Robotics 1

June 11, 2025

Exercise 1

Consider the 12 possible sequences of Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ about moving axes. Find which of these sequences are associated to a singular transformation $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ when $\boldsymbol{\phi} = \mathbf{0}$ and explain why. Out of the remaining sequences, determine which one produces $\boldsymbol{T} = \boldsymbol{I}$ when $\boldsymbol{\phi} = \mathbf{0}$. For this Euler sequence:

a) determine the values of $\boldsymbol{\phi}$ that correspond to a singular matrix $\boldsymbol{T}$;

b) in a singularity, find an angular velocity $\boldsymbol{\omega}$ that cannot be represented by any $\dot{\boldsymbol{\phi}}$;

c) when $\boldsymbol{\phi} = (\pi/3, 0, -\pi/2)$ [rad], determine $\dot{\boldsymbol{\phi}}$ that realizes $\boldsymbol{\omega} = (1, -1, 1)$ [rad/s].

Exercise 2

The ABB CBR 15000 collaborative robot (also named *GoFa*, for 'Go Faster') is shown in Fig. 1 and in the accompanying Extra Sheet. This robot has has six revolute joints but *no* spherical wrist. Define the set of Denavit–Hartenberg (DH) frames that satisfies the following conditions:

- the DH frame RF_0 is on the floor at the robot base, as shown in the Extra Sheet;
- the origin O_6 of frame RF_6 should be placed at the center of the end-effector flange, with the axis $\boldsymbol{z}_6$ in the approach direction;
- twist angles are all non-negative ($\alpha_i \geq 0$, for $i = 1, \ldots, 6$);
- positive joint rotations should be consistent with those indicated by the manufacturer (see Extra Sheet).

Draw the frames and fill in the corresponding table of DH parameters directly on the Extra Sheet. When the robot is in the configuration shown therein, provide also the numerical values of the joint variables q_i, for $i = 1, \ldots, 6$. *[Hint: you should find values $q_i \in [0, \pi]$ for all six joints!]* Evaluate numerically the direct kinematics in this configuration and verify the correctness of the obtained end-effector position and orientation with respect to the base frame RF_0.

Figure 1: The ABB CBR 15000 collaborative robot.

Exercise 3

With reference to the DH assignment in Exercise 2, consider the last three joints q_4, q_5 and q_6 as assigned. For a desired position ${}^0\boldsymbol{p}_{4,d} = (p_{4x,d}, p_{4y,d}, p_{4z,d})$ of the origin O_4 of the DH frame RF_4, determine the inverse kinematics solutions for the first three joints in symbolic form. *[Hint: use algebraic transformations!]*. Consider only the regular case.

Let now the desired pose of the last DH frame RF_6 be

$$ {}^0\boldsymbol{T}_{6,d} = \begin{pmatrix} {}^0\boldsymbol{R}_{6,d} & {}^0\boldsymbol{p}_{6,d} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & -101 \\ -1 & 0 & 0 & 550 \\ 0 & 1 & 0 & 599 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

with lengths expressed in [mm], while the last three joints are assigned as $(q_4, q_5, q_6) = (\pi/2, -\pi/2, 0)$ [rad]. Using the previously obtained inverse kinematics formulas, compute the numerical values of (at least) one solution (q_1, q_2, q_3) for the first three joints. At the end, verify if and when the correct desired pose ${}^0\boldsymbol{T}_{6,d}$ is realized by a complete solution $\boldsymbol{q} \in \mathbb{R}^6$.

Exercise 4

A 2R planar robot should move from $\boldsymbol{q}_s = (0, \pi/4)$ to $\boldsymbol{q}_f = (\pi/2, -\pi/2)$ [rad] in a coordinated time T and with a smooth profile for $t \in [0, T]$ satisfying zero boundary conditions on velocity and acceleration. If the maximum absolute joint velocity and joint acceleration are $\boldsymbol{V}_{max} = (V_{max,1}, V_{max,2}) = (2, 3.5)$ [rad/s] and, respectively, $\boldsymbol{A}_{max} = (A_{max,1}, A_{max,2}) = (3, 6)$ [rad/s^2], determine the minimum motion time T^*.

Suppose now that at $t = T^*/4$ an emergency is detected and the robot should come as soon as possible to a complete stop. Which would be the new motion profile and the minimum time instant T_s at which $\dot{\boldsymbol{q}}(T_s) = \mathbf{0}$? Which is the final reached configuration $\boldsymbol{q}(T_s)$? In this situation, sketch the overall position, velocity and acceleration profiles for $t \in [0, T_s]$.

[4 hours; open books]

Solution

June 11, 2025

Exercise 1

Singularities in the Euler sequence of rotations around moving axes occur when two of the axes align. This corresponds to situations where the inverse transformation from a rotation matrix $\boldsymbol{R}$ to the set of angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ fails to have two regular solutions and become (partly) undefined — only the sum or difference of two of the three angles are determined. At the same time, the differential mapping from $\dot{\boldsymbol{\phi}}$ to the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ is not surjective in a singularity $\boldsymbol{\phi}_s$ — there exist angular velocities that cannot be generated by any $\dot{\boldsymbol{\phi}}$. Therefore, the six sequences

$$XYX \quad XZX \quad YXY \quad YZY \quad ZXZ \quad ZYZ$$

become automatically singular when the (second) angle β in the sequence is zero (and therefore when $\boldsymbol{\phi} = \mathbf{0}$) since the first and third axes will be coincident. Out of the remaining six sequences, when seeking for the condition

$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{T}(\mathbf{0})\dot{\boldsymbol{\phi}} = \boldsymbol{I}\,\dot{\boldsymbol{\phi}},$$

the following four sequences

$$YXZ \qquad YZX \qquad ZXY \qquad ZYX$$

are certainly to be excluded since the first column in the matrix $\boldsymbol{T}(\boldsymbol{\phi})$, which is always constant, is not $(1\ \ 0\ \ 0)^T$ (as part of the necessary identity matrix). Similarly, the second column in the sequence XZY does not become $(0\ \ 1\ \ 0)^T$ when $\alpha = 0$. For the last remaining sequence XYZ, we have

$$\boldsymbol{\omega} = \begin{pmatrix} \begin{matrix}1\\0\\0\end{matrix} & \boldsymbol{R}_X(\alpha)\begin{pmatrix}0\\1\\0\end{pmatrix} & \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\begin{pmatrix}0\\0\\1\end{pmatrix} \end{pmatrix} \dot{\boldsymbol{\phi}} = \begin{pmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{pmatrix} \dot{\boldsymbol{\phi}} = \boldsymbol{T}(\alpha.\beta)\dot{\boldsymbol{\phi}},$$

with $\boldsymbol{T}(0,0) = \boldsymbol{I}$ and $\det \boldsymbol{T}(\alpha,\beta) = \cos\beta$, so that the singularity occurs for $\beta = \pm\,\pi/2$. In such a singularity,

$$\boldsymbol{T}(\alpha, \pm\,\pi/2) = \begin{pmatrix} 1 & 0 & \pm\,1 \\ 0 & \cos\alpha & 0 \\ 0 & \sin\alpha & 0 \end{pmatrix} = \boldsymbol{T}_s(\alpha).$$

Therefore, angular velocities of the form

$$\boldsymbol{\omega} = \rho \begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{T}_s(\alpha)\right\} \qquad \forall \rho \neq 0,$$

cannot be realized by any possible choice of $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})$. Finally, at $\boldsymbol{\phi} = (\alpha, \beta, \gamma) = (\pi/3, 0, -\pi/2)$ (the third angle γ is irrelevant), the angular velocity $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z) = (1, -1, 1)$ is realized by

$$\dot{\boldsymbol{\phi}} = \boldsymbol{T}^{-1}(\pi/3, 0) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.3660 \\ 1.3660 \end{pmatrix}.$$

Exercise 2

An assignment of DH frames for the ABB *GoFa* robot that satisfies all the given conditions is shown in Fig. 2. The associated DH parameters are reported in Tab. 1. The numerical values of the linear DH parameters, expressed in [mm], are taken from the left side view of the robot (see the Extra Sheet). The angular values of the joint variables correspond to the configuration shown in Fig. 2.

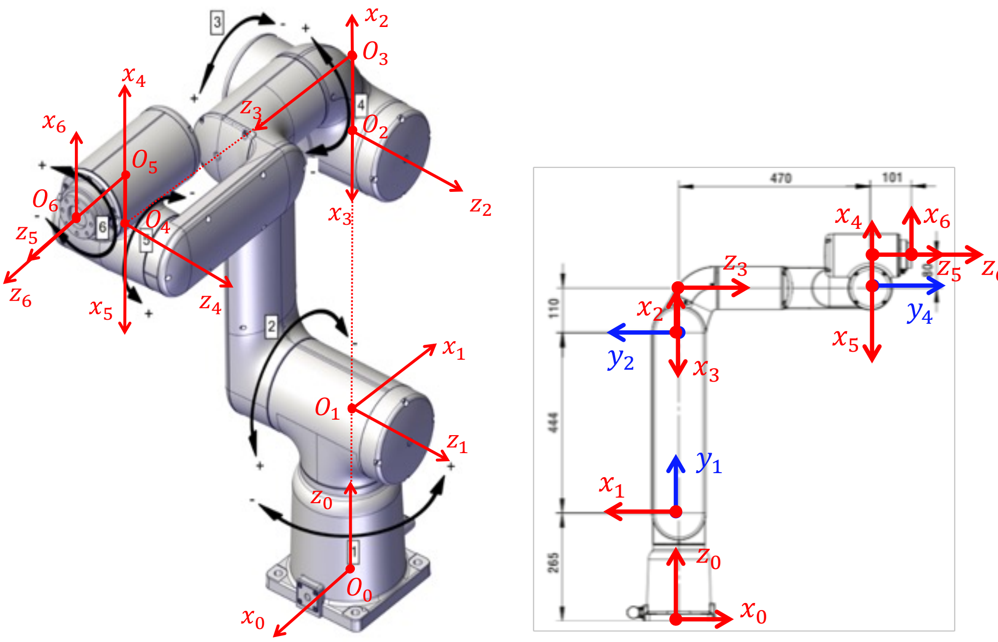

Figure 2: Assignment of DH frames for the ABB CBR 15000 robot (3D and left side views).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 265$	$q_1 = \pi$
2	0	$a_2 = 444$	0	$q_2 = \pi/2$
3	$\pi/2$	$a_3 = -110$	0	$q_3 = \pi$
4	$\pi/2$	0	$d_4 = 470$	$q_4 = \pi$
5	$\pi/2$	$a_5 = -80$	0	$q_5 = \pi$
6	0	0	$d_6 = 101$	$q_6 = \pi$

Table 1: Table of DH parameters for the frame assignment in Fig. 2. Lengths are expressed in [mm]. The values of the joint variables (in blue) correspond to the configuration shown in Fig. 2.

Building the DH homogeneous transformation matrices and computing numerically the direct kinematics with the data in Tab. 1 yields

$$ {}^0\boldsymbol{T}_6 = \begin{pmatrix} 0 & 0 & 1 & 571 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 899 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

which corresponds to what expected: $\boldsymbol{x}_6$ is oriented as $\boldsymbol{z}_0$, $\boldsymbol{y}_6$ is oriented as $-\boldsymbol{y}_0$, and $\boldsymbol{z}_6$ is oriented as $\boldsymbol{x}_0$; moreover, the origin O_6 is in the plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$ at an height $d_1 + a_2 + |a_3| + |a_5| = 899$ mm along $\boldsymbol{z}_0$ and at a distance $d_4 + d_6 = 571$ mm from O_0 along $\boldsymbol{x}_0$.

The only alternative feasible DH frame assignment (not pursued herafter) is with the $\boldsymbol{x}_2$ axis pointing downward. This would have led to the following changes in the DH table: $a_2 = -444$, $q_2 = -\pi/2$, $q_3 = 0$.

Exercise 3

Using the values in Tab. 1, the position of the origin O_4 is computed as

$$ {}^0\boldsymbol{p}_{4,\text{hom}}(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3)\,{}^3\boldsymbol{A}_4(q_4)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix} c_1\,(a_2c_2+a_3c_{23}+d_4s_{23})\\ s_1\,(a_2c_2+a_3c_{23}+d_4s_{23})\\ d_1+a_2s_2+a_3s_{23}-d_4c_{23}\\ 1\end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{p}_4(q_1,q_2,q_3)\\ 1\end{pmatrix}. $$

Note that this position is independent from the joint variable q_4. The inverse kinematics problem is then expressed by three nonlinear equations in the three unknowns $\boldsymbol{q}_{123} = (q_1, q_2, q_3)$,

$$ {}^0\boldsymbol{p}_{4,d} = \begin{pmatrix} p_{4x,d}\\ p_{4y,d}\\ p_{4z,d}\end{pmatrix} = {}^0\boldsymbol{p}_4(q_1,q_2,q_3), \tag{1} $$

and is addressed as follows.

Squaring and summing the three equations in (1) (after moving the constant d_1 to the left-hand side) gives

$$ \begin{aligned} p_{4x,d}^2+p_{4y,d}^2+(p_{4z,d}-d_1)^2 &= (a_2c_2+a_3c_{23}+d_4s_{23})^2+(a_2s_2+a_3s_{23}-d_4c_{23})^2\\ &= a_2^2+a_3^2+d_4^2+2a_2a_3\,(c_2c_{23}+s_2s_{23})+2a_2d_4\,(c_2s_{23}-s_2c_{23})\\ &= a_2^2+a_3^2+d_4^2+2a_2a_3c_3+2a_2d_4s_3. \end{aligned} $$

This can be rearranged in the general form

$$ a\sin q_3 + b\cos q_3 = c, \tag{2} $$

with

$$ a = 2a_2d_4, \qquad b = 2a_2a_3, \qquad c = p_{4x,d}^2+p_{4y,d}^2+(p_{4z,d}-d_1)^2-(a_2^2+a_3^2+d_4^2). $$

The transcendental equation (2) has been studied in the lecture slides (InverseKinematics.pdf, slide #12). From there, we know that this equation has (one or two) real solutions if and only if

$$ \Delta = a^2+b^2-c^2 \geq 0 \qquad \Rightarrow \qquad 4a_2^2(d_4^2+a_3^2) \geq \left(p_{4x,d}^2+p_{4y,d}^2+(p_{4z,d}-d_1)^2-(a_2^2+a_3^2+d_4^2)\right)^2 \tag{3} $$

Under condition (3), the solutions to (2) are computed via an algebraic transformation as

$$ q_3^{+/-} = 2\arctan\frac{a\pm\sqrt{\Delta}}{b+c}. \tag{4} $$

Further, from the first two equations in (1) we obtain for each of the two solutions q_3^+ and q_3^- a corresponding

$$ q_1 = \text{ATAN2}\left\{\gamma p_{4y,d}, \gamma p_{4x,d}\right\}, \qquad \text{with } \gamma = \text{sgn}\,(a_2c_2+a_3c_{23}+d_4s_{23}) = \{+,-\}, \tag{5} $$

excluding the singular case, i.e., when $\gamma = \text{sgn}\,(0) := 0$.

Finally, for each solution pair (q_1, q_3) the trigonometric functions in (1) are expanded and the first two equations combined to get

$$ \begin{aligned} c_1p_{4x,d}+s_1p_{4y,d} &= a_2c_2+a_3\,(c_2c_3-s_2s_3)+d_4\,(s_2c_3+c_2s_3)\\ p_{4z,d}-d_1 &= a_2s_2+a_3\,(s_2c_3+c_2s_3)-d_4\,(c_2c_3-s_2s_3)\,. \end{aligned} $$

These are rearranged in a linear system of two equations in the two unknowns c_2 and s_2 as

$$ \begin{pmatrix} a_2+a_3c_3+d_4s_3 & d_4c_3-a_3s_3\\ a_3s_3-d_4c_3 & a_2+a_3c_3+d_4s_3\end{pmatrix}\begin{pmatrix}c_2\\ s_2\end{pmatrix} = \begin{pmatrix} c_1p_{4x,d}+s_1p_{4y,d}\\ p_{4z,d}-d_1\end{pmatrix} \quad\Leftrightarrow\quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{6} $$

Provided that $\det \boldsymbol{A} = (a_2 + a_3 c_3 + d_4 s_3)^2 + (a_3 s_3 - d_4 c_3)^2 \neq 0$, the unique solution $\boldsymbol{x} = (c_2, s_2) = \boldsymbol{A}^{-1}\boldsymbol{b}$ leads to

$$q_2 = \text{ATAN2}\{s_2, c_2\}. \tag{7}$$

Summarizing, there is a total of four inverse kinematics solutions (out of singularities) for the positioning of the origin O_4 of frame RF_4.

Consider now the second part of the problem. Since

$${}^0\boldsymbol{T}_6(\boldsymbol{q}) = {}^0\boldsymbol{T}_4(q_1, q_2, q_3, q_4)\,{}^4\boldsymbol{T}_6(q_5, q_6)$$

and the last three joint angles have been assigned as $\boldsymbol{q}_{456} = (q_4, q_5, q_6) = (\pi/2, -\pi/2, 0)$ [rad], from the desired pose of the last frame RF_6 one obtains

$$\begin{aligned} {}^0\boldsymbol{T}_4(q_1, q_2, q_3, \pi/2) &= \begin{pmatrix} s_1 & c_1 s_{23} & c_1 c_{23} & c_1 \left(a_2 c_2 + a_3 c_{23} + d_4 s_{23}\right) \\ -c_1 & s_1 s_{23} & s_1 c_{23} & s_1 \left(a_2 c_2 + a_3 c_{23} + d_4 s_{23}\right) \\ 0 & -c_{23} & s_{23} & d_1 + a_2 s_2 + a_3 s_{23} - d_4 c_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= {}^0\boldsymbol{T}_{6,d}\,{}^4\boldsymbol{T}_6^{-1}(-\pi/2, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 470 \\ 0 & 0 & 1 & 599 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{8}$$

The last column of ${}^0\boldsymbol{T}_4$ contains the position vector ${}^0\boldsymbol{p}_4(q_1, q_2, q_3)$ for which the inverse kinematics problem has been solved in symbolic form. From the above identity, we have the numerical input data

$$p_{4x,d} = 0 \qquad p_{4y,d} = 470 \qquad p_{4z,d} = 599 \quad \text{[mm]}.$$

Therefore, substituting also the constant DH parameters ($d_1 = 265$, $a_2 = 444$, $a_3 = -110$, $d_4 = 470$ [mm]), we compute the numerical solutions following the above procedure.

In eq. (2), it is

$$a = 417360 \qquad b = c = -97680 \qquad \Rightarrow \qquad \Delta = a^2 > 0,$$

and so from (4)

$$q_3^+ = -2.6818 \qquad q_3^- = 0 \quad \text{[rad]}.$$

From (5), we obtain the two values for q_1

$$q_1^+ = \frac{\pi}{2} \quad (\text{when } \gamma > 0) \qquad q_1^- = -\frac{\pi}{2} \quad (\text{when } \gamma < 0) \qquad \text{[rad]}.$$

For all four combinations (q_1, q_3), we solve the linear system (6) and compute the corresponding q_2 from (7):

$$q_2^{++} = -0.3351 \qquad q_2^{+-} = \frac{\pi}{2} \qquad q_2^{-+} = \frac{\pi}{2} \qquad q_2^{--} = -2.8065 \quad \text{[rad]}.$$

The four solutions for the first three joints are thus:

$$\boldsymbol{q}_{123}^{+++} = \begin{pmatrix} \pi/2 \\ -0.3351 \\ -2.6818 \end{pmatrix} \quad \boldsymbol{q}_{123}^{++-} = \begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \end{pmatrix} \quad \boldsymbol{q}_{123}^{--+} = \begin{pmatrix} -\pi/2 \\ \pi/2 \\ -2.6818 \end{pmatrix} \quad \boldsymbol{q}_{123}^{---} = \begin{pmatrix} -\pi/2 \\ -2.8065 \\ 0 \end{pmatrix} \text{[rad]}. \tag{9}$$

Note also that a correct association with the signs of γ (evaluated for the solutions pairs (q_2, q_3)) has been made for q_1 in (5) since

$$\gamma(\boldsymbol{q}_{123}^{+++}) > 0 \qquad \gamma(\boldsymbol{q}_{123}^{++-}) > 0 \qquad \gamma(\boldsymbol{q}_{123}^{--+}) < 0 \qquad \gamma(\boldsymbol{q}_{123}^{---}) < 0.$$

It can be easily verified that all four inverse kinematics solutions (9) provide the same desired value for the position of O_4:

$${}^0\boldsymbol{p}_4(\boldsymbol{q}_{123}^{+++}) = {}^0\boldsymbol{p}_4(\boldsymbol{q}_{123}^{++-}) = {}^0\boldsymbol{p}_4(\boldsymbol{q}_{123}^{--+}) = {}^0\boldsymbol{p}_4(\boldsymbol{q}_{123}^{---}) = \begin{pmatrix} 0 \\ 470 \\ 599 \end{pmatrix} = {}^0\boldsymbol{p}_{4,d}.$$

Note however that three of these solutions do not comply with the rotation matrix ${}^0\boldsymbol{R}_4$ in (8). In fact, the element $(1,1)$ ($s_1 = 1$) requires $q_1 = \pi/2$. Moreover, the element $(3,3)$ ($s_{23} = 1$) requires that $q_2 + q_3 = \pi/2$. The only solution satisfying these conditions is $\boldsymbol{q}_{123}^{++-}$. In fact, it is

$$
{}^0\boldsymbol{T}_4(\boldsymbol{q}_{123}^{++-}, \pi/2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 470 \\ 0 & 0 & 1 & 599 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Accordingly, using the fixed values of $\boldsymbol{q}_{456}$, it is

$$
{}^0\boldsymbol{T}_4(\boldsymbol{q}_{123}^{++-}, \pi/2)\,{}^4\boldsymbol{T}_6(-\pi/2, 0) = {}^0\,\boldsymbol{T}_{6,d},
$$

as desired. On the other hand, the remaining three solutions to the inverse kinematics problem for O_4 will generate an end-effector pose different from ${}^0\boldsymbol{T}_{6,d}$.

Exercise 4

In order to satisfy the zero boundary conditions on the first two time derivatives, the smooth trajectories of the two robot joints are chosen as quintic polynomials in the form

$$
q_i(t) = q_{s,i} + \Delta_i \left(6\left(\frac{t}{T}\right)^5 - 15\left(\frac{t}{T}\right)^4 + 10\left(\frac{t}{T}\right)^3 \right) \qquad t \in [0, T] \qquad i = 1, 2, \tag{10}
$$

with $q_{s,1} = 0$, $q_{s,2} = \pi/4$, $\Delta_1 = q_{f,1} - q_{s.1} = \pi/2$, $\Delta_2 = q_{f,2} - q_{s,2} = -3\pi/4$ [rad]. The associated velocity, acceleration, and jerk are

$$
\begin{aligned}
\dot{q}_i(t) &= \frac{30\,\Delta_i}{T} \left(\left(\frac{t}{T}\right)^4 - 2\left(\frac{t}{T}\right)^3 + \left(\frac{t}{T}\right)^2 \right) \\
\ddot{q}_i(t) &= \frac{60\,\Delta_i}{T^2} \left(2\left(\frac{t}{T}\right)^3 - 3\left(\frac{t}{T}\right)^2 + \frac{t}{T} \right) \\
\dddot{q}_i(t) &= \frac{60\,\Delta_i}{T^3} \left(6\left(\frac{t}{T}\right)^2 - 6\left(\frac{t}{T}\right) + 1 \right).
\end{aligned}
$$

The acceleration is zero at $t = 0$, $t = T$ and $t = T/2$, the latter being the half-time instant at which the velocity reaches its maximum absolute value. Thus

$$
|\dot{q}_i(T/2)| = \max_{t \in [0,T]} |\dot{q}_i(t)| = \frac{30\,|\Delta_i|}{16\,T} \le V_{max,i} \qquad \Rightarrow \qquad T \ge \frac{30\,|\Delta_i|}{16\,V_{max,i}} = T_{v,i}.
$$

The jerk is zero at the roots of the quadratic equation in $\tau = t/T$

$$
6\tau^2 - 6\tau + 1 = 0 \qquad \Rightarrow \qquad \tau_{1,2} = \frac{1}{2} \pm \frac{1}{2\sqrt{3}} \qquad \Rightarrow \qquad t_{1,2} = \frac{T}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right),
$$

namely at two symmetric instants with respect to the half-time $T/2$, where the acceleration reaches its maximum absolute value. We have

$$
|\ddot{q}_i(t_{1,2})| = \max_{t \in [0,T]} |\ddot{q}_i(t)| = \frac{60\,|\Delta_i|}{T^2} \left| 2\tau_{1,2}^3 - 3\tau_{1,2}^2 + \tau_{1,2} \right| = \frac{60\,\Delta_i}{T^2}\,|k| \le A_{max,i} \quad \Rightarrow \quad T \ge \sqrt{\frac{60\,|\Delta_i|}{A_{max,i}}\,|k|} = T_{a,i},
$$

where $k = \pm\,0.0962$. Plugging in the data, we have

$$
T_{v,1} = 1.4726 \qquad T_{v,2} = 1.2622 \qquad T_{a,1} = 1.7387 \qquad T_{a,2} = 1.5057 \;\; \text{[s]},
$$

and thus

$$
T^* = \max\{T_{v,1}, T_{v,2}, T_{a,1}, T_{a,2}\} = T_{a,1} = 1.7387 \text{ s}.
$$

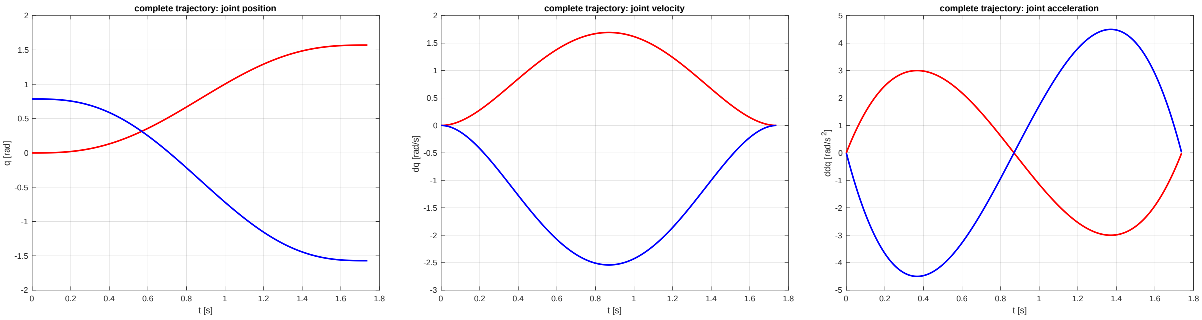

Figure 3: The complete motion of the robot joints (joint 1 in red, joint 2 in blue) for $T^* = 1.7387$ s: position [left], velocity [center], and acceleration [right].

When the complete trajectory is executed, the evolution of the position, velocity and acceleration of the two joints is shown in Fig. 3.

Suppose now that an emergency is detected at $t = T_0 = T^*/4 = 0.4347$ s. In order for the robot to stop its motion as soon as possible, the maximum feasible deceleration $-A_{max} < 0$ is applied to a joint if its current velocity is $\dot{q}(T_0) > 0$ and the maximum feasible acceleration $A_{max} > 0$ is applied if its current velocity is $\dot{q}(T_0) < 0$. This command is applied until the joint stops; then, the acceleration drops to zero and the joint remains at rest. In general, the stopping instants for the two joints will be different (thus, not coordinated), as well as the traveled distance from the instant T_0 at which emergency begins until each joint eventually stops. Moreover, a discontinuity will be present in the acceleration profiles.

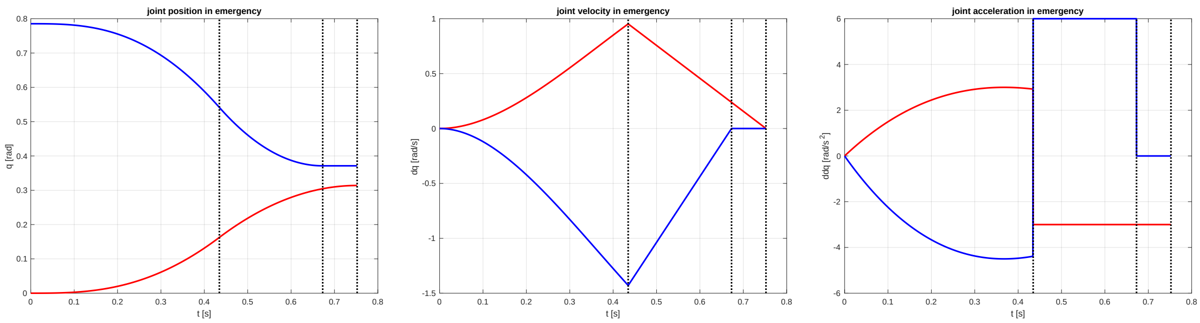

Figure 4: The motion of the robot joints (joint 1 in red, joint 2 in blue) in response to the emergency stop: position [left], velocity [center], and acceleration [right]. The dotted vertical lines correspond to the instants $T_0 = T^*/4 = 0.4347$ s (emergency begins), $T_{s2} = 0.6729$ s (when the second joint stops), and $T_{s1} = 0.7523$ s (when also the first joint stops).

We evaluate the planned quintic polynomial trajectory (10) and its first derivative at $t = T_0$ and obtain

$$\boldsymbol{q}_0 = \boldsymbol{q}(T_0) = \begin{pmatrix} 0.1626 \\ 0.5415 \end{pmatrix} \text{[rad]} \qquad \dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}(T_0) = \begin{pmatrix} 0.9529 \\ -1.4293 \end{pmatrix} \text{[rad/s]}.$$

Thus, the first joint will decelerate with $\ddot{q}_1 = -A_{max,1} = -3$ rad/s^2 for $\Delta T_{s1} = \dot{q}_1(T_0)/A_{max,1} = 0.3176$ s and stops at $T_{s1} = T_0 + \Delta T_{s1} = 0.7523$ s. On the other hand, the second joint will accelerate with $\ddot{q}_2 = A_{max,2} = 6$ rad/s^2 for $\Delta T_{s2} = |\dot{q}_2(T_0)|/A_{max,2} = 0.2382$ s and stops at $T_{s2} = T_0 + \Delta T_{s2} = 0.6729$ s (sooner than the first joint). As a result, the minimum stopping time for the robot is

$$T_s = \max\{T_{s1}, T_{s2}\} = T_{s1} = 0.7523\,\text{s} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}(T_s) = \boldsymbol{0}.$$

Accordingly, the final rest configuration reached by the robot is

$$\boldsymbol{q}(T_s) = \boldsymbol{q}_0 + \frac{1}{2}\begin{pmatrix} \dot{q}_1(T_0)\,\Delta T_{s1} \\ \dot{q}_2(T_0)\,\Delta T_{s2} \end{pmatrix} = \begin{pmatrix} 0.1626 \\ 0.5415 \end{pmatrix} + \begin{pmatrix} 0.1513 \\ -0.1702 \end{pmatrix} = \begin{pmatrix} 0.3139 \\ 0.3713 \end{pmatrix} \text{[rad]}.$$

Note also that when the second joint stops (at $T_{s2} = 0.6729$ s), the first joint position and velocity are

$$q_1(T_{s2}) = 0.3045 \text{ rad} \qquad \dot{q}_1(T_{s2}) = 0.2382 \text{ rad/s}.$$

The overall joint motion in this emergency situation is shown in Fig. 4.

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