Robotics I

A: preferred for 6 credits

January 12, 2010

Exercise 1

Consider the Cartesian path defined by

$$\boldsymbol{p} = \boldsymbol{p}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} R\cos s \\ R\sin s \\ h\,s \end{pmatrix}, \quad s \in [0, +\infty)$$

where $R > 0$ and $h > 0$. This path is a spiral around the $\boldsymbol{z}$-axis. Define a timing law $s = s(t)$ having a *trapezoidal speed* profile in $t \in [0, T]$, for a given and sufficiently large final time $T > 0$, such that the resulting planned trajectory $\boldsymbol{p}_d(t) = \boldsymbol{p}(s(t))$ satisfies the following conditions:

- $\dot{\boldsymbol{p}}_d(0) = \dot{\boldsymbol{p}}_d(T) = \boldsymbol{0}$;
- $\|\dot{\boldsymbol{p}}_d(t)\| \leq V$, for a given $V > 0$;
- $\|\ddot{\boldsymbol{p}}_d(t)\| \leq A$, for a given and sufficiently large $A > 0$.

Provide in particular the reached height $z_d(T)$ in closed form.

Moreover, define a *coordinated motion* for the *orientation* along the above path, by specifying a moving frame that has its $\boldsymbol{x}_o$ axis always pointing and orthogonal to the central axis of the spiral (the $\boldsymbol{z}$-axis) and its $\boldsymbol{z}_o$ always parallel to it. What is the maximum value reached by the norm of the angular velocity, $\|\boldsymbol{\omega}\|$, associated to the planned trajectory?

Finally, evaluate the solution found for the following numerical data:

$$R = 0.3 \text{ [m]}, \quad h = 0.1 \text{ [m]}, \quad V = 1 \text{ [m/s]}, \quad A = 5 \text{ [m/s}^2\text{]}, \quad T = 4 \text{ [s]}.$$

Exercise 2A

Extend the design of an input-output linearizing (and decoupling) trajectory controller presented in the textbook, as well as in class, for the case of a unicycle to the kinematic model of a front-wheel driven car-like vehicle. This control design should allow a suitable point B attached to the car-like vehicle to reproduce exactly (in nominal conditions) and to track in a stable way (in presence of non-persistent disturbances) *any continuous reference trajectory*, possibly having velocity discontinuities. Provide the full expression of the control law, analyzing its singularities (if any), and of the resulting closed-loop system. Discuss the pros and cons of this control approach, in particular with respect to the presence of obstacles in the vicinity of the reference trajectory.

[150 minutes; open books]

Solutions

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Exercise 1

The velocity vector along the path is given by

$$\dot{\boldsymbol{p}}_d = \frac{d\boldsymbol{p}_d(t)}{dt} = \frac{d\boldsymbol{p}(s)}{ds}\frac{ds(t)}{dt} = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix} \dot{s},$$

and thus

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{R^2+h^2}\,|\dot{s}(t)|.$$

The constraint $\|\dot{\boldsymbol{p}}_d(t)\| \leq V$ on the Cartesian velocity becomes

$$|\dot{s}(t)| \leq \frac{V}{\sqrt{R^2+h^2}} =: V_{\max}$$

for the speed profile $\dot{s}$.

The acceleration vector along the path is given by

$$\ddot{\boldsymbol{p}}_d = \frac{d^2\boldsymbol{p}_d(t)}{dt^2} = \frac{d\boldsymbol{p}(s)}{ds}\ddot{s}(t) + \frac{d^2\boldsymbol{p}(s)}{ds^2}\dot{s}^2(t) = \begin{pmatrix} -R\sin s \\ R\cos s \\ h \end{pmatrix} \ddot{s} + \begin{pmatrix} -R\cos s \\ -R\sin s \\ 0 \end{pmatrix} \dot{s}^2,$$

and thus

$$\|\ddot{\boldsymbol{p}}_d(t)\| = \sqrt{(R^2+h^2)\,\ddot{s}^2(t) + (R\,\dot{s}^2(t))^2}.$$

The constraint $\|\ddot{\boldsymbol{p}}_d(t)\| \leq A$ on the Cartesian acceleration can be rewritten as

$$(R^2+h^2)\,\ddot{s}^2(t) \leq A^2 - (R\,\dot{s}^2(t))^2$$

for the acceleration profile $\ddot{s}$. Since this constraint has to be satisfied for all $t \in [0, T]$, one should consider the worst case, i.e., $|\dot{s}| = V_{\max}$. We obtain

$$|\ddot{s}(t)| \leq \sqrt{\frac{A^2 - (\frac{RV^2}{R^2+h^2})^2}{R^2+h^2}} =: A_{\max}.$$

In order to have a feasible $A_{\max} > 0$, the value of A should be sufficiently large, i.e.,

$$A > \frac{RV^2}{R^2+h^2}. \tag{1}$$

At this stage, given the total time T and the computed limits $V_{\max}$ and $A_{\max}$, the timing law with trapezoidal speed profile is fully specified. In particular, we have for the acceleration/deceleration interval time

$$T_s = \frac{V_{\max}}{A_{\max}} = \frac{V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}.$$

In order to have a complete trapezoidal profile (with at least one instant where $V_{\max}$ is reached), the total time T should be sufficiently large, i.e.,

$$T \geq 2T_s = \frac{2V}{\sqrt{A^2 - (\frac{RV^2}{R^2+h^2})^2}}. \tag{2}$$

The total displacement of the parameter s at time $t = T$ is then

$$s_{\max} := s(T) = (T - T_s)V_{\max} = TV_{\max} - \frac{V_{\max}^2}{A_{\max}} = \frac{TV}{\sqrt{R^2 + h^2}} - \frac{V^2}{\sqrt{(R^2 + h^2)A^2 - \frac{(RV^2)^2}{R^2+h^2}}}.$$

Therefore, the reached height at the final time $t = T$ is

$$z_d(T) = h\, s(T) = h\, s_{\max}.$$

For completeness, we compute also the curvature of the given parametric path:

$$\kappa(s) = \frac{\left\| \dfrac{d\boldsymbol{p}}{ds} \times \dfrac{d^2\boldsymbol{p}}{ds^2} \right\|}{\left\| \dfrac{d\boldsymbol{p}}{ds} \right\|^3} = \frac{R}{R^2 + h^2}.$$

Indeed, $\kappa(s)$ is constant for all s and collapses to $1/R$ for $h = 0$.

For planning the requested orientation trajectory, which has to be coordinated with the position trajectory, we define a moving frame as a function of the same parameter s. This is given by

$$\boldsymbol{R}(s) = \begin{pmatrix} \boldsymbol{x}_o(s) & \boldsymbol{y}_0(s) & \boldsymbol{z}_o(s) \end{pmatrix} = \begin{pmatrix} -\cos s & \sin s & 0 \\ -\sin s & -\cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this moving frame is *not* the Frenet frame associated to the parametrized path. Using the notations $\boldsymbol{p}'(s) = d\boldsymbol{p}(s)/ds$ and $\boldsymbol{p}''(s) = d^2\boldsymbol{p}(s)/ds^2$, the Frenet frame is specified as

$$\begin{aligned} \boldsymbol{R}_{\text{Frenet}}(s) &= \begin{pmatrix} \boldsymbol{t}(s) & \boldsymbol{n}(s) & \boldsymbol{b}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{p}'(s)}{\|\boldsymbol{p}'(s)\|} & \dfrac{\boldsymbol{p}''(s)}{\|\boldsymbol{p}''(s)\|} & \boldsymbol{t}(s) \times \boldsymbol{n}(s) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{R}{\sqrt{R^2+h^2}} \sin s & -\cos s & \frac{h}{\sqrt{R^2+h^2}} \sin s \\ \frac{R}{\sqrt{R^2+h^2}} \cos s & -\sin s & -\frac{h}{\sqrt{R^2+h^2}} \cos s \\ \frac{h}{\sqrt{R^2+h^2}} & 0 & \frac{R}{\sqrt{R^2+h^2}} \end{pmatrix}. \end{aligned}$$

In fact, the two frames coincide (modulo a rotation of $\pi/2$ around the $\boldsymbol{z}$-axis) only when $h = 0$.

Setting $\boldsymbol{R}_d(t) = \boldsymbol{R}(s(t))$, the angular velocity vector is computed from

$$\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_d \boldsymbol{R}_d^T = \dot{s}(t) \begin{pmatrix} \sin s(t) & \cos s(t) & 0 \\ -\cos s(t) & \sin s(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\cos s(t) & -\sin s(t) & 0 \\ \sin s(t) & -\cos s(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}(t) & 0 \\ \dot{s}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As expected (being the rotation of the moving frame only around the $\boldsymbol{z}$-axis and counterclockwise),

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{s}(t) \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{\omega}\| = |\dot{s}(t)|,$$

and the maximum value of the norm of the angular velocity vector is obviously $V_{\max}$.

With the given numerical data, which satisfy both inequalities (1) and (2), we obtain:

$$V_{\max} = \sqrt{10} = 3.1623, \quad A_{\max} = 4\sqrt{10} = 12.6491, \quad T_s = 0.25,$$
$$s_{\max} = 3.75\sqrt{10} = 11.8585, \quad z_d(T) = 0.375\sqrt{10} = 1.1859\,.$$

In the following, we show plots of the planned trajectory obtained in Matlab (code available).

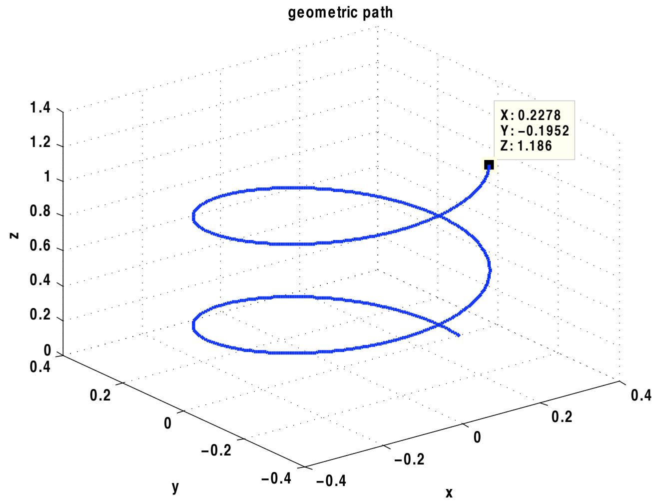

Figure 1: The spiral Cartesian trajectory (with coordinates of the final reached point at time $T = 4$ s)

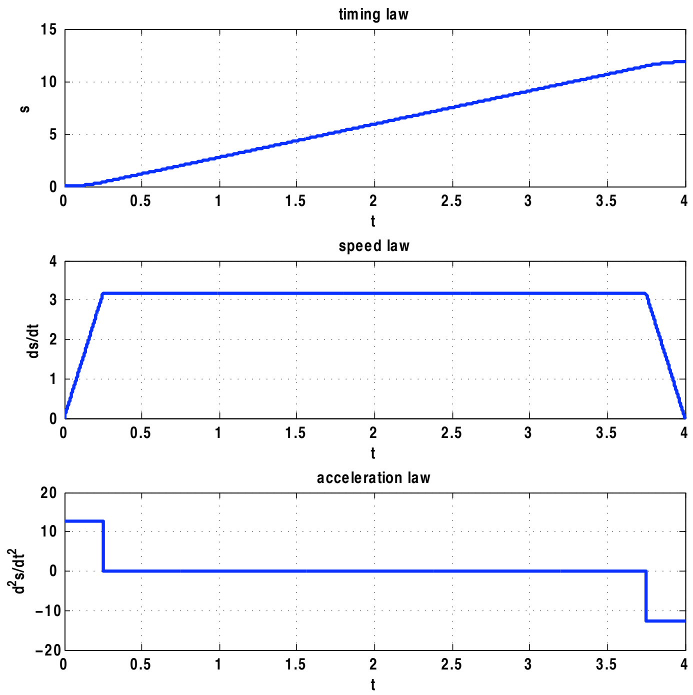

Figure 2: Timing law: Path parameter $s(t)$, speed $\dot{s}(t)$, and acceleration $\ddot{s}(t)$

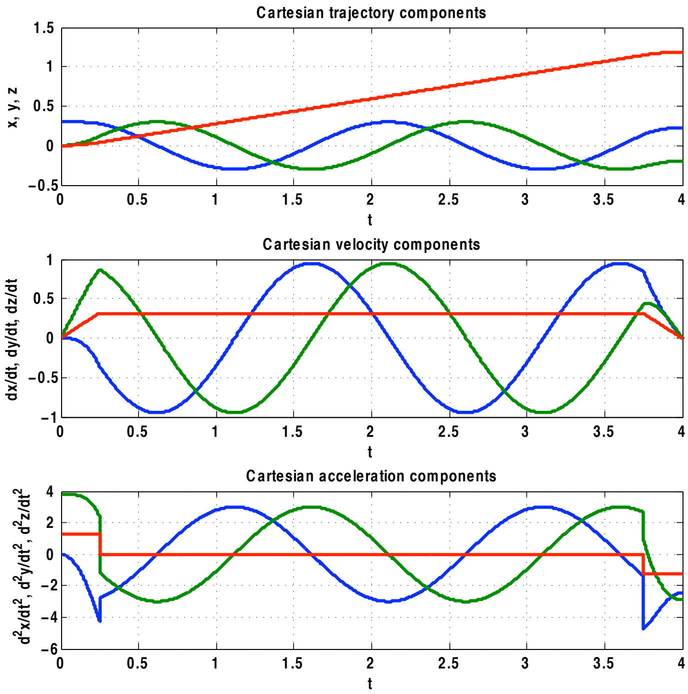

Figure 3: Components of Cartesian trajectory: Position, velocity, and acceleration (x in blue, y in green, z in red)

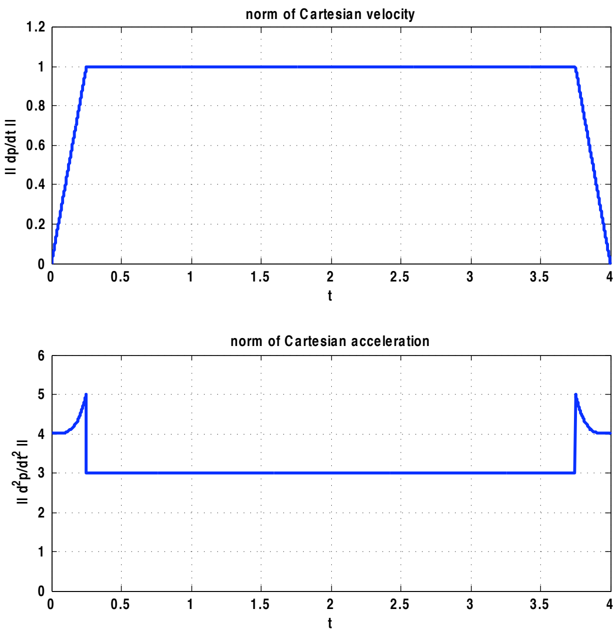

Figure 4: Norms of the Cartesian velocity and acceleration: The given bounds $\|\dot{\boldsymbol{p}}_d(t)\| \leq 1$ and $\|\ddot{\boldsymbol{p}}_d(t)\| \leq 5$ are always satisfied during motion

Exercise 2A

The kinematic model of a front-wheel driven *car-like* vehicle is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi\, u_1 \\ \sin\theta\cos\phi\, u_1 \\ \dfrac{\sin\phi}{\ell}\, u_1 \\ u_2 \end{pmatrix}, \tag{3}$$

where (x, y) are the coordinates of the rear wheel, θ is the absolute orientation of the vehicle (w.r.t. the $\boldsymbol{x}$ reference axis), ϕ is the steering angle of the front wheel (w.r.t. the car orientation), u_1 is the driving velocity of the front wheel and u_2 is its steering velocity.

Mimicking the trajectory controller design for the unicycle, a point B can be chosen at a distance $|b| > 0$ (b itself can be either positive or negative) from the front wheel along the direction of its absolute orientation, as given by the angle $\theta + \phi$. In this way, the velocity of point B will be affected directly by both commands u_1 and u_2. The position of B is thus given by

$$\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \ell \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + b \begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix}. \tag{4}$$

Differentiating once (4) w.r.t. time and using (3), we obtain

$$\begin{aligned} \begin{pmatrix} \dot{x}_B \\ \dot{y}_B \end{pmatrix} &= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \ell\,\dot{\theta} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} + b\,(\dot{\theta}+\dot{\phi}) \begin{pmatrix} -\sin(\theta+\phi) \\ \cos(\theta+\phi) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\phi) - \frac{b}{\ell}\sin\phi\sin(\theta+\phi) & -b\,\sin(\theta+\phi) \\ \sin(\theta+\phi) + \frac{b}{\ell}\sin\phi\cos(\theta+\phi) & b\,\cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \boldsymbol{T}(\theta,\phi) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

Since

$$\det \boldsymbol{T}(\theta,\phi) = b \neq 0,$$

matrix $\boldsymbol{T}$ (the so-called decoupling matrix of the system) can be inverted at any configuration. The input-output linearizing and decoupling control is then globally defined as

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \boldsymbol{T}^{-1}(\theta,\phi) \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad (5) \\ &= \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\frac{1}{b}\sin(\theta+\phi) - \frac{1}{\ell}\sin\phi\cos(\theta+\phi) & \frac{1}{b}\cos(\theta+\phi) - \frac{1}{\ell}\sin\phi\sin(\theta+\phi) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \end{aligned}$$

where v_x and v_y are two auxiliary inputs to be defined for asymptotically stable trajectory tracking purposes.

The closed-loop system, which is still partly nonlinear, is described by

$$\begin{pmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ \dfrac{\sin\phi}{\ell}\,(v_x\cos(\theta+\phi) + v_y\sin(\theta+\phi)) \\ \left(\dfrac{v_y}{b} - \dfrac{v_x\sin\phi}{\ell}\right)\cos(\theta+\phi) - \left(\dfrac{v_x}{b} + \dfrac{v_y\sin\phi}{\ell}\right)\sin(\theta+\phi) \end{pmatrix}, \tag{6}$$

showing that the velocity commands v_x and v_y independently drive the x and y components of the velocity of point B. For a given *continuous* reference trajectory $\boldsymbol{p}_{B,d}(t) = \left(x_{B,d}(t) \quad y_{B,d}(t) \right)^T$, an asymptotically (actually, exponentially) stable tracking is obtained by choosing in (5)

$$v_x = \dot{x}_{B,d} + k_x(x_{B,d} - x_B), \qquad v_y = \dot{y}_{B,d} + k_y(y_{B,d} - y_B),$$

with $k_x > 0$ and $k_y > 0$.

The main advantage of this control design stands in its simplicity for tracking very general output trajectories. In fact, the underlying path can also have tangent discontinuities which can be executed without stopping the motion of point B. Such behavior may occur even in the presence of geometric cusps, since an instantaneous reversal of the velocity of point B is still feasible. On the other hand, the choice of a suitable value of b is critical. A small value of $|b|$ will lead to high control efforts in the presence of path tangent discontinuities to be crossed at non-negligible speed or, more in general, when sharp directional changes are required. A large value of $|b|$ will instead increase the actual area "spanned" around the nominal output trajectory by the vehicle body during motion. This should be taken into account for collision avoidance of nearby obstacles.

Moreover, as an additional issue with respect to the simpler case of a unicycle, it would be interesting to study the effect of choosing negative values for b and of varying its ratio to the car length ℓ. In summary, an investigation of the properties of boundedness of the evolution of (θ, ϕ) (the so-called *zero dynamics* of the closed-loop system (6)) should be conducted when the point B is commanded so as to exactly reproduce some specific classes of (complex) reference trajectories.

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