Robotics 1

Midterm Test — November 18, 2022

Exercise 1

Consider the rotation matrix

$$\boldsymbol{R}_d = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix}.$$

Find, if possible, all angle-axis pairs $(\theta, \boldsymbol{r})$ that provide the desired orientation $\boldsymbol{R}_d$. At the end, check your results by verifying that $\boldsymbol{R}(\theta, \boldsymbol{r}) = \boldsymbol{R}_d$.

Exercise 2

The end-effector of a robot undergoes a change of orientation between an initial $\boldsymbol{R}_i$ and a final $\boldsymbol{R}_f$, as specified by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\frac{\sqrt{3}}{2} \\ -1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Provide a minimal representation of the relative rotation between the initial and the final orientation using YXY Euler angles $(\alpha_1, \alpha_2, \alpha_3)$. At the end, check your solutions by performing the direct computation.

Exercise 3

A DC motor is used to move a link of length $L = 0.7$ [m], as shown in Fig. 1. The motor mounts on its axis an absolute encoder and uses as transmission elements an Harmonic Drive having a flexspline with $N_{FS} = 160$ teeth and a gear with two toothed wheels of radius $r_1 = 2$ and $r_2 = 4$ [cm], respectively.

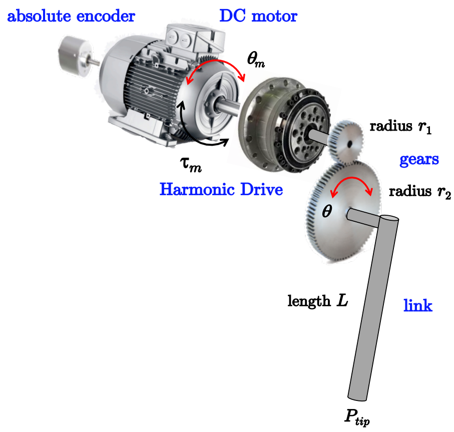


Figure 1: The actuation arrangement of a single link.

- Compute the reduction ratio $n_r > 1$ of the transmission system. Which is the direction of rotation of the link when the motor angular position θ_m is turning counterclockwise?
- Determine the resolution of the absolute encoder that allows distinguishing two link tip positions that are $\Delta r = 0.1$ [mm] away. What should be the minimum number of tracks N_t of the encoder?
- If the link has an angular range $\Delta\theta_{max} = 180°$, how many turns of the motor are needed to cover the entire range? With a multi-turn absolute encoder, what is the minimum number of bits for counting all these turns?
- If the motor inertia is $J_m = 1.2 \cdot 10^{-4}$ [kgm^2], determine the optimal value of the link inertia J_l around the axis at its base which minimizes the motor torque τ_m needed for a desired link acceleration $\ddot{\theta}$. What is then the value of τ_m (in [Nm]) for $\ddot{\theta} = 7$ [rad/s^2]?

Exercise 4

A large 6R robot manipulator is mounted on the ceiling of an industrial cell and holds firmly a cylindric object in its jaw gripper. The world frame RF_w of the cell is placed on the floor, at about the cell center. The robot base frame RF_0 is defined by ${}^w\boldsymbol{T}_0$, while its end-effector frame RF_e has the origin O_e at the center of the grasped object. The robot direct kinematics is expressed in symbolic form by ${}^0\boldsymbol{T}_e(\boldsymbol{q})$, in terms of the joint variables $\boldsymbol{q}$. A camera is placed in the cell and its frame RF_c, having the origin O_c at the center of the image plane and the $\boldsymbol{z}_c$ unit vector along the focal axis of the camera, is defined by ${}^w\boldsymbol{T}_c$.

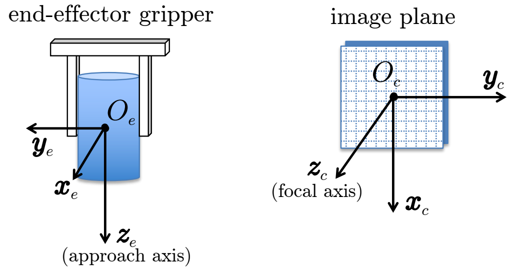


Figure 2: Definition of frames RF_e and RF_c for the considered task.

Figure 2 details the placement of the end-effector frame RF_e and of the camera frame RF_c. The robot should hold the object in front of the camera, with the major axis of the cylinder aligned to the camera focal axis and its center at a distance $d > 0$ from O_c. Define the task kinematics equation, to be solved for the joint variables $\boldsymbol{q}$, when the transformation matrices and the object-camera offset are given by

$$ {}^w\boldsymbol{T}_0 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 3.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^w\boldsymbol{T}_c = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad d = 1 \text{ [m]}. $$

Discuss also whether the robot is kinematically redundant for the task or not.

Exercise 5

For the spatial RPR robot of Fig. 3, complete the assignment of Denavit-Hartenberg (DH) frames and fill in the associated table of parameters. The origin of the last frame should be placed at the point P. Moreover, the frame assignment should be such that all constant DH parameters are *non-negative* and the value of the joint variables q_i, $i = 1, 2, 3$, are *strictly positive* in the shown configuration. Compute then the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of point P.

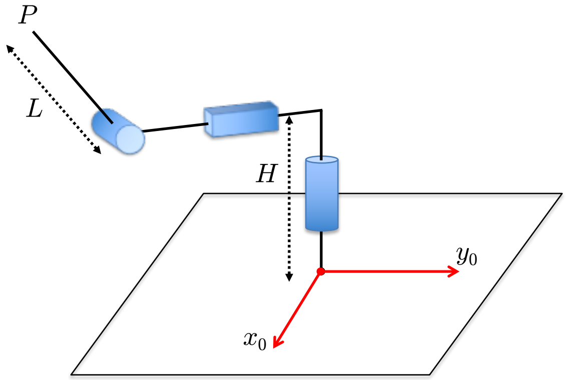


Figure 3: A spatial RPR robot.

Exercise 6

For the spatial RPR robot of Fig. 3, provide the closed-form expression of the inverse kinematics for the position $\boldsymbol{p}$ of point P. Assuming for simplicity that the joints have unlimited ranges, how many inverse kinematics solutions are there in the regular case? Compute the numerical values of all inverse solutions $\boldsymbol{q}$ when $\boldsymbol{p} = (3, 4, 1.5)$ [m] and the geometric parameters of the robot are $H = L = 1$ [m]. Check the solutions!

[180 minutes, open books]

Solution

November 18, 2022

Exercise 1

It is easy to verify that $\boldsymbol{R}_d \in SO(3)$. Denoting by r_{ij} the elements of $\boldsymbol{R}_d$, since the matrix is symmetric, it is

$$\sin\theta = \frac{1}{2}\sqrt{(r_{12}-r_{21})^2 + (r_{13}-r_{31})^2 + (r_{23}-r_{32})^2} = 0.$$

We are in a singular case for the inverse problem of extracting an angle and axis from a rotation matrix. Moreover,

$$\cos\theta = \frac{\text{trace}\{\boldsymbol{R}_d\} - 1}{2} = -1 \qquad \Rightarrow \qquad \theta = \pi \quad (\text{or } -\pi, \text{ which is the same angle}).$$

Therefore, a solution exists for $\boldsymbol{r}$ and we shall use the special formulas

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{\dfrac{r_{11}+1}{2}} \\ \pm\sqrt{\dfrac{r_{22}+1}{2}} \\ \pm\sqrt{\dfrac{r_{33}+1}{2}} \end{pmatrix} = \begin{pmatrix} \pm\dfrac{1}{\sqrt{6}} \\ \pm\dfrac{2}{\sqrt{6}} \\ \pm\dfrac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \pm 0.4082 \\ \pm 0.8165 \\ \pm 0.4082 \end{pmatrix},$$

where the correct combinations of signs (among the 8 possibilities) should be determined so as to guarantee that the remaining three equalities in $\boldsymbol{R}_d = 2\boldsymbol{r}\boldsymbol{r}^T - \boldsymbol{I}$ hold:

$$2\,r_x r_y = r_{12} = \frac{2}{3}, \qquad 2\,r_x r_z = r_{13} = -\frac{1}{3}, \qquad 2\,r_y r_z = r_{23} = -\frac{2}{3}.$$

By coding this logic, one obtains the two solutions

$$\boldsymbol{r}_1 = \begin{pmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{pmatrix}, \qquad \boldsymbol{r}_2 = \begin{pmatrix} -0.4082 \\ -0.8165 \\ 0.4082 \end{pmatrix} = -\boldsymbol{r}_1.$$

Using

$$\boldsymbol{R}(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta,$$

we can check that $\boldsymbol{R}(\theta,\boldsymbol{r}_1) = \boldsymbol{R}(\theta,\boldsymbol{r}_2) = \boldsymbol{R}_d$ is satisfied.

Exercise 2

The relative rotation ${}^i\boldsymbol{R}_f$ between the initial orientation $\boldsymbol{R}_i$ and the final orientation $\boldsymbol{R}_f$ is computed as

$${}^i\boldsymbol{R}_f = \boldsymbol{R}_i^T \boldsymbol{R}_f = \begin{pmatrix} 0 & 0 & -1 \\ 0.5 & -\dfrac{\sqrt{3}}{2} & 0 \\ -\dfrac{\sqrt{3}}{2} & -0.5 & 0 \end{pmatrix}.$$

On the other hand, the rotation matrix associated to a minimal representation with YXY Euler angles $(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\begin{aligned}
\boldsymbol{R}_{YXY}(\alpha_1,\alpha_2,\alpha_3) &= \boldsymbol{R}_Y(\alpha_1)\boldsymbol{R}_X(\alpha_2)\boldsymbol{R}_Y(\alpha_3) = \\
&= \begin{pmatrix} \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 1 & 0 \\ -\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \cos\alpha_3 & 0 & \sin\alpha_3 \\ 0 & 1 & 0 \\ -\sin\alpha_3 & 0 & \cos\alpha_3 \end{pmatrix} \\
&= \begin{pmatrix} c_1 c_3 - s_1 c_2 s_3 & s_1 s_2 & c_1 s_3 + s_1 c_2 c_3 \\ s_2 s_3 & c_2 & -s_2 c_3 \\ -s_1 c_3 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{pmatrix},
\end{aligned}$$

where the usual shorthand notation has been used (e.g., $c_i = \cos\alpha_i$). The inverse representation problem, namely finding all triples $(\alpha_1, \alpha_2, \alpha_3)$ of YXY Euler angles such that

$$\boldsymbol{R}_{YXY}(\alpha_1, \alpha_2, \alpha_3) = {}^i\boldsymbol{R}_f, \tag{1}$$

can be solved in closed form (up to singular cases). Denote by r_{ij} the elements of ${}^i\boldsymbol{R}_f$. Taking advantage of the simpler expressions in the second column (viz., second row) of $\boldsymbol{R}_{YXY}$, one has from eq. (1)

$$c_2 = r_{22}, \qquad s_2 = \pm\sqrt{r_{12}^2 + r_{32}^2} \qquad \Rightarrow \qquad \alpha_2 = \text{ATAN2}\left\{s_2, c_2\right\},$$

yielding the two (symmetric) values $\alpha_2^{(I),(II)} = \pm 2.6180$ [rad]. Since $s_2 = \pm\, 0.5 \neq 0$, the problem at hand is regular and computations can be carried out also for the other two angles. We have:

$$s_1 = \frac{r_{12}}{s_2}, \qquad c_1 = \frac{r_{32}}{s_2} \qquad \Rightarrow \qquad \alpha_1 = \text{ATAN2}\left\{s_1, c_1\right\},$$

and

$$s_3 = \frac{r_{21}}{s_2}, \qquad c_3 = \frac{-r_{23}}{s_2} \qquad \Rightarrow \qquad \alpha_3 = \text{ATAN2}\left\{s_3, c_3\right\}.$$

Depending on the sign chosen for s_2, there are again two solutions for each angle. We obtain

$$\alpha_1^{(I)} = \pi, \quad \alpha_1^{(II)} = 0 \qquad \text{and} \qquad \alpha_3^{(I)} = \frac{\pi}{2}, \quad \alpha_3^{(II)} = -\frac{\pi}{2} \ \text{[rad]}.$$

As a result, the two (regular) solutions of the problem are:

$$\boldsymbol{\alpha}^{(I)} = \begin{pmatrix} \pi \\ \frac{5\pi}{6} \\ \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 3.1416 \\ 2.6180 \\ 1.5708 \end{pmatrix}, \qquad \boldsymbol{\alpha}^{(II)} = \begin{pmatrix} 0 \\ -\frac{5\pi}{6} \\ -\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -2.6180 \\ -1.5708 \end{pmatrix} \ \text{[rad]}.$$

It is easy to check that

$$\boldsymbol{R}_i \boldsymbol{R}_{YXY}(\boldsymbol{\alpha}^{(I)}) = \boldsymbol{R}_i \boldsymbol{R}_{YXY}(\boldsymbol{\alpha}^{(II)}) = \boldsymbol{R}_f.$$

Exercise 3

The reduction ratio n_r of the entire transmission is the product of the reduction ratios n_{HD} of the Harmonic Drive and n_g of the spur gear:

$$n_r = n_{HD} \cdot n_g = \frac{N_{FS}}{2} \cdot \frac{r_2}{r_1} = 80 \cdot 2 = 160.$$

Both transmission elements invert on the output axis the direction of rotation of their input axis. As a result, the angular position θ of the link is turning is the same direction (positive counterclockwise) of the angular position θ_m of the motor.

A linear variation $\Delta r = 1 \cdot 10^{-4}$ [m] in position at the tip of the link corresponds to an angular variation $\Delta\theta$ at the base. Therefore, the needed resolution $\Delta\theta_m$ at the motor side (where the absolute encoder is mounted) is

$$\Delta\theta_m = \Delta\theta \cdot n_r = \frac{\Delta r}{L} \cdot n_r = 1.4286 \cdot 10^{-4} \cdot 160 = 0.0229 \ \text{[rad]} \ (= 1.31°).$$

Being the resolution of an absolute encoder equal to $\Delta = 2\pi/2^{N_t}$, the request $\Delta \leq \Delta\theta_m$ implies that the minimum number of tracks N_t is the integer

$$N_t = \left\lceil \log_2\left(\frac{2\pi}{\Delta\theta_m}\right) \right\rceil = \lceil 8.1027 \rceil = 9.$$

In order to cover the entire range $\Delta\theta_{max}$ (in degrees) of link angular motion, the number of motor turns is

$$n_{turns} = \frac{\Delta\theta_{max} \cdot n_r}{360°} = \frac{180° \cdot 160}{360°} = 80.$$

For counting this number of turns, the minimum number of devoted bits N_{mt} in a multi-turn absolute encoder should be

$$N_{mt} = \lceil \log_2 80 \rceil = 7.$$

Finally, the optimal value of the link inertia J_l is computed from the optimal value of the reduction ratio:

$$n_r = \sqrt{\frac{J_l}{J_m}} \qquad \Rightarrow \qquad J_l = J_m \cdot n_r^2 = 1.2 \cdot 10^{-4} \cdot 160^2 = 3.0720 \text{ [kgm}^2\text{]}.$$

The motor torque τ_m needed for obtaining a desired link acceleration $\ddot{\theta} = 7$ [rad/s^2] is then

$$\tau_m = J_m \ddot{\theta}_m + \frac{1}{n_r} J_l \ddot{\theta} = \left(J_m n_r + \frac{J_l}{n_r} \right) \ddot{\theta} = (2 J_m n_r) \ddot{\theta} = 0.0384 \cdot 7 = 0.2688 \text{ [Nm]}.$$

Exercise 4

The kinematic identity describing the task is given by

$$^w\boldsymbol{T}_0 \, {}^0\boldsymbol{T}_e(\boldsymbol{q}) = {}^w\boldsymbol{T}_c \, {}^c\boldsymbol{T}_e, \tag{2}$$

in which the desired pose of the robot end-effector in the world frame is equivalently expressed passing through the robot or through the camera, respectively the left-hand side or the right-hand side of (2). Since the unit axes $\boldsymbol{z}_e$ and $\boldsymbol{z}_c$ should be aligned and in the opposite direction ($\boldsymbol{z}_c = -\boldsymbol{z}_e$) and the offset between O_c and O_e should be only along $\boldsymbol{z}_c$, an homogeneous matrix that defines the correct pose of the end-effector, as seen from the camera frame[1], is given by

$$^c\boldsymbol{T}_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } d = 1 \text{ [m]}. \tag{3}$$

Note that this choice is not unique: it corresponds to aligning also the $\boldsymbol{x}_e$ unit vector of the end-effector frame with the unit vector $\boldsymbol{x}_c$ of the camera frame. However, such alignment is not necessary and one may choose to have an arbitrary angle $\alpha \in (\pi, \pi]$ between these two vectors. As a result, also the more general homogeneous matrix

$$^c\boldsymbol{T}_e(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ -\sin\alpha & -\cos\alpha & 0 & 0 \\ 0 & 0 & -1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } d = 1 \text{ [m]}, \tag{4}$$

satisfies the task[2]. Since there is one parameter left free of choice in defining a desired 3D pose, the task is 5-dimensional and the 6R robot has one degree of redundancy in realizing this task (in fact, the task involves *positioning* and *pointing* in 3D).

Given $^w\boldsymbol{T}_0$ and $^w\boldsymbol{T}_c$, one obtains from (2) and (3)

$$^0\boldsymbol{T}_e(\boldsymbol{q}) = \left({}^w\boldsymbol{T}_0\right)^{-1} {}^w\boldsymbol{T}_c \, {}^c\boldsymbol{T}_e = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 2.2929 \\ 0 & -1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2.2071 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

[1] The same description holds as seen from the end-effector frame since in this case $^e\boldsymbol{T}_c = \left({}^c\boldsymbol{T}_e\right)^{-1} = {}^c\boldsymbol{T}_e$, due to the task symmetry.

[2] With $\alpha = \pi$, the unit vectors $\boldsymbol{y}_e$ and $\boldsymbol{y}_c$ would be aligned.

which is the requested task kinematics equation to be solved for $\boldsymbol{q}$ (i,e., the formulation of the inverse kinematics problem for the 6R robot). A similar equation is found when using (4) in place of (3).

Exercise 5

The (unique) DH frame assignment for the RPR robot of Fig. 3 satisfying all requests is shown in Fig. 4. The corresponding DH parameters are reported in Tab. 1.

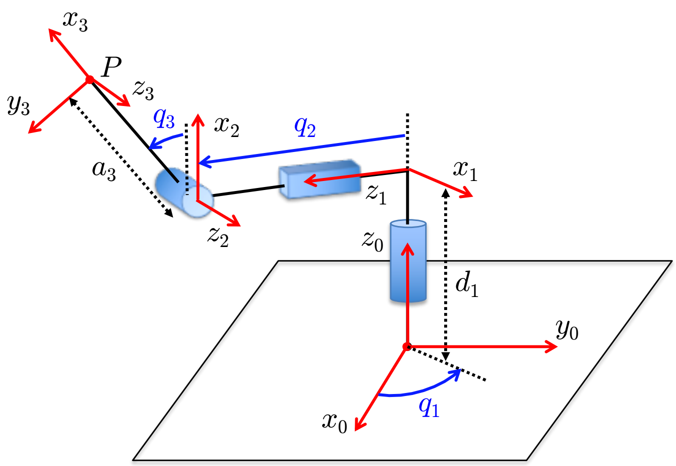


Figure 4: DH frames for the spatial RPR robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = H > 0$	$q_1 > 0$
2	$\pi/2$	0	$q_2 > 0$	$\pi/2$
3	0	$a_3 = L > 0$	0	$q_3 > 0$

Table 1: DH parameters corresponding to the frames in Fig. 4. The signs attributed to the joint variables refer to the shown robot configuration.

From the associated homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \boldsymbol{A}_2(q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we compute

$$\boldsymbol{p}_{hom} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = \boldsymbol{A}_1(q_1) \left(\boldsymbol{A}_2(q_2) \left(\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right) \right),$$

yielding the direct kinematics of the position of point P as

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} s_1 (q_2 + a_3 s_3) \\ -c_1 (q_2 + a_3 s_3) \\ d_1 + a_3 c_3 \end{pmatrix}. \qquad (5)$$

Exercise 6

Consider the direct kinematics (5), with assigned desired values for the components p_x, p_y, and p_z for the position vector $\boldsymbol{p}$ on the left-hand side. From the third equation, one has

$$c_3 = \frac{p_z - d_1}{a_3} \qquad \Rightarrow \qquad s_3 = \pm\sqrt{1 - c_3^2}.$$

Provided that $c_3 \in [-1, 1]$, two symmetric solutions are found for q_3, each corresponding to a sign chosen for s_3:

$$q_3^{(o)} = \text{ATAN2}\{|s_3|, c_3\}, \qquad q_3^{(i)} = \text{ATAN2}\{-|s_3|, c_3\} = -q_3^{(o)}. \tag{6}$$

The solution $q_3^{(o)}$ has the forearm (link 3) bent *outward* from the base joint axis, while with $q_3^{(i)}$ the forearm is bent *inward*. When $c_3 = \pm 1$, the two solutions in (6) collapse into a singleton $q_3 = 0$ (for $c_3 = 1$, link 3 is vertical and points *upward*) or $q_3 = \pi$ (for $c_3 = -1$, link 3 is vertical and points *downward*). These two situations are a *singularity* for the solution q_3. When $|c_3| > 1$, the inverse kinematics problem has no solution because the desired position $\boldsymbol{p}$ of point P is outside the reachable workspace of the robot.

Next, squaring and summing the first two equations in (5) yields

$$p_x^2 + p_y^2 = (q_2 + a_3 s_3)^2 \geq 0.$$

If this quantity is strictly positive, we can extract the root and substitute it in place of the common factor in the right-hand side of the first two kinematic equations in (5) so as to obtain

$$p_x = \pm s_1 \sqrt{p_x^2 + p_y^2}, \qquad -p_y = \pm c_1 \sqrt{p_x^2 + p_y^2},$$

which involve only the unknown q_1 and the input data. Then, two solutions are obtained for q_1,

$$q_1 = \text{ATAN2}\left\{ \frac{p_x}{\pm\sqrt{p_x^2 + p_y^2}}, \frac{-p_y}{\pm\sqrt{p_x^2 + p_y^2}} \right\},$$

depending on the upper or lower sign chosen for the square root in both arguments (and independently from the signs in the solution (6) for q_3). Actually, since this computation is performed only when $p_x^2 + p_y^2 > 0$, one can simplify the expression of the solutions as

$$q_1^{(f)} = \text{ATAN2}\{p_x, -p_y\}, \qquad q_1^{(b)} = \text{ATAN2}\{-p_x, +p_y\}. \tag{7}$$

In the solution $q_1^{(f)}$ the base of the robot *faces* point P, whereas with $q_1^{(b)}$ the base is rotated by π and the robot is giving the *back* to point P. If $p_x^2 + p_y^2 = 0$, i.e., the desired position of point P is on the axis of joint 1, q_1 is undefined and there are infinite solutions to the inverse kinematics problem (*singular* case).

Two possible ways can be followed to determine the variable q_2 of the prismatic joint.

First method. Add the first two equations in (5), weighted respectively by s_1 and $-c_1$:

$$s_1 p_x - c_1 p_y = q_2 + a_3 s_3.$$

From this, using the previously obtained results for s_1, c_1 and s_3, we have

$$q_2 = s_1 p_x - c_1 p_y \mp a_3 \sqrt{1 - c_3^2} = \pm\sqrt{p_x^2 + p_y^2} \mp \sqrt{a_3^2 - (p_z - d_1)^2}. \tag{8}$$

Note that the argument of the last square root in (8) is always non-negative (otherwise the desired position $\boldsymbol{p}$ of point P would be outside the reachable workspace, as already noted). There are four combinations of possible signs to be chosen in eq. (8), resulting in four solutions for q_2 in the regular case, each corresponding to one of the alternative solutions for q_1 and for q_3. When the solution for q_3 is in singularity, meaning that $a_3^2 = (p_z - d_1)^2$, only two solutions are left for q_2. The same occurs when the solution for q_1 is in singularity ($p_x = p_y = 0$). At the intersection of the singularities, there is only one solution, namely $q_2 = 0$.

Second method. Square and sum all three equations in (5), after having moved d_1 to the left in the third one. This leads to

$$p_x^2 + p_y^2 + (p_z - d_1)^2 = (q_2 + a_3 s_3)^2 + (a_3 c_3)^2 = q_2^2 + a_3^2 + 2 a_3 s_3 q_2.$$

This is a polynomial equation of second degree in the unknown q_2, which can be rewritten in the form

$$q_2^2 + 2b\, q_2 - c = 0,$$

with

$$b = a_3 s_3 = \pm\sqrt{a_3^2 - (p_z - d_1)^2}, \qquad c = p_x^2 + p_y^2 + (p_z - d_1)^2 - a_3^2.$$

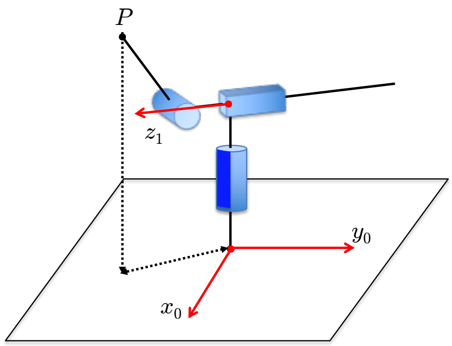
Accordingly, we obtain two pairs of solutions (one pair for each sign chosen for b)

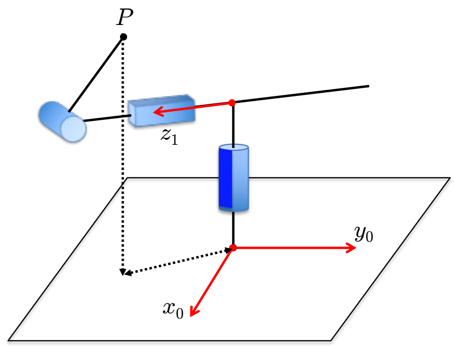
$$\begin{aligned} q_2^{(++/+-)} &= b \pm \sqrt{b^2 + c} = \sqrt{a_3^2 - (p_z - d_1)^2} \pm \sqrt{p_x^2 + p_y^2} \\ q_2^{(-+/--)} &= -b \pm \sqrt{b^2 + c} = -\sqrt{a_3^2 - (p_z - d_1)^2} \pm \sqrt{p_x^2 + p_y^2}. \end{aligned} \tag{9}$$

The two eqs. (9) are clearly equivalent to eq. (8). When $b = 0$, only two solutions are left. When $b = c = 0$ simultaneously, $q_2 = 0$ is the only solution.

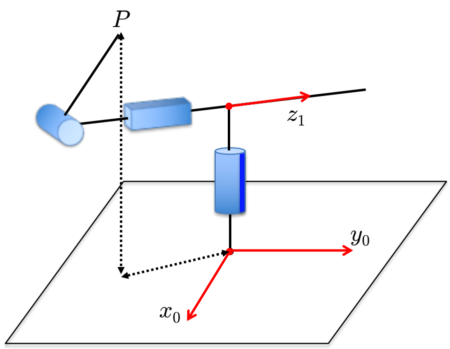
The four generic solutions in the regular case are summarized below, each having a sketch of the associated robot configuration (the front part of the robot base, where $\boldsymbol{z}_1$ is pointing, is shown in dark blue).

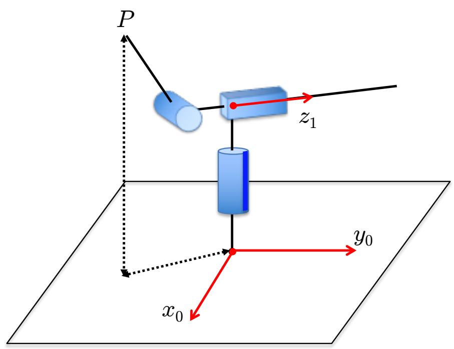
$$\boldsymbol{q}^{(1)} = \begin{pmatrix} q_1^{(f)} \\ q_2^{(+-)} \\ q_3^{(o)} \end{pmatrix} \text{ (base facing, forearm outward)} \qquad \boldsymbol{q}^{(2)} = \begin{pmatrix} q_1^{(f)} \\ q_2^{(++)} \\ q_3^{(i)} \end{pmatrix} \text{ (base facing, forearm inward)}$$





$$\boldsymbol{q}^{(3)} = \begin{pmatrix} q_1^{(b)} \\ q_2^{(--)} \\ q_3^{(o)} \end{pmatrix} \text{ (base backing, forearm outward)} \qquad \boldsymbol{q}^{(4)} = \begin{pmatrix} q_1^{(b)} \\ q_2^{(-+)} \\ q_3^{(i)} \end{pmatrix} \text{ (base backing, forearm inward)}$$





Consider now the given numerical data. Since $d_1 = H = 1$ and $a_3 = L = 1$ [m], the four (regular) solutions for $\boldsymbol{p} = (3, 4, 1.5)$ are:

$$\boldsymbol{q}^{(1)} = \begin{pmatrix} 2.4981 \\ 4.1340 \\ 1.0472 \end{pmatrix}, \ \boldsymbol{q}^{(2)} = \begin{pmatrix} 2.4981 \\ 5.8660 \\ -1.0472 \end{pmatrix}, \ \boldsymbol{q}^{(3)} = \begin{pmatrix} -0.6435 \\ -5.8660 \\ 1.0472 \end{pmatrix}, \ \boldsymbol{q}^{(4)} = \begin{pmatrix} -0.6435 \\ -4.1340 \\ -1.0472 \end{pmatrix} \ \text{[rad/m/rad]}.$$

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