Robotics 1

Midterm Test — November 21, 2025

Exercise 1

Compute the two axis/angle representations $(\boldsymbol{r}_i, \theta_i)$, $i = 1, 2$, corresponding to the rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Verify your results by evaluating $\boldsymbol{R}(\boldsymbol{r}_i, \theta_i)$ with the found solutions. Provide then the corresponding quaternion $\mathcal{Q} = \{\eta, \boldsymbol{\epsilon}\}$. Why is this $\mathcal{Q}$ unique?

Exercise 2

a) Figure 1 shows a sketch of a transmission with three engaged toothed gears having radius r_1, r_2, and r_3 and inertia J_1, J_2, and J_3 around their respective rotation axis. A motor can apply a torque τ to the first gear. If the last gear has to accelerate with $\ddot{\theta}_3 = a$, write the expression of the torque τ that the motor should provide (neglect all dissipative effects). If the transmission ratio n_1 of the first pair of gears has already been chosen, which is the optimal transmission ratio n_2^* of the second pair of gears that minimizes τ for a desired a?

b) Consider the following numerical data: $r_1 = 0.002$, $r_2 = 0.01$, $r_3 = 0.00667$ [m]; $J_1 = 5 \cdot 10^{-6}$, $J_2 = 2 \cdot 10^{-7}$, $J_3 = 4.5 \cdot 10^{-6}$ [kgm^2]. Provide the value of τ (in Nm) that is required to obtain $a = 10$ rad/s^2. Then, for the given n_1, compare this torque value with the one needed when using n_2^* instead of n_2.

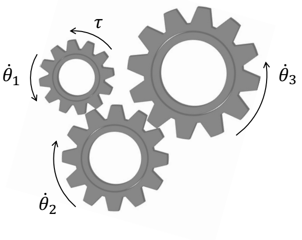

Figure 1: A transmission with three toothed gears.

Exercise 3

Consider the 3R spatial robot in Fig. 2, with the base and end-effector frames specified therein. On the sheet distributed separately, draw the standard Denavit–Hartenberg (DH) frames associated to the parameters in Tab. 1. Accordingly, write the elements in the homogeneous transformation matrix ${}^3\boldsymbol{T}_E$.

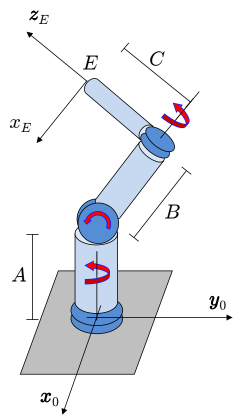

Figure 2: A 3R spatial robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	A	q_1
2	$\pi/2$	0	0	q_2
3	0	0	0	q_3

Table 1: DH table of parameters for the robot in Fig. 2.

Exercise 4

With reference to Fig. 3, a 3R planar robot is mounted on the ceiling of a room of height h, at a distance A from the left wall. At a distance B from the same wall, a collecting container is placed on the floor, tilted by an angle β. The shape of the container is approximately an inverted triangle of height L, with a rectangular opening of width δ at the top. The robot holds a rectangular object in the gripper and should place it at the top of the container with the right orientation. The object has width δ, fitting exactly in the opening of the container with zero clearance. Using 4×4 homogeneous transformation matrices, define the kinematic equation of this robotic task in the world frame RF_w. Derive then the expression of the desired end-effector pose ${}^0\boldsymbol{T}_{E,d}$ that should be realized by the direct kinematics ${}^0\boldsymbol{T}_E(\boldsymbol{q})$ of the robot.

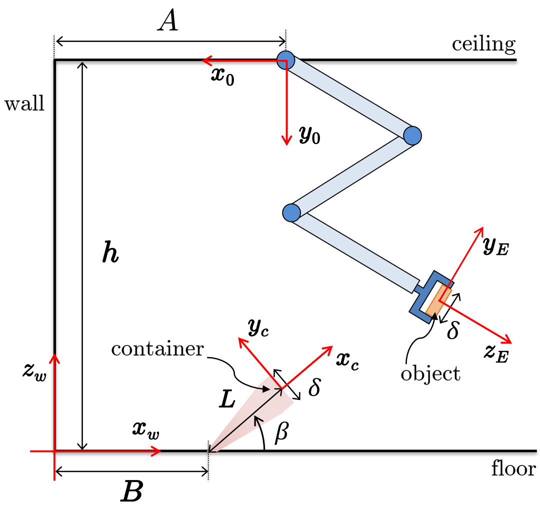

Figure 3: Set up of the robotic task in Exercise 4.

Exercise 5

a) For the RRP planar robot in Fig. 4, assign the Denavit–Hartenberg reference frames so that: *i)* all axes x_i, $i = 1, 2, 3$, are in the plane of motion; *ii)* all constant DH parameters are nonnegative; *iii)* the origin of RF_3 is at the end-effector point $P = (p_x, p_y)$ and z_3 is in the approach direction. On the sheet distributed separately, draw the frames and complete then the corresponding DH table of parameters. With the resulting joint variables $\boldsymbol{q} = (q_1, q_2, q_3)$, write the direct kinematics in the form $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ for $\boldsymbol{r} = (p_x, p_y, \phi)$, where ϕ is the angle of the approach axis z_3 of the last frame with respect to x_0.

b) For a desired $\boldsymbol{r}_d$, solve in closed form the inverse kinematics problem of this robot, assuming that joints have unlimited range. In doing so, identify the conditions for the existence and the number of solutions.

c) Define all quantities needed for implementing the Gradient method for solving the inverse kinematics problem. Find the robot configurations in which this numerical method may get stuck with an error $\boldsymbol{e}^k = \boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q}^k) \neq \boldsymbol{0}$ at the k-th iteration.

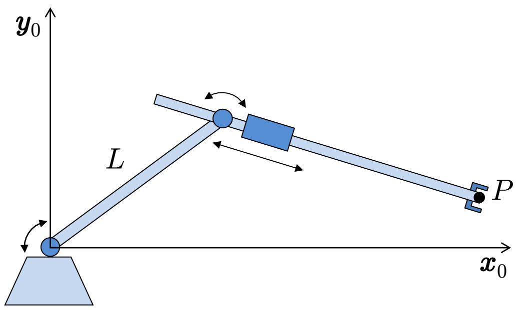

Figure 4: A RRP planar robot.

[180 minutes, open books]

Solution

November 21, 2025

Exercise 1

The assigned rotation matrix $\boldsymbol{R}$ has the elements

$$R_{12} = R_{21} = 0 \qquad R_{13} = R_{31} = 0 \qquad R_{23} = R_{32} = -\frac{1}{\sqrt{2}},$$

so that $\sin\theta = 0$ and we are in a singular situation. Since $R_{11} = -1$, it is $\cos\theta = -1$ and therefore $\theta = \pm\pi$. Replacing the values of the two trigonometric functions in $\boldsymbol{R}(\boldsymbol{r},\theta) = \boldsymbol{R}$ yields

$$2\,\boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{I} = \boldsymbol{R},$$

from which

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{(R_{11}+1)/2} \\ \pm\sqrt{(R_{22}+1)/2} \\ \pm\sqrt{(R_{33}+1)/2} \end{pmatrix} = \begin{pmatrix} 0 \\ \pm\sqrt{(\sqrt{2}-1)/(2\sqrt{2})} \\ \pm\sqrt{(\sqrt{2}+1)/(2\sqrt{2})} \end{pmatrix},$$

with

$$r_x r_y = \frac{R_{12}}{2} = 0 \qquad r_x r_z = \frac{R_{13}}{2} = 0 \qquad r_y r_z = \frac{R_{23}}{2} = -\frac{1}{2\sqrt{2}} < 0.$$

Only the last identity provides information on the correct signs to be chosen for r_y and r_z, which should be indeed opposite. Thus, the two axis/angle solutions are

$$\boldsymbol{r}_1 = \begin{pmatrix} 0 \\ \sqrt{(\sqrt{2}-1)/(2\sqrt{2})} \\ -\sqrt{(\sqrt{2}+1)/(2\sqrt{2})} \end{pmatrix} = \begin{pmatrix} 0 \\ 0.3827 \\ -0.9239 \end{pmatrix} \qquad \theta_1 = \pi$$

and

$$\boldsymbol{r}_2 = -\boldsymbol{r}_1 = \begin{pmatrix} 0 \\ -\sqrt{(\sqrt{2}-1)/(2\sqrt{2})} \\ \sqrt{(\sqrt{2}+1)/(2\sqrt{2})} \end{pmatrix} = \begin{pmatrix} 0 \\ -0.3827 \\ 0.9239 \end{pmatrix} \qquad \theta_2 = -\theta_1 = -\pi.$$

Using these numerical values, it is easy to check from the definition

$$\boldsymbol{R}(\boldsymbol{r},\theta) = \boldsymbol{r}\boldsymbol{r}^T + (\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta$$

that

$$\boldsymbol{R}(\boldsymbol{r}_1,\theta_1) = \boldsymbol{R}(\boldsymbol{r}_2,\theta_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -0.7071 & -0.7071 \\ 0 & -0.7071 & 0.7071 \end{pmatrix} = \boldsymbol{R}.$$

The quaternion representation of the given $\boldsymbol{R}$ is easily found from the axis/angle representation as

$$\mathcal{Q} = \{\eta, \boldsymbol{\varepsilon}\} = \{\cos(\theta_1/2), \sin(\theta_1/2)\,\boldsymbol{r}_1\} = \{\cos(\theta_2/2), \sin(\theta_2/2)\,\boldsymbol{r}_2\} = \{0, (0, 0.3827, -0.9239)\}.$$

This value is unique, also in this singular case, being

$$\cos(\theta_1/2) = \cos(\theta_2/2) = \cos(\pm\pi/2) = 0$$

and

$$\sin(\theta_2/2)\,\boldsymbol{r}_2 = \sin(-\theta_1/2)\,(-\boldsymbol{r}_1) = -\sin(\theta_1/2)\,(-\boldsymbol{r}_1) = \sin(\theta_1/2)\,\boldsymbol{r}_1 = \sin(\pi/2)\,\boldsymbol{r}_1\,(=\boldsymbol{r}_1).$$

Exercise 2

a) The transmission ratios of the two pairs of gears are respectively[1]

$$n_1 = \frac{r_2}{r_1} \qquad \text{and} \qquad n_2 = \frac{r_3}{r_2},$$

with the total transmission ratio being $n = n_1 n_2 = r_3/r_1 = 3.335$. From the balance of the applied motor torque with the inertial torques referred at the rotation axis of the first gear, we have

$$\tau = J_1 \ddot{\theta}_1 + \frac{1}{n_1}\big(J_2 \ddot{\theta}_2 + \frac{1}{n_2} J_3 \ddot{\theta}_3\big). \tag{1}$$

From the transmission ratios, we have also

$$\ddot{\theta}_1 = n_1 \ddot{\theta}_2 \qquad \text{and} \qquad \ddot{\theta}_2 = n_2 \ddot{\theta}_3.$$

Substituting in eq. (1) and setting $\ddot{\theta}_3 = a$, we obtain

$$\tau = \big(J_1 n_1 n_2 + J_2 \frac{n_2}{n_1} + J_3 \frac{1}{n_1 n_2}\big) a. \tag{2}$$

If n_1 is already assigned, eq. (2) can be reorganized as

$$\tau = \Big(\big(J_1 n_1 + \frac{J_2}{n_1}\big)\, n_2 + \big(\frac{J_3}{n_1}\big)\, \frac{1}{n_2}\Big) a. \tag{3}$$

The value of n_2 that minimizes τ for given a and inertias J_i, $i = 1, 2, 3$, is obtained then by setting

$$\frac{\partial \tau}{\partial n_2} = 0 \qquad \Rightarrow \qquad \big(J_1 n_1 + \frac{J_2}{n_1}\big) - \big(\frac{J_3}{n_1}\big)\, \frac{1}{n_2^2} = 0,$$

which is solved by

$$n_2^* = \sqrt{\frac{J_3}{J_1 n_1^2 + J_2}}. \tag{4}$$

This is indeed a minimum for τ since

$$\frac{\partial^2 \tau}{\partial n_2^2} = \frac{2 J_3}{n_1 n_2^3} > 0.$$

Using (4) in (3), the expression of the optimal torque becomes in this case

$$\tau^* = \frac{2\sqrt{J_3 (J_1 n_1^2 + J_2)}}{n_1}\, a. \tag{5}$$

b) Using the numerical data, being $n_1 = r_2/r_1 = 5$ and $n_2 = r_3/r_2 = 0.667$, we obtain from (2) a torque value $\tau = 1.805 \cdot 10^{-4}$ Nm. Instead, from (4) we have

$$n_2^* = 0.1896,$$

which is more than three times smaller than n_2. With the transmission ratio n_2^*, the optimal torque (5) becomes

$$\tau^* = \frac{2\sqrt{4.5 \cdot 10^{-6}\,(5 \cdot 10^{-6} \cdot 25 + 2 \cdot 10^{-7})}}{5}\, 10 = 9.49 \cdot 10^{-5} \text{ Nm},$$

which is about 50% smaller than before.

Exercise 3

The DH frames corresponding to Tab. 1 for the 3R spatial robot of Fig. 2 are shown in Fig. 5. Note that these frames are uniquely defined, once the DH parameters and the base frame have been specified. Also shown are the joint variables $q_i = \theta_i$, $i = 1, 2, 3$, in the current configuration, in which $q_3 = 0$.

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[1] These relations hold no matter if the n_i are greater, equal, or less than 1.

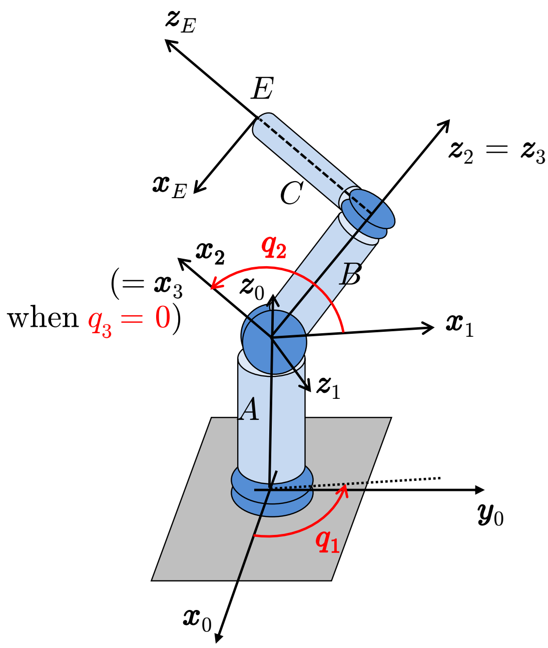

Figure 5: The DH frames for the 3R spatial robot of Fig. 2 corresponding to Tab. 1.

The third (and last) link of the robot carries two frames attached to it, the third DH frame RF_3 and the end-effector frame. Accordingly, the constant homogeneous transformation matrix from frame RF_3 to frame RF_E is

$$ {}^3\boldsymbol{T}_E = \begin{pmatrix} 0 & 0 & 1 & C \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & B \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Exercise 4

We define the following 4×4 homogeneous transformations matrices.

- World frame to robot base frame

$$ {}^w\boldsymbol{T}_0 = \begin{pmatrix} -1 & 0 & 0 & A \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

- World frame to container frame (at the top of it)

$$ {}^w\boldsymbol{T}_c = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & B + L\cos\beta \\ 0 & 0 & -1 & 0 \\ \sin\beta & \cos\beta & 0 & L\sin\beta \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

- Robot base frame to end-effector frame

$$ {}^0\boldsymbol{T}_E(\boldsymbol{q}) = {}^0\boldsymbol{T}_3(\boldsymbol{q})\,{}^3\boldsymbol{T}_E = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

where an asterisk stands for a generic (configuration-dependent) nonzero element. This matrix represents the direct kinematics of the 3R planar robot (whose derivation is not requested by the problem), combined with a constant rotation from the last DH frame to the end-effector frame (having z_E in the approach direction and y_E in the slide direction of the gripper).

- Robot end-effector frame to container frame

$$
{}^E\boldsymbol{T}_c = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

which specifies the way the object held by the end-effector gripper should be placed in order to fit into the opening of the container, namely with both frame origins centered and with axes z_E and x_c aligned and opposite.

The kinematic equation describing the task in the world frame is then

$$
{}^w\boldsymbol{T}_0\, {}^0\boldsymbol{T}_E(\boldsymbol{q})\, {}^E\boldsymbol{T}_c = {}^w\boldsymbol{T}_c, \tag{6}
$$

which represent the loop closure going directly to the top of the container or passing through the robot. From (6), we isolate the robot direct kinematics as

$$
{}^0\boldsymbol{T}_E(\boldsymbol{q}) = {}^w\boldsymbol{T}_0^{-1}\, {}^w\boldsymbol{T}_c\, {}^E\boldsymbol{T}_c^{-1} = {}^0\boldsymbol{T}_{E,d}.
$$

The desired pose is then computed as

$$
\begin{aligned}
{}^0\boldsymbol{T}_{E,d} &= \begin{pmatrix} -1 & 0 & 0 & A \\ 0 & 0 & -1 & h \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta & 0 & B + L\cos\beta \\ 0 & 0 & -1 & 0 \\ \sin\beta & \cos\beta & 0 & L\sin\beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sin\beta & \cos\beta & A - B - L\cos\beta \\ 0 & \cos\beta & \sin\beta & h - L\sin\beta \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

The matrix structure of this desired pose is consistent with the structure of matrix ${}^0\boldsymbol{T}_E(\boldsymbol{q})$, allowing the possible solution of the inverse kinematics problem using the three joints of this robot.

Exercise 5

a) The requested DH frame assignment for the PPR robot in Fig. 4 is shown in Fig. 6. The corresponding DH parameters are given in Tab. 2.

i	α_i	a_i	d_i	θ_i
1	0	L	0	q_1
2	$\pi/2$	0	0	q_2
3	0	0	q_3	0

Table 2: DH table of parameters for the frame assignment in Fig. 6.

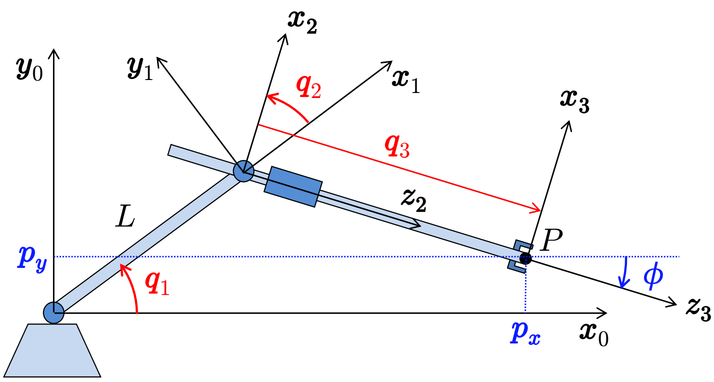

Figure 6: DH frame assignment for the PPR robot in Fig. 4.

Accordingly, the DH homogeneous transformation matrices are (using the shorthand notation):

$$ {}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & -s_1 & 0 & Lc_1 \\ s_1 & c_1 & 0 & Ls_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Then, the direct kinematics is described by

$$ {}^0\boldsymbol{T}_3(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_{12} & 0 & s_{12} & Lc_1 + q_3 s_{12} \\ s_{12} & 0 & -c_{12} & Ls_1 - q_3 c_{12} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Therefore, the requested compact form of the direct kinematics is

$$ \boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} Lc_1 + q_3 s_{12} \\ Ls_1 - q_3 c_{12} \\ q_1 + q_2 - \frac{\pi}{2} \end{pmatrix}. \tag{7} $$

The third equation follows from the fact that x_3 (parallel to x_2) makes an angle $q_1 + q_2$ with x_0 —see the first column of ${}^0\boldsymbol{R}_3(\boldsymbol{q})$; the direction of the approach axis z_3 is then rotated by $-90°$ (counterclockwise) —see also the third column of ${}^0\boldsymbol{R}_3(\boldsymbol{q})$.

b) For an assigned value $\boldsymbol{r}_d = (p_{xd}, p_{yd}, \phi_d)$ to vector $\boldsymbol{r}$ in (7), the inverse kinematics problem is addressed as follows. From the third equation, we have $q_1 + q_2 = \phi_d + \pi/2$. Replacing this argument in s_{12} and c_{12} of the first two equations, and using trigonometric identities, yields

$$ \begin{aligned} p_{xd} &= Lc_1 + q_3 c_{\phi_d} \\ p_{yd} &= Ls_1 + q_3 s_{\phi_d}. \end{aligned} \tag{8} $$

Isolating the terms in q_1, then squaring and adding the two equations in (8) results in

$$ (p_{xd} - q_3 c_{\phi_d})^2 + (p_{yd} - q_3 s_{\phi_d})^2 = (Lc_1)^2 + (Ls_1)^2 = L^2. $$

Expanding terms, one obtains the second-order polynomial equation in q_3

$$ q_3^2 - 2\left(p_{xd} c_{\phi_d} + p_{yd} s_{\phi_d}\right) q_3 + \left(p_{xd}^2 + p_{yd}^2 - L^2\right) = 0. $$

Its two solutions are

$$ q_3^{(1),(2)} = \left(p_{xd} c_{\phi_d} + p_{yd} s_{\phi_d}\right) \pm \sqrt{\Delta}, \tag{9} $$

with

$$ \Delta = \left(p_{xd} c_{\phi_d} + p_{yd} s_{\phi_d}\right)^2 - \left(p_{xd}^2 + p_{yd}^2 - L^2\right). $$

Provided that the discriminant $\Delta \geq 0$, the two solutions are both real (coincide if $\Delta = 0$). It can easily be shown that the discriminant simplifies also to

$$\Delta = L^2 - \left(p_{xd} s_{\phi_d} - p_{yd} c_{\phi_d}\right)^2 .$$

Thus, the condition for existence of solution(s) is

$$L \geq \left|p_{xd} s_{\phi_d} - p_{yd} c_{\phi_d}\right| . \tag{10}$$

When (10) is satisfied, reorganize the equations (8) using each of the two solutions (9) as

$$s_1 = \frac{p_{yd} - q_3^{(i)} s_{\phi_d}}{L} \qquad c_1 = \frac{p_{xd} - q_3^{(i)} c_{\phi_d}}{L}, \qquad i = 1, 2,$$

from which

$$q_1^{(i)} = \text{atan2}\left\{p_{yd} - q_3^{(i)} s_{\phi_d}, p_{xd} - q_3^{(i)} c_{\phi_d}\right\}, \qquad i = 1, 2. \tag{11}$$

Finally,

$$q_2^{(i)} = \phi_d + \frac{\pi}{2} - q_1^{(i)}, \qquad i = 1, 2. \tag{12}$$

Some care has to be used in handling (12), as the result should be mapped in the principal interval $(-\pi, \pi]$.

Summarizing, the inverse kinematics problem has two solutions $\boldsymbol{q}^{(1)}$ and $\boldsymbol{q}^{(2)}$ from (9), (11) and (12) in the regular case (when $\Delta > 0$), one solution $\boldsymbol{q}^{(1)} = \boldsymbol{q}^{(2)}$ in the singular case (when $\Delta = 0$), and no solution if condition (10) is violated.

Figure 7 shows graphically the two solutions for the case $L = 1$, $p_{xd} = 2.5$, $p_{yd} = \sqrt{3}/2 = 0.8660$ [m], and $\phi_d = 0$ rad. Since $\Delta = 0.25$, the two computed solutions in this regular case are

$$\boldsymbol{q}^{(1)} = \begin{pmatrix} 2.0944 \\ -0.5236 \\ 3 \end{pmatrix} = \begin{pmatrix} 120° \\ -30° \\ 3 \end{pmatrix} \qquad \boldsymbol{q}^{(2)} = \begin{pmatrix} 1.0472 \\ 0.5236 \\ 2 \end{pmatrix} = \begin{pmatrix} 60° \\ 30° \\ 2 \end{pmatrix} \qquad \text{[rad/rad/m] or [deg/deg/m].}$$

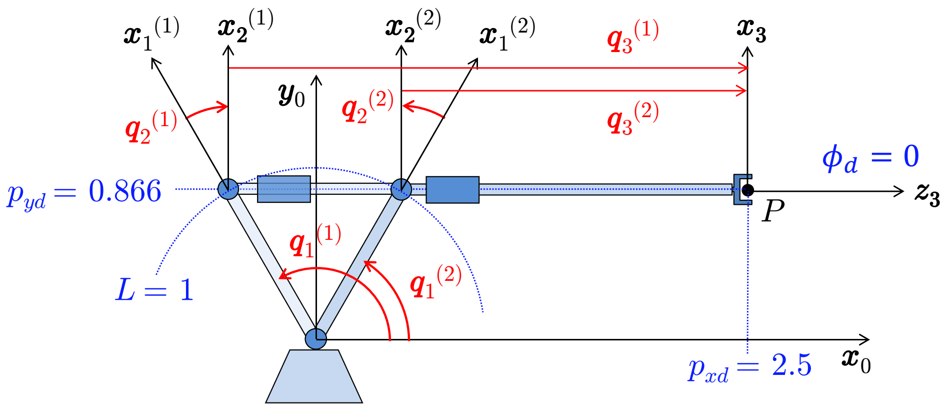

Figure 7: Example of two regular inverse kinematic solutions for the RRP robot.

c) To implement the Gradient method for solving the inverse kinematics problem of this robot with the iteration

$$\boldsymbol{q}^{k+1} = \boldsymbol{q}^k + \alpha_k \boldsymbol{J}^T(\boldsymbol{q}^k)\left(\boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q}^k)\right), \tag{13}$$

for a suitable $\alpha_k > 0$, we need the direct kinematics function $\boldsymbol{f}(\boldsymbol{q})$, given by (7), and its analytic Jacobian, which is computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -Ls_1 + q_3 c_{12} & q_3 c_{12} & s_{12} \\ Lc_1 + q_3 s_{12} & q_3 s_{12} & -c_{12} \\ 1 & 1 & 0 \end{pmatrix}. \tag{14}$$

This matrix has determinant

$$\det \boldsymbol{J}(\boldsymbol{q}) = L s_2,$$

and is singular when $q_2 = 0$ or π. Evaluating the transpose of the Jacobian matrix (14), say, at the singularity $\boldsymbol{q}_s = (q_1, 0, q_3)$ with generic values of q_1 and q_3, yields

$$\boldsymbol{J}_s^T = \boldsymbol{J}^T(\boldsymbol{q}_s) = \begin{pmatrix} -Ls_1 + q_3 c_1 & Lc_1 + q_3 s_1 & 1 \\ q_3 c_1 & q_3 s_1 & 1 \\ s_1 & -c_1 & 0 \end{pmatrix}.$$

It is easy to see that the null space of matrix $\boldsymbol{J}_s^T$ is given by all vectors in $\mathbb{R}^3$ of the form

$$\mathcal{N}(\boldsymbol{J}_s^T) = \{\boldsymbol{e}_\gamma\} = \gamma \begin{pmatrix} c_1 \\ s_1 \\ -q_3 \end{pmatrix} \qquad \forall \gamma,$$

i.e., such that $\boldsymbol{J}_s^T \boldsymbol{e}_\gamma = \boldsymbol{0}$. Then, if the robot is in $\boldsymbol{q}_s$ and $\boldsymbol{e}_k = \boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q}^k) \in \mathcal{N}(\boldsymbol{J}_s^T) - \{\boldsymbol{0}\}$, the gradient update in (13) vanishes and $\boldsymbol{q}^{k+1} = \boldsymbol{q}^k$ (the method is stuck with a nonzero task error). Similar reasoning holds for the other class of singular configurations having $q_2 = \pi$ rad.

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