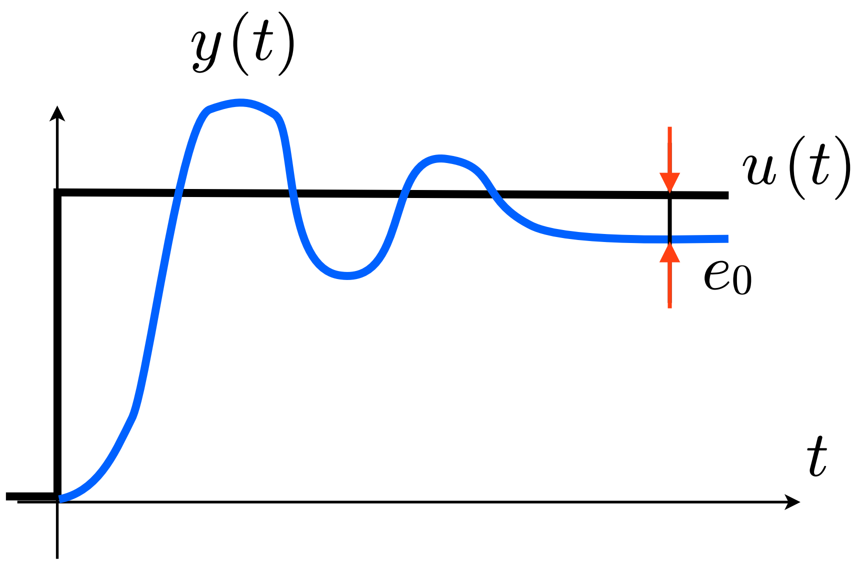
			Laplace transform
order 0 (step function)	$\delta_{-1}(t)$		$\frac{1}{s}$
order 1 (ramp function)	$t\delta_{-1}(t)$		$\frac{1}{s^2}$
order 2 (quadratic function)	$\frac{t^2}{2}\delta_{-1}(t)$		$\frac{1}{s^3}$

Steady-state specifications - system type

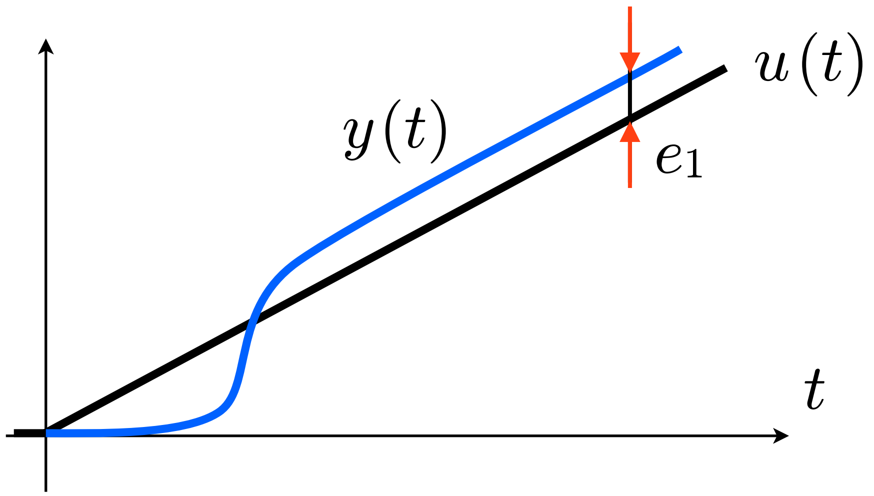
Def a system is of **type k** if its steady-state response to a canonical input of order k differs from the input by a **non-zero constant** or,

equivalently,

if the tracking error at steady-state (output minus input) is constant and different from zero.



type 0



type 1

We apply this definition to a **feedback control system** where the input is the reference signal r and the output is the controlled output y and we look for conditions which guarantee that a feedback system is of type k

alternative definition:

a system is of **type k** if the tracking error at steady state to an order $k-1$ input is 0

An asymptotically stable (**negative) unit feedback** control system is of **type** k

if and only if

the **open-loop system** $L(s)$ has k **poles** in $s = 0$

Basic ideas for proof:

- we assume that closed-loop system is asymptotically stable (by hypothesis)
- the tracking error at steady-state is constant and non-zero if and only if there are k zeros in $s = 0$ in the transfer function from the reference to the tracking error, that is the sensitivity function $S(s)$
- we can apply the final value theorem
- the zeros of $S(s)$ coincide with the poles of the loop function $L(s)$ since if $L(s) = N_L(s)/D_L(s)$ then

$$S(s) = \frac{1}{1 + L(s)} = \frac{D_L(s)}{D_L(s) + N_L(s)}$$

if $L(s)$ has $k > 0$ poles in $s = 0$, we factor the denominator as $D_L(s) = s^k \, D'_L(s)$

with $D'_L(0) \neq 0$ i.e. with no roots in $s = 0$ in $D'_L(s)$

let K_L be the loop function generalized gain

the sensitivity function is then rewritten as $$S(s) = \frac{s^k D'_L(s)}{s^k D'_L(s) + N_L(s)}$$

poles with real part < 0 ← **Hyp.** closed-loop system is asymptotically stable

- reference order $\ell < k$ (system type)

$$e_\ell = \lim_{s\to 0} sS(s)r(s) = \lim_{s\to 0} s\frac{s^k D'_L(s)}{s^k D'_L(s) + N_L(s)}\frac{1}{s^{\ell+1}} = \left.\frac{s^{k-\ell} D'_L(s)}{s^k D'_L(s) + N_L(s)}\right|_{s=0} = 0$$

- reference order $\ell = k$

$$e_\ell = \lim_{s\to 0} sS(s)r(s) = \lim_{s\to 0} s\frac{s^k D'_L(s)}{s^k D'_L(s) + N_L(s)}\frac{1}{s^{\ell+1}} = \frac{D'_L(0)}{N_L(0)} = \frac{1}{K_L}$$

- reference order $\ell > k$

$$e_y(s) = sS(s)r(s) = \frac{s^k D'_L(s)}{s^k D'_L(s) + N_L(s)}\frac{1}{s^{\ell+1}} = \frac{D'_L(s)}{s^k D'_L(s) + N_L(s)}\frac{1}{s^{\ell-k+1}}$$

thus the steady state will have polynomial contributions and will not tend to 0 as t increases

Steady-state specifications - system type (summary)

Define with K_P and K_C the **generalized gain** respectively of the plant and the controller, therefore the generalized gain of the loop function $L(s)$ is $K_L = K_P K_C$

- order $\ell = 0$ reference

$$\text{steady state error } e_0 = S(0) = \begin{cases} \frac{1}{1+K_L} & \text{if} \quad \text{system type} = 0 \\ 0 & \text{if} \quad \text{system type} \geq 1 \end{cases} \longleftarrow \text{final value th.}$$

since the presence of 1 or more roots in $s = 0$ in the denominator $D_L(s)$ of the loop function makes the numerator of $S(s)$ become zero

- order $\ell \geq 1$ reference

$$\text{steady state error } e_\ell = \begin{cases} \infty & \text{if} \quad \text{system type} < \ell & \longleftarrow \textbf{no} \text{ final value th.} \\ \frac{1}{K_L} & \text{if} \quad \text{system type} = \ell & \longleftarrow \text{final value th.} \\ 0 & \text{if} \quad \text{system type} > \ell & \longleftarrow \text{final value th.} \end{cases}$$

Summarizing table: tracking error (error w.r.t. the reference)

	error	System type			
Input order		0	1	2	3
0	$\delta_{-1}(t)$	$\frac{1}{1+K_L}$	0	0	0
1	$t\delta_{-1}(t)$	$+\infty$	$\frac{1}{K_L}$	0	0
2	$\frac{t^2}{2}\delta_{-1}(t)$	$+\infty$	$+\infty$	$\frac{1}{K_L}$	0
3	$\frac{t^3}{3!}\delta_{-1}(t)$	$+\infty$	$+\infty$	$+\infty$	$\frac{1}{K_L}$

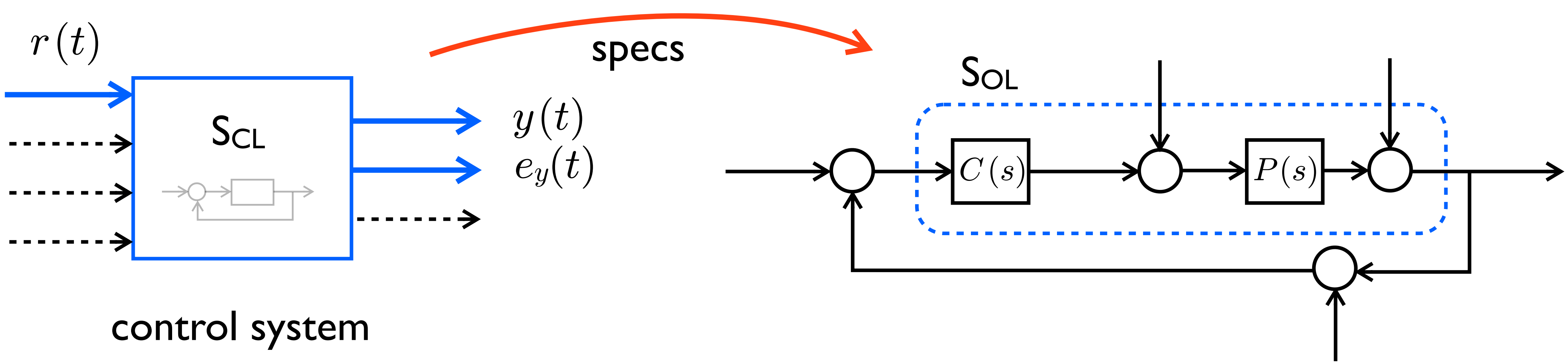
Therefore

we can define the refrence-to-output specifications in terms of **system type** and value of maximum allowable tracking error or, equivalently,

- presence of the sufficient number of poles in $s = 0$ in the open-loop system
- absolute value of the open-loop gain K_L sufficiently large in order to guarantee the maximum allowed error

$$|e_k| \leq e_{kmax} \Longleftrightarrow \begin{cases} \frac{1}{|1+K_L|} \leq e_{kmax} & \Leftrightarrow \quad |1+K_L| \geq \frac{1}{e_{kmax}} \quad \text{if} \quad \text{Type } 0 \\ \frac{1}{|K_L|} \leq e_{kmax} & \Leftrightarrow \quad |K_L| \geq \frac{1}{e_{kmax}} \quad \text{if} \quad \text{Type } k \geq 1 \end{cases}$$

We have translated the **closed-loop specifications in equivalent open-loop ones**



Steady-state specifications - disturbance

The disturbance is just another - undesired - input.

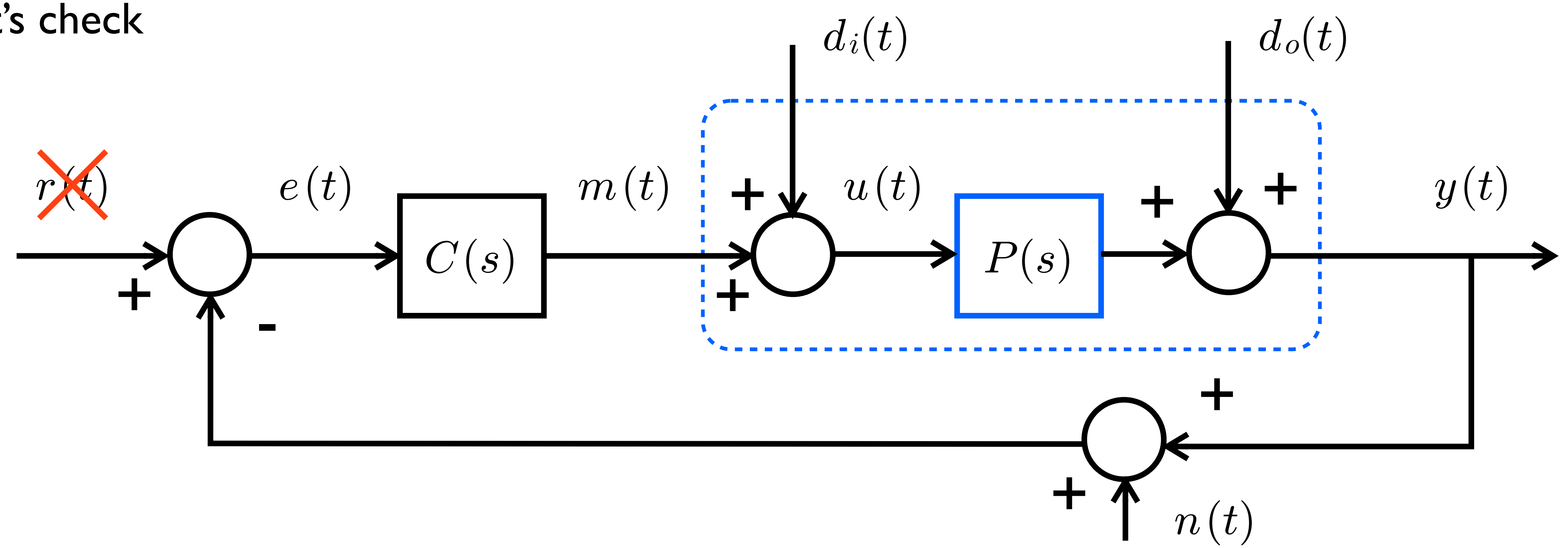
Let us consider the **constant disturbance input** case and use the same basic principle as for the reference.

> To make an asymptotically stable control system controlled output insensible (**astatic**), at steady-state, to a **constant input** d_i or d_o, we need to ensure the presence of a **pole at** $s = 0$ in the **forward path before** the entering point of the disturbance.

This is true for any **constant disturbance on the forward path**.

Note that nothing is said for the measurement noise n

Let's check



constant unit disturbances and measurement noise

$$d_i(s) = d_o(s) = n(s) = \frac{1}{s}$$

being

$$y(s) = T(s)r(s) + P(s)S(s)d_i(s) + S(s)d_o(s) - T(s)n(s)$$

we have (setting the reference to zero) the following steady-state responses w.r.t. input unit steps

$$y_{\text{ss}} = [P(s)S(s)]_{s=0} + S(0) - T(0)$$

Hyp.
closed-loop system asymptotically stable

therefore we need to compute the value of the terms

$d_i \longrightarrow y_{\text{ss}}$ $\quad [P(s)S(s)]_{s=0}$

$d_o \longrightarrow y_{\text{ss}}$ $\quad S(0)$

$n \longrightarrow y_{\text{ss}}$ $\quad$ - $T(0)$

define $\quad C(s) = \dfrac{N_C(s)}{D_C(s)} \quad P(s) = \dfrac{N_P(s)}{D_P(s)} \quad L(s) = \dfrac{N_L(s)}{D_L(s)} = \dfrac{N_C(s)N_P(s)}{D_C(s)D_P(s)}$

we are going to analyze the different cases

$d_o \longrightarrow y_{ss}$

- to have no steady-state contribution to the output $y_{\rm ss}$ from a constant output disturbance $d_{\rm o}$ we need to have $S(0) = 0$ that is, being

$$S(s) = \frac{1}{1+L(s)} = \frac{D_L(s)}{D_L(s)+N_L(s)} = \frac{D_C(s)D_P(s)}{D_L(s)+N_L(s)} = \frac{D_C(s)D_P(s)}{D_P(s)D_P(s)+N_C(s)N_P(s)}$$

the zeros of the sensitivity function $S(s)$ coincide with the poles of the loop function $L(s)$ so we will have $S(0) = 0$ (i.e., $s = 0$ is a zero of $S(s)$) if and only if we have at least one pole at the origin in the open-loop system $L(s)$ (and for this disturbance, this is equivalent to requiring the presence of at least a pole in $s = 0$ before the entry point of the disturbance).

so either a pole in $s = 0$ is already present in the plant or we need to introduce it in the controller (necessary part of the controller to cancel out the effect of the constant disturbance $d_{\rm o}$ at steady-state on the output).

if no pole in $s = 0$ is present in the loop function we have a steady-state effect of a constant unit output disturbance d_o given by

$$y_{\text{ss}} = S(0) = \frac{1}{1 + K_L} = \frac{1}{1 + K_C K_P}$$

so a high-gain controller will reduce the effect of the given disturbance provided the system remains asymptotically stable

$d_i \longrightarrow y_{\text{ss}}$

- for the steady-state contribution to the output y_{ss} of a constant input disturbance d_i note that

$$P(s)S(s) = \frac{N_P(s)D_C(s)}{D_P(s)D_P(s) + N_C(s)N_P(s)}$$

therefore in order to get a zero contribution at steady-state we can either

- have a pole in $s = 0$ in $C(s)$ (ahead of the entry point of the disturbance) or
- have a zero in $s = 0$ in $P(s)$ but this leads also to a zero steady-state contribution of a constant reference to the output, i.e. zero gain $T(0) = 0$ while we would like this gain to be as close to 1 as possible (avoid when possible but the plant is given, so no choice)

otherwise we have a finite non-zero contribution given by

$$\frac{K_P}{1 + K_P K_C} \qquad \text{if } P(s) \text{ has no poles in } 0$$

$$\frac{1}{K_C} \qquad \text{if } P(s) \text{ has poles in } 0$$

proof as exercise

$n \longrightarrow y_{\text{ss}}$

- for the steady-state contribution to the output y_{ss} of a constant measurement noise n note that

$$T(s) = \frac{N_P(s)N_C(s)}{D_P(s)D_P(s) + N_C(s)N_P(s)}$$

therefore in order to get a zero contribution at steady-state we can either

- have a zero in $s = 0$ in $P(s)$ and/or $C(s)$ but this leads also to a zero steady-state contribution of a constant reference to the output, i.e. zero gain $T(0) = 0$ while we would like this gain to be as close to 1 as possible (avoid when possible, i.e. do not add a zero in $s = 0$ in $P(s)$)

otherwise we have a finite non-zero contribution given by

$$\frac{K_P K_C}{1 + K_P K_C} \qquad \text{if } L(s) \text{ has no poles in } 0$$

$$1 \qquad \text{if } L(s) \text{ has poles in } 0$$

High-gain through K_c makes things worse w.r.t. noise $n(t)$

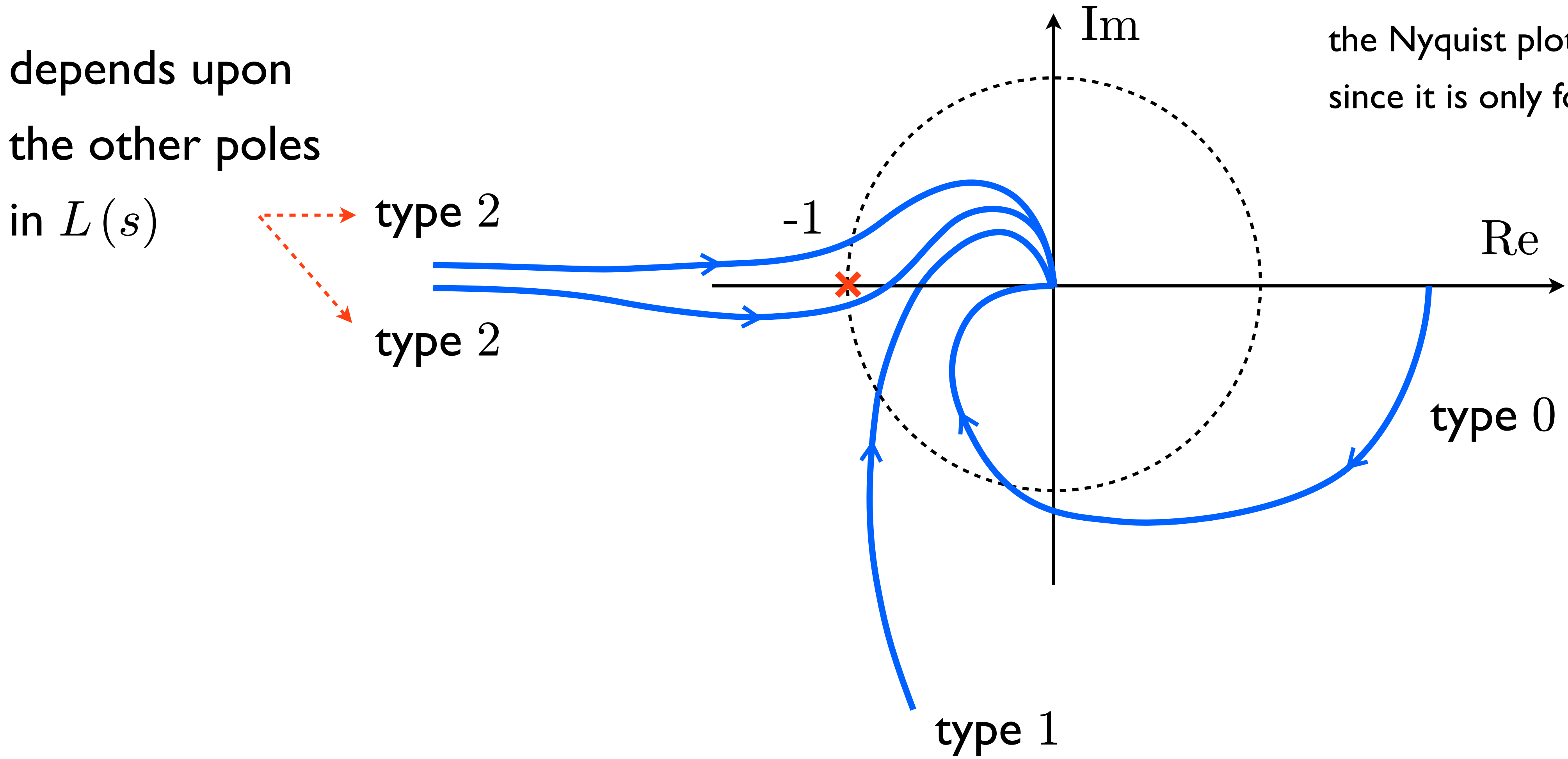
A constant measurement noise is a bias in the transducer and it represents a serious flaw in the measurement chain

Effect of integrators on stability

The previous analysis has shown that, provided the control system remains asymptotically stable, adding integrators in the forward path has beneficial effects on the steady-state behavior of the closed-loop system.

However integrators in the open-loop system have a **destabilizing effect** on the closed-loop as shown in the following Nyquist plot or equivalently by noting the introduced lag on the phase due to the integrators (-$\pi/2$ for each pole in 0).

In the design process we will introduce the minimum necessary number of integrators



the Nyquist plots shown are not complete since it is only for ω in $(0^+, +\infty)$

Other steady-state requirements

- asymptotic **tracking** of a sinusoidal reference. Let the reference be (for positive t)

$$r(t) = \sin \bar{\omega} t \qquad \text{with Laplace transform} \qquad r(s) = \frac{\bar{\omega}}{s^2 + \bar{\omega}^2}$$

in order to asymptotically track this reference the controlled output $y(t)$ needs to tend asymptotically to the reference $r(t)$ or, equivalently, the tracking error $e_y(t) = r(t) - y(t)$ needs to tend to zero as t tends to infinity.

Recalling that the transfer function from the reference to the error is

$$\frac{e_y(s)}{r(s)} = \frac{r(s) - y(s)}{r(s)} = 1 - T(s) = S(s)$$

and that, for an asymptotically stable system, the steady-state response to a sinusoidal is

$$e_{\text{ss}}(t) = |S(j\bar{\omega})| \sin(\bar{\omega} t + \angle S(j\bar{\omega}))$$

it is clear that, in order to achieve zero asymptotic error we need, at the specific input frequency $\bar{\omega}$, to be able to ensure that

$$|S(j\bar{\omega})| = 0 \qquad \Longleftrightarrow \qquad S(s)\Big|_{s=j\bar{\omega}} = 0$$

that is the sensitivity function must have a pure imaginary zero (and its conjugate) at the frequency of the input signal $\bar{\omega}$

from the previous analysis we also know that the zeros of the sensitivity function coincide with the poles of the open-loop function (in a unit feedback scheme), therefore the necessary condition becomes (sufficiency only if the closed loop system is asymptotically stable)

> in order to guarantee **asymptotic tracking** of a sinusoid with frequency $\bar{\omega}$ in an asymptotically stable feedback system, the open-loop system needs to have a pair of conjugate poles in $s = \pm j\bar{\omega}$

Being $L(s) = C(s)P(s)$ and assuming that the plant has no poles in $s = \pm j\bar{\omega}$ the controller needs to be of the form

$$C(s) = \frac{N_C(s)}{(s^2 + \bar{\omega}^2)D'_C(s)}$$

Note that this leads to 0 error at steady state.

A less stringent requirement would be asking a small error at steady state or, equivalently, a small value of the sensitivity function magnitude $|S(j\bar{\omega})|$.

Furthermore this small error requirement can be achieved over a frequency range while zero steady-state error no.

- asymptotic rejection of a sinusoidal disturbance (similarly)

$$d_1(t) = \sin \bar{\omega} t \qquad d_i \longrightarrow y_{\text{ss}} \qquad |P(j\bar{\omega})S(j\bar{\omega})| = 0 \qquad [P(s)S(s)]_{s=j\bar{\omega}} = 0$$

$$d_2(t) = \sin \bar{\omega} t \qquad d_o \longrightarrow y_{\text{ss}} \qquad |S(j\bar{\omega})| = 0 \qquad S(s)\Big|_{s=j\bar{\omega}} = 0$$

Assuming that the plant has no poles in $s = \pm j\bar{\omega}$ the controller needs to be of the form

$$C(s) = \frac{N_C(s)}{(s^2 + \bar{\omega}^2)D'_C(s)}$$

- again adding poles in $\pm j\bar{\omega}$ has a destabilising effect on the closed loop
- we are able to nullify the effect of a sinusoidal disturbance (or track a sinusoidal reference) at a finite number of distinct frequencies, not in a frequency range (see the Sensitivity function in the Performance lecture if the goal is to attenuate disturbances in a frequency range $[\omega_1, \, \omega_2]$)

Transient specifications

We already know how to characterize the transient and therefore we can define requirements on the closed-loop dynamic behavior in terms of

- poles (and zeros) location in the complex plane (time constants, damping coefficients, natural frequencies)
- particular quantities defined on the step response (rise-time, overshoot and settling time)

We can also define two quantities in the frequency domain related to the transient behavior

- bandwidth B_3
- resonant peak M_r

which are related to the rise time and the overshoot establishing interesting connections between time and frequency domain characterization of the transient

Bandwidth

for the typical magnitude plots encountered so far, we define the bandwidth B_3 as the first frequency such that for all frequencies greater than the bandwidth the magnitude is attenuated by a factor greater than $1/\sqrt{2}$ w.r.t. its value in $\omega = 0$

$$B_3 : \qquad |W(jB_3)| = \frac{|W(j0)|}{\sqrt{2}}$$

and being $20 \log_{10} \left(\frac{1}{\sqrt{2}} \right) \approx -3\, dB$

$$B_3 : \qquad |W(jB_3)|_{dB} = |W(j0)|_{dB} - 3$$

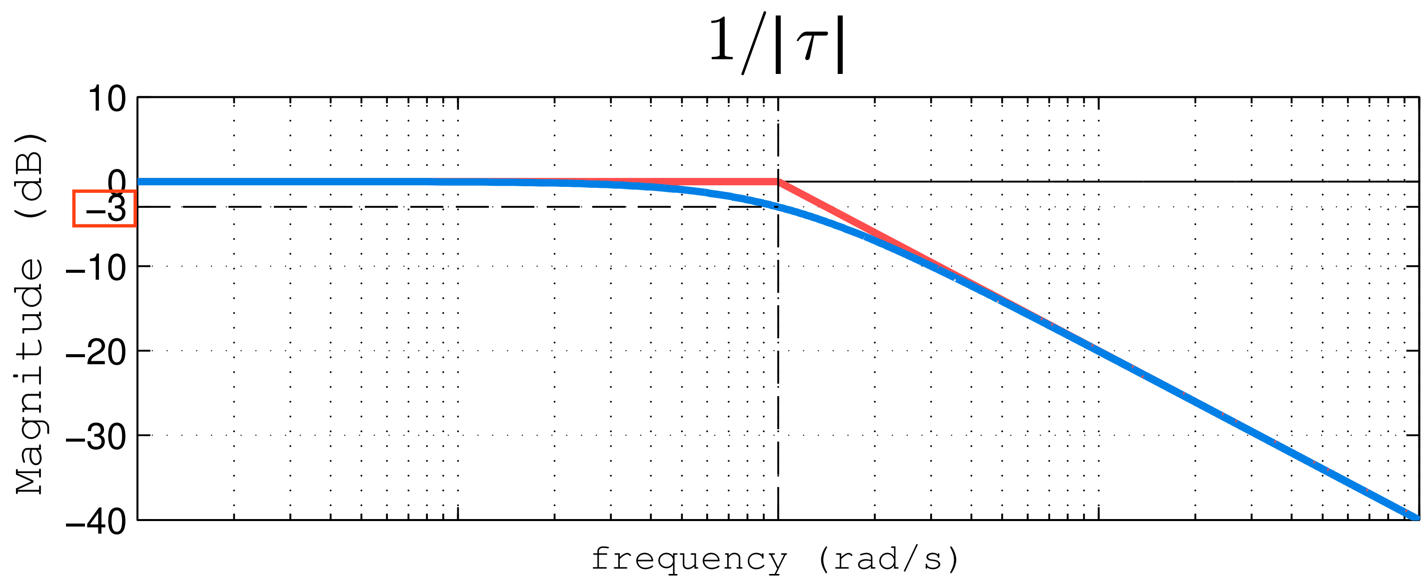
- characterizes the filtering capacities of the dynamical system
- is relative to the static gain $|W(j0)|$

simplest example

$$W(s) = \frac{K}{1+\tau s}$$ asymptotically stable system (therefore $\tau > 0$)

magnitude plot

normalized w.r.t. $|K|_{dB}$



being
$$\begin{aligned}|W(j\omega)|_{dB} - |W(j0)|_{dB} &= |W(j\omega)|_{dB} - |K|_{dB} \\ &= |K|_{dB} + |1/(1+j\omega\tau)|_{dB} - |K|_{dB} \\ &= |1/(1+j\omega\tau)|_{dB}\end{aligned}$$

and
$$|1 + j\tau/|\tau|\,|_{dB} = 20\log_{10}\sqrt{2} \approx 3\,dB$$

the bandwidth coincides with the cutoff frequency
$$B_3 = \frac{1}{\tau}$$

Resonant peak

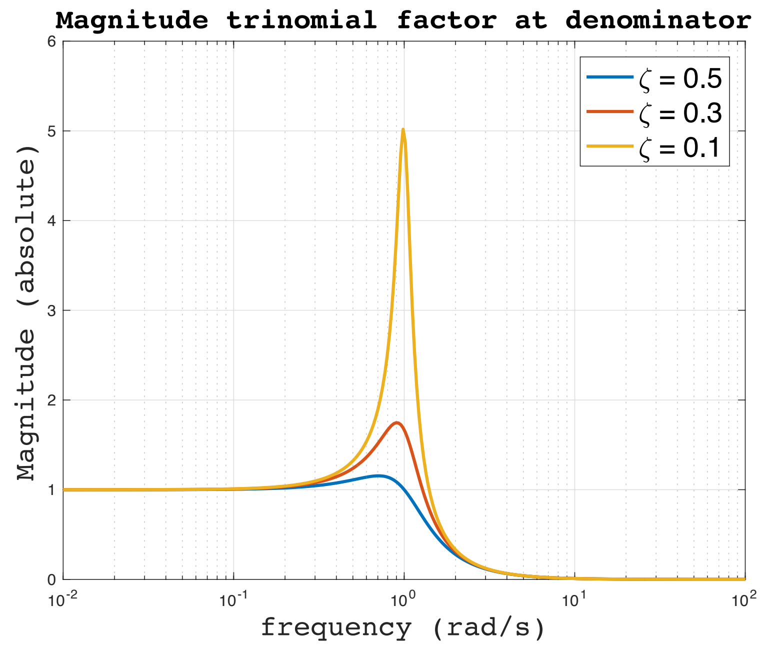
we define the resonant peak M_r as the maximum value of the frequency response magnitude referred to its value in $\omega = 0$

$$M_r = \frac{\max |W(j\omega)|}{|W(j0)|}$$

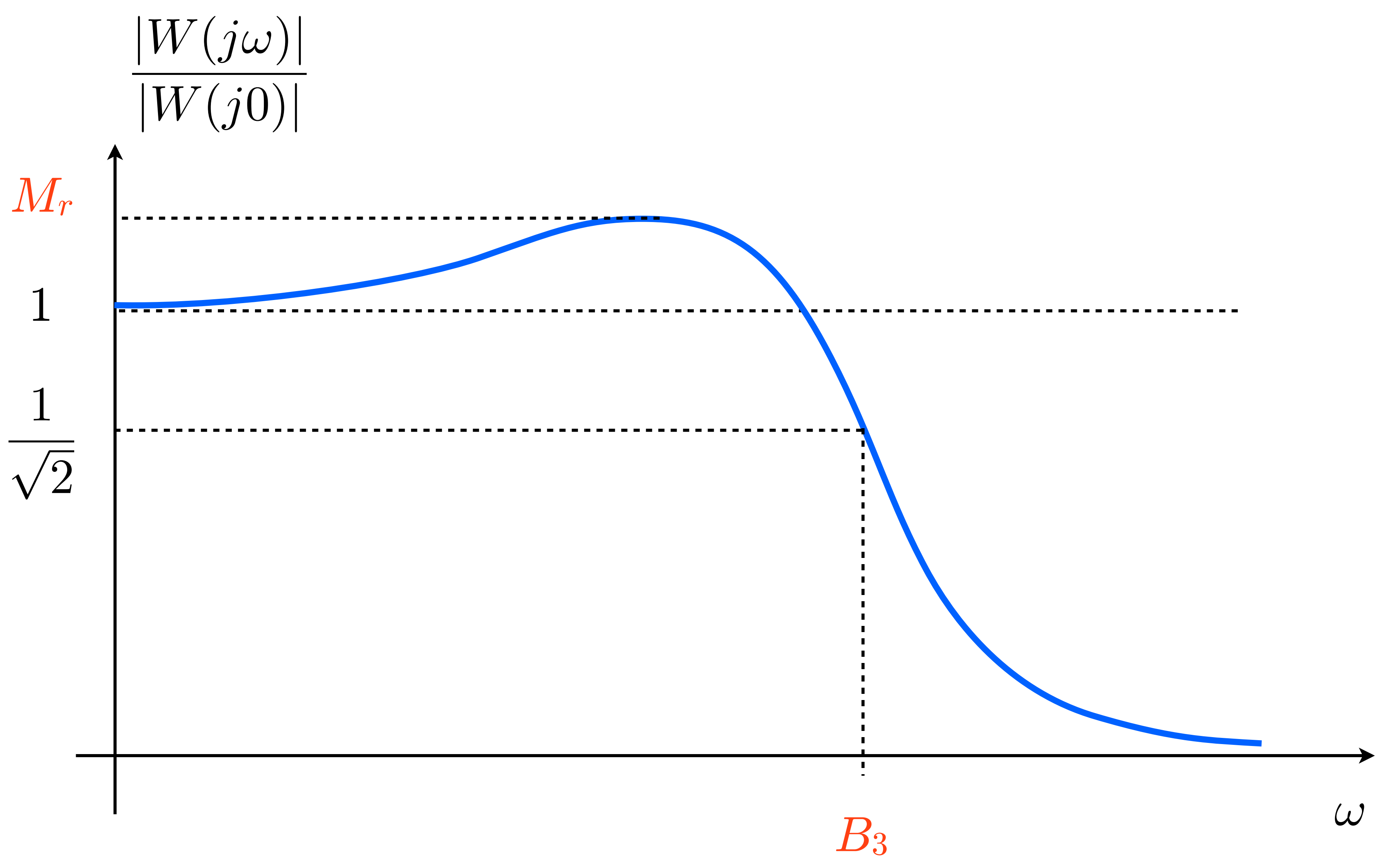
or in dB

$$M_r|_{dB} = \max |W(j\omega)|_{dB} - |W(j0)|_{dB}$$

a high resonant peak indicates that the system behaves similarly to a second order system with low damping coefficient



on a plot with normalized magnitude (not in dB) the bandwidth and resonance peak are as shown



Relationships

typically (with some exceptions)

$$B_3 \, t_r \approx \text{constant}$$

higher bandwidth (higher frequency components of the input signal are not attenuated and therefore are allowed to go through) leads to smaller rise time (faster system response)

$$\frac{1 + M_p}{M_r} \approx \text{constant}$$

higher resonant peak (as if we had a second order system with lower damping coefficient) leads to higher overshoot (the oscillation damps out slower)

very useful relationships in order to understand the connections between time and frequency domain response characteristics

Transient specifications

- we may want to ensure a maximum rise time $t_{r,\max}$

$$t_r \leq t_{r,\max} \qquad \Longleftrightarrow \qquad B_3 \geq B_{3,\min}$$

this may be achieved by ensuring a sufficiently high bandwidth (greater than some value $B_{3,\min}$)

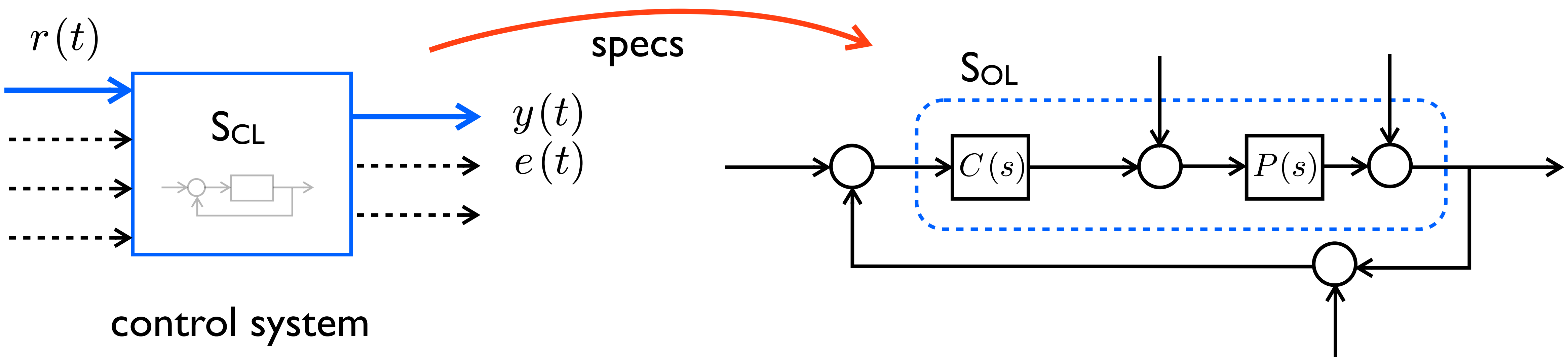
- we may want to ensure a maximum overshoot $M_{p,\max}$

$$M_p \leq M_{p,\max} \qquad \Longleftrightarrow \qquad M_r \leq M_{r,\max}$$

this may be achieved by ensuring a sufficiently low resonant peak (smaller than some value $M_{r,\max}$)

Transient specifications

we want to relate some transient specifications on the closed-loop system (control system) to some characteristics of the open-loop system (easier to guarantee)



bandwidth B_3 (and rise time t_r) → ω_c crossover frequency

resonant peak M_r (and overshoot M_p) → PM phase margin

closed loop **open loop**

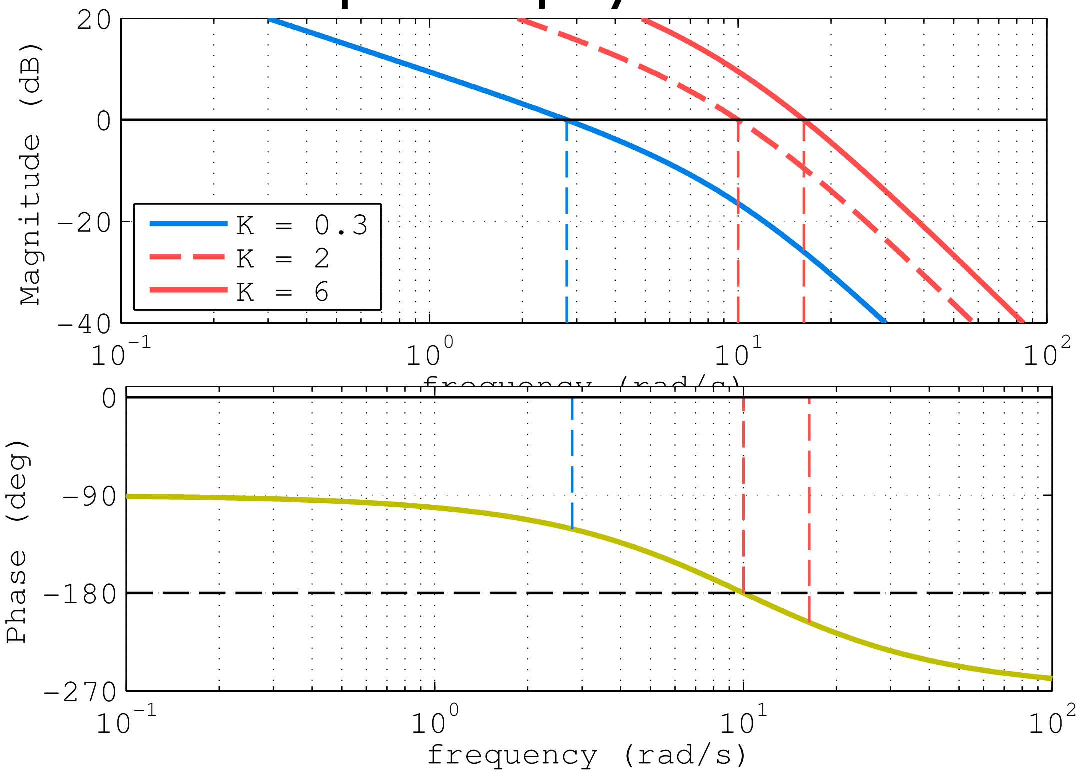
we can see these typical (with some exceptions) relationships through an example

$$F(s) = \frac{10\,K}{s(1+s/10)^2}$$

open-loop system

comparison for increasing values of K

open loop system



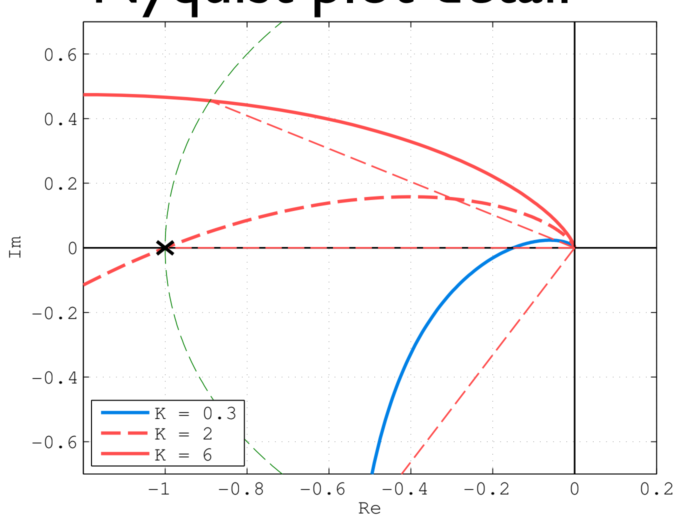
PM and M_r relationship

open-loop phase margin PM decreases

&

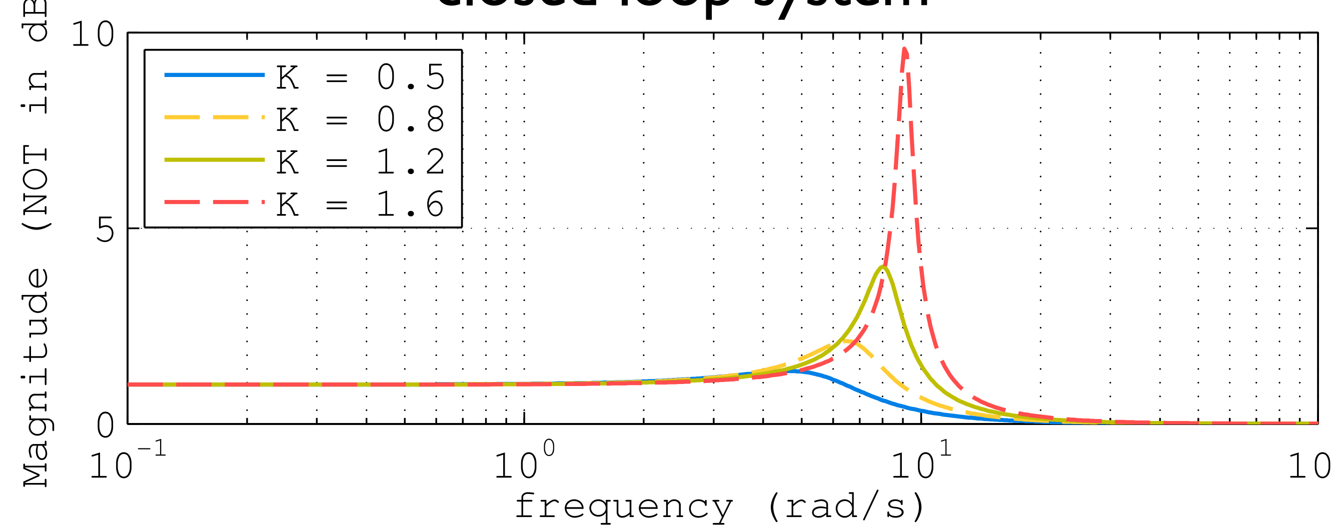
closed-loop resonant peak M_r increases

Nyquist plot detail



recall that if the Nyquist plot goes through the critical point then the closed-loop system has pure imaginary poles (zero damping and thus infinite resonant peak)

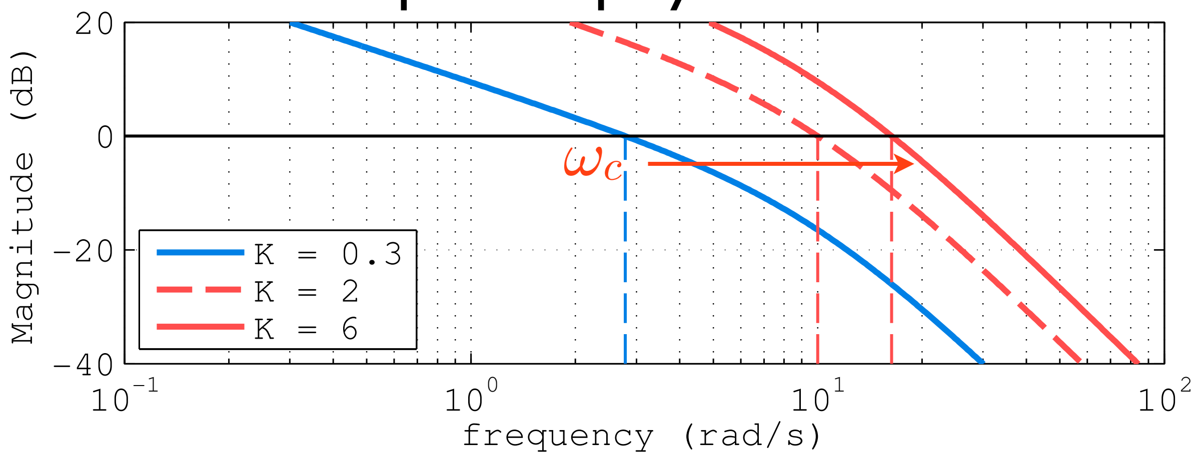
closed loop system



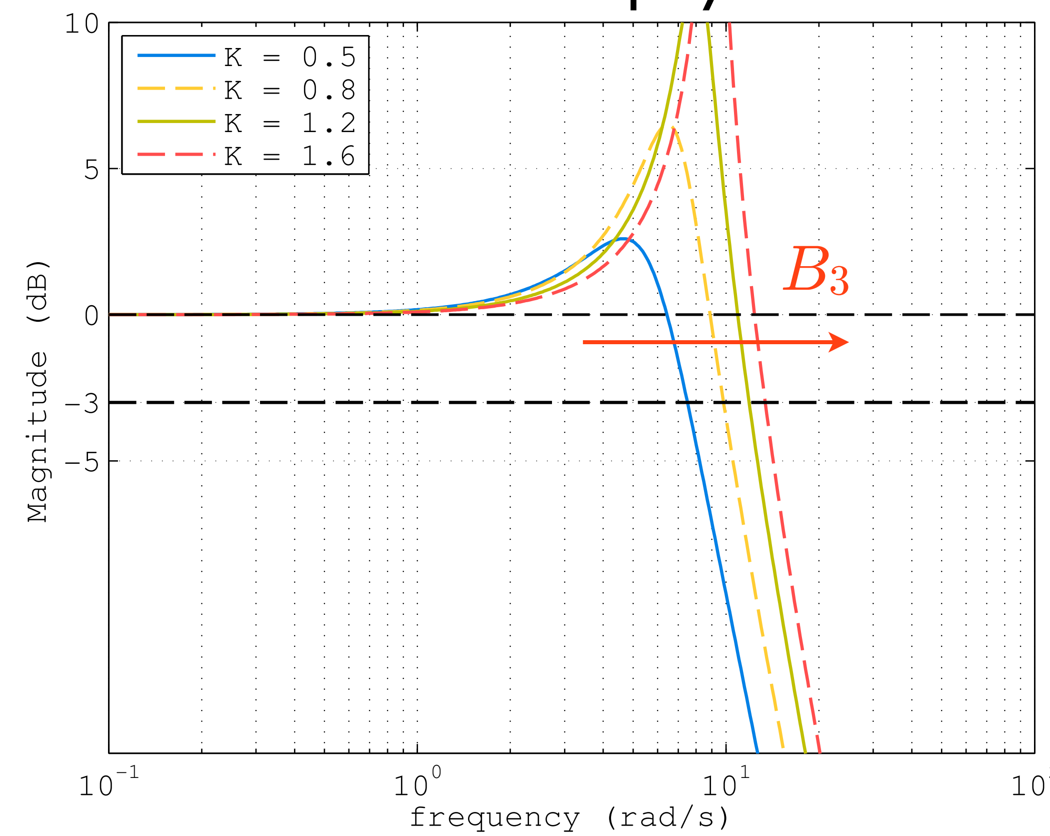
same open-loop system comparison for increasing values of K

ω_c and B_3 relationship

open loop system



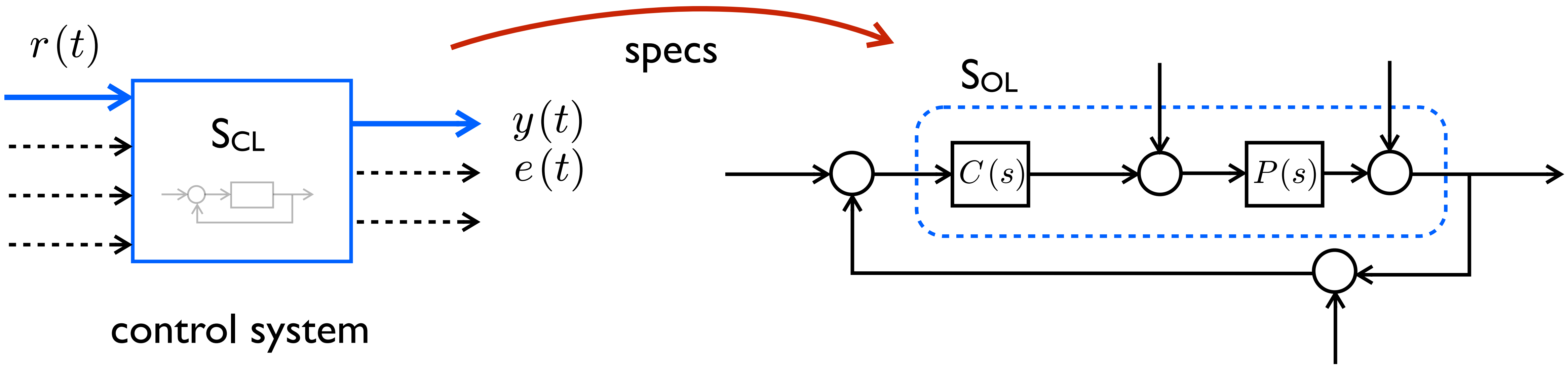
closed loop system



as the open-loop crossover frequency ω_c increases the closed-loop bandwidth B_3 increases

- also evident from the complementary sensitivity $T(s)$ approximate magnitude plot (B_3 slightly larger than ω_c) as shown in Performance slides

Transient specifications: from closed loop to open loop



$t_r \leq t_{r,\max} \iff B_3 \geq B_{3,\min}$ $\qquad$ $\omega_c \geq \omega_{c,\min}$

bandwidth B_3 (and rise time t_r) → ω_c crossover frequency

$M_p \leq M_{p,\max} \iff M_r \leq M_{r,\max}$ $\qquad$ $PM \geq PM_{\min}$

resonant peak M_r (and overshoot M_p) → PM phase margin

closed-loop system $\qquad$ **open-loop system**