Robotics 1

March 20, 2026

Exercise 1

The collaborative robot ECR5 by Efort is shown in Fig. 1. The robot has six revolute joints and a non-spherical wrist. Assign a set of frames according to the standard Denavit-Hartenberg (DH) convention in such a way that *all twist angles are nonnegative*. Place the origin of the first DH frame on the ground and of last DH frame at the center of the final flange. Draw the frames directly on the distributed extra sheet in the clearest possible way. Build the corresponding table of parameters and specify the numerical values of the nonzero parameters based on the data sheet.

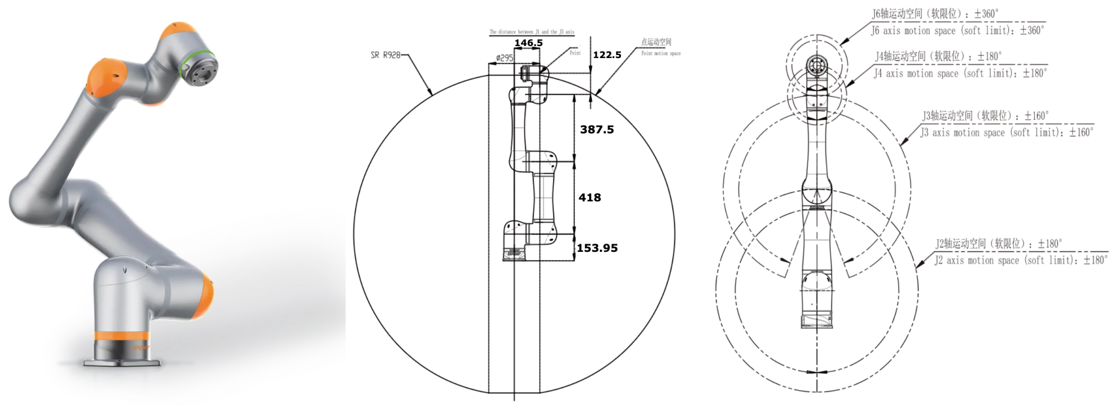

Figure 1: The 6R robot Efort ECR5 and drawings taken from the data sheet

Exercise 2

The direct kinematics of a 3-dof robot is given by

$$\boldsymbol{p} = \begin{pmatrix} d_2 \sin q_1 + (a_2 \cos q_2 + a_3 \cos(q_2 + q_3)) \cos q_1 \\ -d_2 \cos q_1 + (a_2 \cos q_2 + a_3 \cos(q_2 + q_3)) \sin q_1 \\ d_1 + a_2 \sin q_2 + a_3 \sin(q_2 + q_3) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

for generic (nonzero) d_1, a_2, d_2, and a_3.

- Compute the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{f}/\partial \boldsymbol{q}$ and find *all* its singular configurations.
- In one of the singular configurations $\boldsymbol{q}_s$, determine a basis for the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$.
- In the same singular configuration $\boldsymbol{q}_s$, find a velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^3$ that cannot be realized. Provide then a joint velocity $\dot{\boldsymbol{q}}_s$ that minimizes the norm of the velocity error $\boldsymbol{e} = \dot{\boldsymbol{p}} - \boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}$.

Exercise 3

Consider the following path planning problem for a generic revolute joint of a robot. A path $q(s)$, parametrized by $s \in [0, 1]$ and with continuity up to the second derivative $q''(s) = d^2q(s)/ds^2$ for all $s \in [0, 1)$, should interpolate the three joint values $q(0) = q_a$, $q(0.5) = q_b$, and $q(1) = q_c$, satisfying the following boundary conditions:

$$q'(0) = q''(0) = 0 \qquad q'(1) = 0.$$

Determine the expression of the solution path in symbolic form. Apply then your result to the joint position data $q_a = 3$, $q_b = 2$, $q_c = 4$ [rad]. What is the numerical value of $q''(1)$ in this case?

[180 minutes (3 hours); open books]

Solution

March 20, 2026

Exercise 1

A possible assignment of DH frames that satisfies the additional requirement of nonnegative twist angles α_i, $i = 1, \dots, 6$, is shown in Fig. 2. The corresponding parameters are given in Tab. 1, where the signs of the constant nonzero parameters a_2, a_3, d_1, and d_4 to d_6 are also specified.

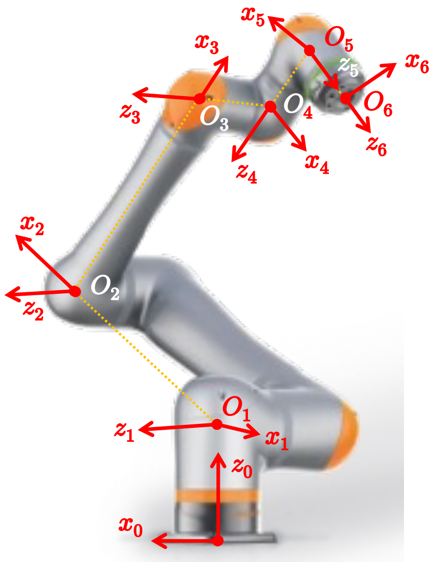

Figure 2: DH frames for the 6R robot Efort ECR5

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	0	$a_2 > 0$	0	q_2
3	0	$a_3 > 0$	0	q_3
4	$\pi/2$	0	$d_4 < 0$	q_4
5	$\pi/2$	0	$d_5 < 0$	q_5
6	0	0	$d_6 > 0$	q_6

Table 1: DH parameters for the frame assignment in Fig. 2

Numerical values of the constant nonzero parameters are read from the data sheet as

$$d_1 = 153.95 \qquad a_2 = 418 \qquad a_3 = 387.5 \qquad d_4 = -146.5 \qquad d_5 = -122.5,$$

whereas only an approximated value can be extracted for $d_6 \approx 110$ (all units are in mm).

Exercise 2

The given direct kinematics is rewritten here using the compact notation (e.g., $s_{23} = \sin(q_2 + q_3)$):

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} d_2 s_1 + (a_2 c_2 + a_3 c_{23})\, c_1 \\ -d_2 c_1 + (a_2 c_2 + a_3 c_{23})\, s_1 \\ d_1 + a_2 s_2 + a_3 s_{23} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

It refers to the end-effector position of a 3R elbow-type robot with a lateral offset d_2 at the shoulder. Differentiating (1) with respect to time, we obtain

$$\dot{\boldsymbol{p}} = \begin{pmatrix} d_2 c_1 - (a_2 c_2 + a_3 c_{23})\, s_1 & -(a_2 s_2 + a_3 s_{23})\, c_1 & -a_3 s_{23} c_1 \\ d_2 s_1 + (a_2 c_2 + a_3 c_{23})\, c_1 & -(a_2 s_2 + a_3 s_{23})\, s_1 & -a_3 s_{23} s_1 \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{2}$$

In order to investigate the singularities of the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$, it is convenient to express the relation (2) in the rotated frame of link 1, eliminating in this way from the matrix the variable q_1 (that does not affect the determinant in any case). Being the rotation matrix

$$\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} d_2 & -(a_2 s_2 + a_3 s_{23}) & -a_3 s_{23} \\ a_2 c_2 + a_3 c_{23} & 0 & 0 \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{pmatrix}$$

and ${}^1\dot{\boldsymbol{p}} = {}^1\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Thus,

$$\det \boldsymbol{J}(\boldsymbol{q}) = \det {}^1\boldsymbol{J}(\boldsymbol{q}) = a_2 a_3 s_3 \,(a_2 c_2 + a_3 c_{23}),$$

which is zero either when *i)* $\sin q_3 = 0$ ($q_3 = 0$ or π), or when *ii)* $a_2 c_2 + a_3 c_{23} = \sqrt{p_x^2 + p_y^2 - d_2^2} = 0$, or when *iii)* both hold ($q_3 = 0$ or π and $q_2 = \pm\pi/2$). Condition *i)* occurs when the second and third link are stretched or folded. Condition *ii)* corresponds to the end-effector placement at a distance d_2 (i.e., the shoulder offset) from the vertical axis of the first joint.

In a singular configuration $\boldsymbol{q}_s = (q_1, q_2, 0)$, where q_2 is different from $\pm\pi/2$, i.e., case *i)*, one has

$${}^1\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} d_2 & -(a_2 + a_3)\, s_2 & -a_3 s_2 \\ (a_2 + a_3)\, c_2 & 0 & 0 \\ 0 & (a_2 + a_3)\, c_2 & a_3 c_2 \end{pmatrix} \quad \Rightarrow \quad \text{rank}\ {}^1\boldsymbol{J}(\boldsymbol{q}_s) = 2$$

$$\Rightarrow \quad \dot{\boldsymbol{q}}_s = \beta \begin{pmatrix} 0 \\ -a_3 \\ a_2 + a_3 \end{pmatrix} \in \mathcal{N}\{{}^1\boldsymbol{J}(\boldsymbol{q}_s)\} = \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}.$$

For case *ii)*, $a_2 c_2 + a_3 c_{23} = 0$, one has

$${}^1\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} d_2 & -(a_2 s_2 + a_3 s_{23}) & -a_3 s_{23} \\ 0 & 0 & 0 \\ 0 & 0 & a_3 c_{23} \end{pmatrix} \quad \Rightarrow \quad \text{rank}\ {}^1\boldsymbol{J}(\boldsymbol{q}_s) = 2 \tag{3}$$

$$\Rightarrow \quad \dot{\boldsymbol{q}}_s = \beta \begin{pmatrix} a_2 s_2 + a_3 s_{23} \\ d_2 \\ 0 \end{pmatrix} = \beta \begin{pmatrix} p_z - d_1 \\ d_2 \\ 0 \end{pmatrix} \in \mathcal{N}\{{}^1\boldsymbol{J}(\boldsymbol{q}_s)\} = \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}.$$

For case *iii)*, when both the above conditions hold, one has for instance $q_2 = \pi/2$, $q_3 = 0$ and so

$${}^1\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} d_2 & -(a_2 + a_3) & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{rank } {}^1\boldsymbol{J}(\boldsymbol{q}_s) = 1$$

$$\Rightarrow \quad \dot{\boldsymbol{q}}_s \in \text{span} \left\{ \begin{pmatrix} a_2 + a_3 \\ d_2 \\ 0 \end{pmatrix}, \begin{pmatrix} a_3 \\ 0 \\ d_2 \end{pmatrix} \right\} = \mathcal{N}\{{}^1\boldsymbol{J}(\boldsymbol{q}_s)\} = \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}.$$

In all three singular cases, it is easy to see that an end-effector velocity $\dot{\boldsymbol{p}}_s \in \mathbb{R}^3$ that cannot be instantaneously realized is always

$${}^1\dot{\boldsymbol{p}}_s = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \notin \mathcal{R}\{{}^1\boldsymbol{J}(\boldsymbol{q}_s)\} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}}_s = \boldsymbol{R}(q_1)\,{}^1\dot{\boldsymbol{p}}_s = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \notin \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\} \tag{4}$$

This is obvious for cases *ii)* and *iii)*; for case *i)*, only the first joint can produce a velocity contribution in the y-direction of frame 1, but this will also generate a nonzero component in the x-direction. To check this, one may also border the singular Jacobians ${}^1\boldsymbol{J}(\boldsymbol{q}_s)$ with ${}^1\dot{\boldsymbol{p}}_s$ (or $\boldsymbol{J}(\boldsymbol{q}_s)$ with $\dot{\boldsymbol{p}}_s$) and verify that a nonsingular 3×3 minor can be extracted.

Finally, consider again the singular case *ii)*. The joint velocity that minimizes the norm of the task velocity error $\boldsymbol{e} = \dot{\boldsymbol{p}} - \boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}$ for the unfeasible $\dot{\boldsymbol{p}}$ assigned in (4) is simply $\dot{\boldsymbol{q}} = \boldsymbol{0}$ (with $\|\boldsymbol{e}\| = 1$). This is rather intuitive in the present case as the zero velocity would imply the requested ${}^1\dot{p}_x = 0$ and ${}^1\dot{p}_z = 0$, while no contribution toward ${}^1\dot{p}_z = 1$ could be obtained by any $\dot{\boldsymbol{q}} \neq \boldsymbol{0}$. This is also the unique solution obtained from the theory of pseudoinverses; using (3) and (4), we have in fact

$$\dot{\boldsymbol{q}} = {}^1\boldsymbol{J}^{\#}(\boldsymbol{q}_s)\,{}^1\dot{\boldsymbol{p}}_s = \begin{pmatrix} * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boldsymbol{0}, \tag{5}$$

where $*$ are irrelevant values here. The structure with a zero second column in the pseudoinverse is a direct consequence of having a second zero row in ${}^1\boldsymbol{J}(\boldsymbol{q}_s)$. In fact, checking the four defining identities of a pseudoinverse, the following can be easily show:

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{0}^T \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}^{\#} = \begin{pmatrix} \boldsymbol{J}_1^{\#} & \boldsymbol{0} \end{pmatrix},$$

which, modulo a permutation of rows, is the property used in (5). Similar arguments also apply to the other two singular cases.

Exercise 3

The path planning problem has a total of six conditions: the passages through three points to be interpolated and the three given boundary conditions on derivatives at the two boundaries. The easiest solution is to use polynomial functions of s. In principle, the problem can be solved by a single polynomial $q(s)$ of degree 5, having six free parameters available. However, to reduce path curvature and possible wandering, a patch of two lower order polynomials is preferable. The first polynomial should satisfy four conditions, namely interpolation of q_a at $s = 0$ and of q_b at $s = 0.5$, and two boundary conditions on derivatives in $s = 0$. A polynomial $q_1(s)$ of degree 3 is then sufficient. Similarly, the second polynomial has three specified conditions, namely interpolation of q_b at $s = 0.5$ and of q_c at $s = 1$ and a single boundary condition on the first derivative in $s = 1$. In addition, the second polynomial should match the values of the first and second derivative reached by $q_1(s)$ at $s = 0.5$, to ensure the continuity requirements. Thus, the number of resulting conditions for the second polynomial is five, and a polynomial $q_2(s)$ of degree 4 is sufficient. As a result, two cubic polynomials, i.e., a spline, are not enough in general. A suitable choice of the form of the two polynomials, a cubic and a quartic, will simplify the symbolic computations.

Let the cubic polynomial be defined as

$$q_1(s) = q_a + (q_b - q_a)\left(\frac{s}{0.5}\right)^3 \qquad s \in [0, 0.5].$$

This polynomial automatically satisfies the four conditions

$$q_1(0) = q_a \qquad q_1'(0) = 0 \qquad q_1''(0) = 0 \qquad q_1(0.5) = q_b.$$

In addition, evaluating its derivatives at the end of the first interval gives

$$q_1'(0.5) = 6\,(q_b - q_a) \qquad q_1''(0.5) = 24\,(q_b - q_a)\,. \tag{6}$$

The quartic polynomial can be defined using the normalized parameter $\sigma = (s - 0.5)/0.5 = 2s - 1$, for $s \in [0.5, 1]$:

$$q_2(\sigma) = q_b + c_1\sigma + c_2\sigma^2 + c_3\sigma^3 + c_4\sigma^4 \qquad \sigma \in [0, 1],$$

with first and second derivative, respectively

$$q_2'(\sigma) = \frac{dq_2}{ds} = \frac{dq_2}{d\sigma}\,\sigma' = 2\,\frac{dq_2}{d\sigma} = 2c_1 + 4c_2\sigma + 6c_3\sigma^2 + 8c_4\sigma^3$$

and

$$q_2''(\sigma) = \frac{d^2q_2}{ds^2} = \frac{d^2q_2}{d\sigma^2}\,(\sigma')^2 = 4\,\frac{d^2q_2}{d\sigma^2} = 8c_2 + 24c_3\sigma + 48c_4\sigma^2.$$

Using (6), the two continuity conditions at $s = 0.5$ ($\sigma = 0$) are

$$q_2'(0) = 2c_1 = 6\,(q_b - q_a) \quad \Rightarrow \quad c_1 = 3\,(q_b - q_a)$$

and

$$q_2''(0) = 8c_2 = 24\,(q_b - q_a) \quad \Rightarrow \quad c_2 = 3\,(q_b - q_a)\,.$$

Therefore, the two boundary conditions for the second polynomial at $s = \sigma = 1$ are written as

$$q_2(1) = q_b + 6\,(q_b - q_a) + c_3 + c_4 = q_c \tag{7}$$

and

$$q_2'(1) = 18\,(q_b - q_a) + 6c_3 + 8c_4 = 0. \tag{8}$$

Solving the linear system (7)–(8) yields

$$c_3 = 15q_a - 19q_b + 4q_c \qquad c_4 = -9q_a + 12q_b - 3q_c.$$

Summarizing, the two polynomials are

$$q_1(s) = q_a + \frac{q_b - q_a}{8}\, s^3 \qquad s \in [0, 0.5]$$

and

$$\begin{aligned} q_2(s) = {} & q_b + 3\,(q_b - q_a)\;(2s-1) + 3\,(q_b - 3q_a)\;(2s-1)^2 \\ & + (15q_a - 19q_b + 4q_c)\;(2s-1)^3 - (9q_a - 12q_b + 3q_c)\;(2s-1)^4 \qquad s \in [0, 5, 1]. \end{aligned}$$

In particular, the second derivative of $q_2(s)$ at $s = 1$ is

$$q_2''(1) = -96q_a + 144q_b - 48q_c \tag{9}$$

Substituting the numerical values $q_a = 3$, $q_b = 2$, and $q_c = 4$ [rad], we have

$$q_1(s) = 3 - 8s^3 \qquad q_2(s) = q_2(s) = 2 - 3\,(2s-1) - 3\,(2s-1)^2 + 23\,(2s-1)^3 - 15\,(2s-1)^4,$$

and from (9)

$$q_2''(1) = -192.$$

* * * * *