Robotics 1

February 4, 2021

There are 8 questions. Provide answers with short texts, completed with drawings and derivations needed for the solutions. Students with confirmed midterm grade should do only the second set of 4 questions.

Question #1 [students without midterm]

The orientation of a rigid body is defined by the axis-angle pair $\boldsymbol{r} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$ and $\theta = \pi/6$ [rad]. Determine the angles (α, β, γ) of the Roll-Pitch-Yaw sequence XYZ of fixed axes that provide the same orientation. Check the correctness of the obtained result. Find the singular cases for this RPY representation and provide an example of an axis-angle pair $(\boldsymbol{r}_s, \theta_s)$ that would fall in this class.

Question #2 [students without midterm]

Figure 1 shows a top view of a planar two-jaw articulated gripper. This robotic gripper has a revolute joint at its base, followed by two independent revolute joints for each jaw (the first joints in the two jaws share the same axis). This 5-dof robotic system has a tree structure for which the usual Denavit-Hartenberg frame assignment can also be applied (to each branch). Define the joint coordinates accordingly, together with the two DH tables. Provide then the symbolic expression of some task variables that are relevant for gripping operations, defined as follows:

- position of the midpoint P_c between the tips of the two jaws;
- distance d between the two tips;
- relative angle α_{rel} of the left jaw w.r.t. the right jaw;
- orientation angle β w.r.t. the $\boldsymbol{x}_0$ axis of the jaw pair (from the right jaw tip to the left one).

When the gripper links have all the same length $L = 0.05$ [m], compute the numerical value of such task variables in the configuration $\boldsymbol{q} = (q_1, q_{r2}, q_{r3}, q_{l2}, q_{l3}) = (-\pi/2, -\pi/2, 3\pi/4, \pi/2, -3\pi/4)$. Subscripts r and l stand respectively for DH variables pertaining to the right or left jaw only.

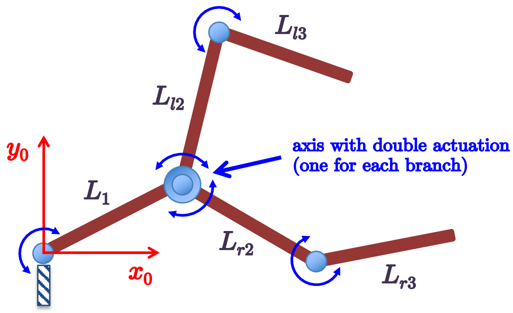


Figure 1: A planar 5-dof two-jaw gripper.

Question #3 [students without midterm]

A planar 2R robot has incremental encoders at the joints measuring the configuration $\boldsymbol{\theta} = (\theta_1, \theta_2)$ used in the computation of its direct kinematics. Because of a bad mounting of the encoders, the two measures are affected by (very) small angular errors δ_1 and δ_2. When using these readings, which of the following statements is correct in terms of Cartesian accuracy of the end-effector position? *A)* there is always an error; *B)* there are configurations at which there may be no error; *C)* the error is always negligible (e.g., below the sensor resolution). Provide a detailed explanation of your answer!

Question #4 [students without midterm]

The prismatic joints of the planar PPR robot in Fig. 2 have bounded ranges, $q_{i,min} \leq q_i \leq q_{i,max}$, for $i = 1, 2$, while the revolute joint q_3 has an unlimited motion range. Draw accurately the primary workspace WS_1 and the secondary workspace WS_2 of this robot, under the following assumption for the third link length: $L < \min\left\{(q_{1,max} - q_{1,min})/2, (q_{2,max} - q_{2,min})/2\right\}$.

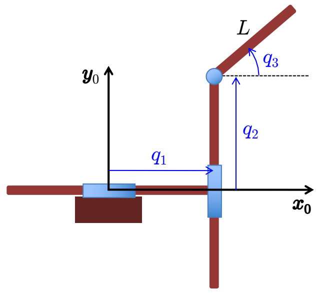


Figure 2: A planar PPR robot.

Question #5 [all students]

The direct kinematics and the initial configuration of a planar RP robot are given by

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{q}^{\{0\}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix},$$

where $0 < \epsilon \ll 1$ is a very small number. Given the following desired end-effector positions,

$$\boldsymbol{p}_{d,\mathrm{I}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{p}_{d,\mathrm{II}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

compute the first iteration (i.e., $\boldsymbol{q}^{\{1\}}$) of a Newton method and of a Gradient method for solving the two inverse kinematics problems. Discuss what happens in each of the four cases when $\varepsilon \to 0$.

Question #6 [all students]

The 3R robotic device in Fig. 3 has joint axes that intersect two by two. The second joint axis is inclined by an angle $\delta \approx 20°$. This structure is mainly intended for pointing the final axis $\boldsymbol{n}$ at a moving target in 3D. Provide the explicit expression of the square angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of this robot. Find the singularities, if any, of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$. Compute the relation between $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and the time derivative $\dot{\boldsymbol{n}}$ of the pointing axis.

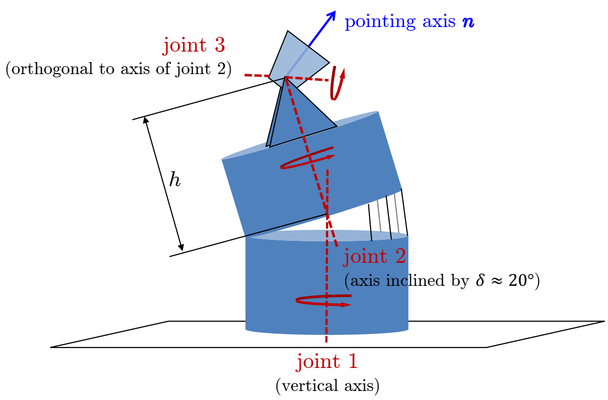


Figure 3: A 3-dof robotic pointing device.

Question #7 [all students]

The desired initial and final orientation of the end effector of a certain robot are specified at time $t = 0$ and $t = T$, respectively by

$$\boldsymbol{R}(0) = \boldsymbol{R}_{in} = \begin{pmatrix} 0.5 & 0 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & -0.5 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{R}(T) = \boldsymbol{R}_{fin} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ -0.5 & -0.5 & -\sqrt{2}/2 \\ 0.5 & 0.5 & -\sqrt{2}/2 \end{pmatrix}.$$

The end effector should start with zero angular velocity and acceleration ($\boldsymbol{\omega}_{in} = \dot{\boldsymbol{\omega}}_{in} = \boldsymbol{0}$, at $t = 0$) and reach the final orientation with angular velocity and acceleration given by

$$\boldsymbol{\omega}(T) = \boldsymbol{\omega}_{fin} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \text{ [rad/s]}, \qquad \dot{\boldsymbol{\omega}}(T) = \dot{\boldsymbol{\omega}}_{fin} = \boldsymbol{0}.$$

Plan a smooth and coordinated trajectory for the end-effector orientation that satisfies all the given boundary conditions for a generic motion time $T > 0$. Setting next $T = 1$, compute at the mid-time instant $t = T/2$ the numerical values of the resulting orientation $\boldsymbol{R}(T/2)$ and angular velocity $\boldsymbol{\omega}(T/2)$ of the robot end effector.

Question #8 [all students]

The planar 3R robot with unitary link lengths shown in Fig. 4 is initially in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\ 11\pi/18,\ -\pi/4)$. Commanded by a joint velocity $\dot{\boldsymbol{q}}(t)$ that uses feedback from the current $\boldsymbol{q}(t)$, the robot should perform a self-motion so as to reach asymptotically the final value $q_{3,fin} = -\pi/2$ for the third joint, while keeping the position of its end-effector always at the same initial point P_{in}. Verify first that such task is feasible. Design then a control scheme that completes the task in a robust way, i.e., by rejecting also possible transient errors and without encountering any singular situation in which the control law is ill conditioned. *Hint: Use an approach based on joint space decomposition.*

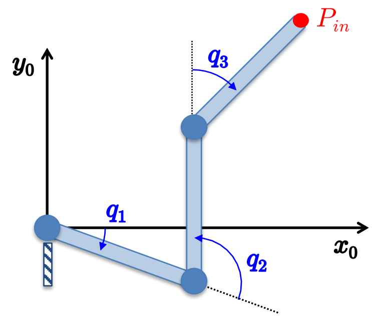


Figure 4: A 3R robot that should perform a self-motion task with constant end-effector position.

[240 minutes (4 hours) for the full exam; open books]

[120 minutes (2 hours) for students with midterm; open books]

Solution

February 4, 2021

Question #1 [students without midterm]

The orientation of a rigid body is defined by the axis-angle pair $\boldsymbol{r} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$ *and* $\theta = \pi/6$ *[rad]. Determine the angles* (α, β, γ) *of the Roll-Pitch-Yaw sequence* XYZ *of fixed axes that provide the same orientation. Check the correctness of the obtained result. Find the singular cases for this RPY representation and provide an example of an axis-angle pair* $(\boldsymbol{r}_s, \theta_s)$ *that would fall in this class.*

Reply #1

The solution requires back and forth transformations among different representations of orientation. From the given axis-angle pair $(\boldsymbol{r}, \theta)$, we compute the rotation matrix

$$\boldsymbol{R}(\boldsymbol{r}, \theta) = \boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta = \begin{pmatrix} 0.9107 & -0.3333 & -0.2440 \\ 0.2440 & 0.9107 & -0.3333 \\ 0.3333 & 0.2440 & 0.9107 \end{pmatrix} = \boldsymbol{R}_d. \tag{1}$$

Three rotations by the angles (α, β, γ) around the sequence of XYZ fixed axes (i.e., of the RPY type) are associated to the rotation matrix

$$\begin{aligned} \boldsymbol{R}_{XYZ}(\alpha, \beta, \gamma) &= \boldsymbol{R}_Z(\gamma)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_X(\alpha) \\ &= \begin{pmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma \\ \cos\beta\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ -\sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}. \end{aligned} \tag{2}$$

For the nonsingular case, when

$$\cos\beta = \pm\sqrt{R_{32}^2 + R_{33}^2} \neq 0,$$

solving $\boldsymbol{R}_{XYZ}(\alpha, \beta, \gamma) = \boldsymbol{R}_d = \{R_{ij}\}$ for (α, β, γ) yields a pair of solutions (one for each sign chosen inside $\cos\beta$):

$$\alpha = \text{ATAN2}\left\{\frac{R_{32}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \beta = \text{ATAN2}\left\{-R_{31}, \cos\beta\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}.$$

Using the data, we have $\cos\beta = \pm\, 0.9428 \neq 0$ and find the two numerical solutions

$$\alpha' = 0.2618, \qquad \beta' = -0.3398, \qquad \gamma' = 0.2618 \qquad \text{[rad]}$$

and

$$\alpha'' = -2.8798, \qquad \beta'' = -2.8018, \qquad \gamma'' = -2.8798 \qquad \text{[rad]}.$$

Plugging any of these triples into (2) returns indeed $\boldsymbol{R}_d$. The singularity of this minimal representation of orientation occurs for $\cos\beta = 0$ (i.e., $\beta = \pm\pi/2$), namely when $R_{32}^2 + R_{33}^2 = 0$. In this case, we can further solve only for the sum $\alpha + \gamma$ (when $\sin\beta = -R_{11} = 1$) or the difference $\alpha - \gamma$ (for $\sin\beta = -1$) of the remaining two angles. For instance, a rotation matrix in this class is

$$\boldsymbol{R}_s = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3}$$

for which $\beta = \pi/2$ and $\alpha - \gamma = -\pi/2$. The rotation matrix $\boldsymbol{R}_s = \{R_{ij}\}$ is in turn nonsingular for the transformation with an axis-angle representation, being

$$\sin\theta = \frac{1}{2}\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2} = 0.8660 \neq 0.$$

Therefore, $\boldsymbol{R}_s$ can be generated by the pair $(\boldsymbol{r}'_s, \theta'_s)$ computed as

$$\theta'_s = \text{ATAN2}\left\{\sin\theta, \frac{R_{11} + R_{22} + R_{33} - 1}{2}\right\} = 2.0944 \text{ [rad]}$$
$$\boldsymbol{r}'_s = \frac{1}{2\sin\theta}\begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix},$$

as well as by the pair $(\boldsymbol{r}''_s, \theta''_s) = (-\boldsymbol{r}'_s, -\theta'_s)$. Note that $\boldsymbol{r}'_s = \boldsymbol{r}$, the same unit axis given in the text as a starting point. This coincidence occurs by pure chance. It is indeed $\theta'_s \neq \theta$, otherwise we would have generated $\boldsymbol{R}_s$ in (3) instead of $\boldsymbol{R}_d$ in (1), obtaining a singular case for the chosen RPY representation. ■

Question #2 [students without midterm]

Figure 1 shows a top view of a planar two-jaw articulated gripper. This robotic gripper has a revolute joint at its base, followed by two independent revolute joints for each jaw (the first joints in the two jaws share the same axis). This 5-dof robotic system has a tree structure for which the usual Denavit-Hartenberg frame assignment can also be applied (to each branch). Define the joint coordinates accordingly, together with the two DH tables. Provide then the symbolic expression of some task variables that are relevant for gripping operations, defined as follows:

- *position of the midpoint P_c between the tips of the two jaws;*
- *distance d between the two tips;*
- *relative angle α_{rel} of the left jaw w.r.t. the right jaw;*
- *orientation angle β w.r.t. the $\boldsymbol{x}_0$ axis of the jaw pair (from the right jaw tip to the left one).*

When the gripper links have all the same length $L = 0.05$ [m], compute the numerical value of such task variables in the configuration $\boldsymbol{q} = (q_1, q_{r2}, q_{r3}, q_{l2}, q_{l3}) = (-\pi/2, -\pi/2, 3\pi/4, \pi/2, -3\pi/4)$. Subscripts r and l stand respectively for DH variables pertaining to the right or left jaw only.

Reply #2

Figure 5 shows the $\boldsymbol{x}_i$ axes of the frames assigned to the articulated gripper according to the Denavit-Hartenberg convention, together with the associated joint variables and all the relevant quantities for defining the task variables. The two DH tables, respectively for the right and the left jaws, are given in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	0	L_1	0	q_1
2	0	L_{r2}	0	q_{r2}
3	0	L_{r3}	0	q_{r3}

i	α_i	a_i	d_i	θ_i
1	0	L_1	0	q_1
2	0	L_{l2}	0	q_{l2}
3	0	L_{l3}	0	q_{l3}

Table 1: Tables of DH parameters for the right and left jaws of the gripper.

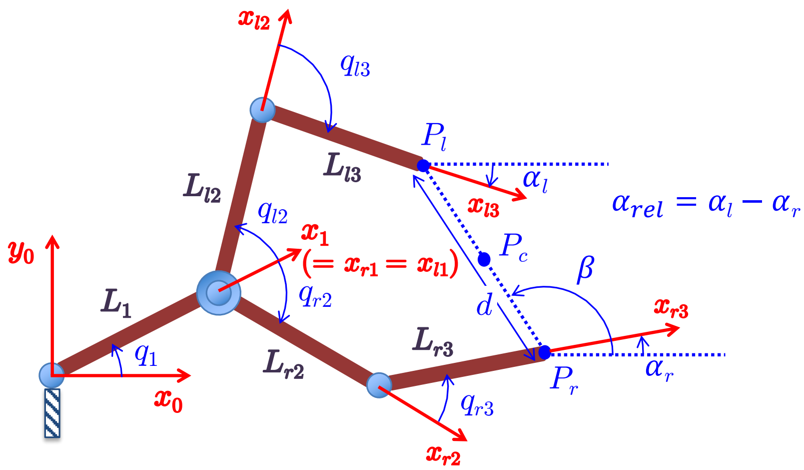


Figure 5: DH frames, with definition of coordinates and task variables for the two-jaw gripper.

Based on these definitions, we compute the direct kinematics of the two tip points P_r and P_l and the final orientations α_r and α_l of the two jaws:

$$\begin{aligned}
\boldsymbol{p}_r &= \begin{pmatrix} L_1 \cos q_1 + L_{r2} \cos(q_1 + q_{r2}) + L_{r3} \cos(q_1 + q_{r2} + q_{r3}) \\ L_1 \sin q_1 + L_{r2} \sin(q_1 + q_{r2}) + L_{r3} \sin(q_1 + q_{r2} + q_{r3}) \end{pmatrix} \\
\boldsymbol{p}_l &= \begin{pmatrix} L_1 \cos q_1 + L_{l2} \cos(q_1 + q_{l2}) + L_{l3} \cos(q_1 + q_{l2} + q_{l3}) \\ L_1 \sin q_1 + L_{l2} \sin(q_1 + q_{l2}) + L_{l3} \sin(q_1 + q_{l2} + q_{l3}) \end{pmatrix} \\
\alpha_r &= q_1 + q_{r2} + q_{r3} \\
\alpha_l &= q_1 + q_{l2} + q_{l3}.
\end{aligned}$$

The considered task variables are then defined as

$$\begin{aligned}
\boldsymbol{p}_c &= \frac{\boldsymbol{p}_l + \boldsymbol{p}_r}{2} \\
d &= \|\boldsymbol{p}_l - \boldsymbol{p}_r\| \\
\alpha_{rel} &= \alpha_l - \alpha_r = q_{l2} + q_{l3} - q_{r2} - q_{r3} \\
\beta &= \text{ATAN2}\left\{p_{ly} - p_{ry}, p_{lx} - p_{rx}\right\}.
\end{aligned}$$

With the given data for link lengths and current configuration, we obtain the numerical values

$$\boldsymbol{p}_c = \begin{pmatrix} 0 \\ -0.08536 \end{pmatrix} \text{ [m]}, \qquad d = 0.02929 \text{ [m]}, \qquad \alpha_{rel} = -\frac{\pi}{2} \text{ [rad]}, \qquad \beta = 0. \qquad \blacksquare$$

Question #3 [students without midterm]

A planar 2R robot has incremental encoders at the joints measuring the configuration $\boldsymbol{\theta} = (\theta_1, \theta_2)$ *used in the computation of its direct kinematics. Because of a bad mounting of the encoders, the two measures are affected by (very) small angular errors* δ_1 *and* δ_2*. When using these readings, which of the following statements is correct in terms of Cartesian accuracy of the end-effector position?* A) *there is always an error;* B) *there are configurations at which there may be no error;*

C) *the error is always negligible (e.g., below the sensor resolution). Provide a detailed explanation of your answer!*

Reply #3

The correct answer is B. Statement A will automatically be false (because of the word 'always'), once we confirm the correctness of B. Statement C seems ambiguous, since no information is provided on the sensor resolution nor on the link lengths, which may both be arbitrary small or large. Therefore, for a given amount of error δ one can consider a robot with sufficiently long links so that the Cartesian accuracy becomes unacceptably large[1]. To show the validity of B, consider the 2R robot in a singular configuration, say the stretched one $\boldsymbol{\theta}^* = (\theta_1, 0)$ for an arbitrary θ_1. The robot Jacobian would become

$$\boldsymbol{J}(\boldsymbol{\theta}^*) = \begin{pmatrix} -\left(l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)\right) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{pmatrix}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} = \begin{pmatrix} -(l_1+l_2)\sin\theta_1 & -l_2 \sin\theta_1 \\ (l_1+l_2)\cos\theta_1 & l_2\cos\theta_1 \end{pmatrix},$$

with rank $\boldsymbol{J}(\boldsymbol{\theta}^*) = 1$. In view of the assumption of small errors $\boldsymbol{\delta}$ for the sensor readings in the joint space, we can use the differential mapping to evaluate their effect on the Cartesian error displacements, i.e., $\Delta\boldsymbol{p} = \boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{\delta}$. Therefore, at $\boldsymbol{\theta}^*$, all sensing errors $\boldsymbol{\delta}^* \in \mathbb{R}^2$ that are in the null space of $\boldsymbol{J}(\boldsymbol{\theta}^*)$ will produce no the Cartesian errors:

$$\boldsymbol{\delta}^* = \epsilon \begin{pmatrix} -l_1 \\ l_1 + l_2 \end{pmatrix}, \quad |\epsilon| \ll 1 \qquad \Rightarrow \qquad \boldsymbol{J}(\boldsymbol{\theta}^*)\boldsymbol{\delta}^* = \boldsymbol{0}. \qquad \blacksquare$$

Question #4 [students without midterm]

The prismatic joints of the planar PPR robot in Fig. 2 have bounded ranges, $q_{i,min} \le q_i \le q_{i,max}$, for $i = 1, 2$, while the revolute joint q_3 has an unlimited motion range. Draw accurately the primary workspace WS_1 and the secondary workspace WS_2 of this robot, under the following assumption for the third link length: $L < \min\{(q_{1,max} - q_{1,min})/2, (q_{2,max} - q_{2,min})/2\}$.

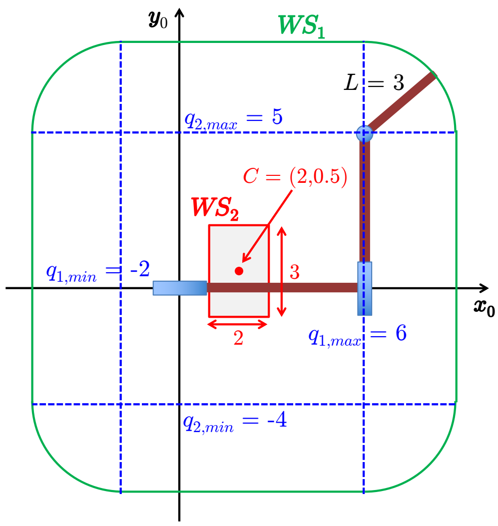


Figure 6: Primary and secondary workspaces for a planar PPR robot with bounded range of the prismatic joints.

[1] Indeed, the definition of a normalized accuracy would scale to the size of the robot. In that case, this argument would not work. But again, the word 'always' largely weakens statement C, just as in case A.

Reply #4

The primary and secondary workspaces of the PPR robot are drawn in Fig. 6, using the following set of representative values:

$$q_{1,min} = -2, \quad q_{1,max} = 6; \qquad q_{2,min} = -4, \quad q_{2,max} = 5; \qquad L = 3 \quad [\text{m}].$$

The ranges of the prismatic joints need not to be symmetric. The assumption on the length L of the third link is satisfied here, allowing the presence of a non-vanishing WS_2, where the robot end-effector can assume any orientation angle in the plane. The outer boundary of WS_1 is a rectangle of side lengths $q_{i,max} - q_{i,min} + 2L$, $i = 1,2$, with corners smoothed by circles of radius L. The outer boundary of WS_2 is a rectangle with sides $q_{i,max} - q_{i,min} - 2L > 0$, $i = 1,2$. ■

Question #5 [all students]

The direct kinematics and the initial configuration of a planar RP robot are given by

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{q}^{\{0\}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix},$$

where $0 < \epsilon \ll 1$ is a very small number. Given the following desired end-effector positions,

$$\boldsymbol{p}_{d,\mathrm{I}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{p}_{d,\mathrm{II}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

compute the first iteration (i.e., $\boldsymbol{q}^{\{1\}}$) of a Newton method and of a Gradient method for solving the two inverse kinematics problems. Discuss what happens in each of the four cases when $\varepsilon \to 0$.

Reply #5

Evaluating the Jacobian of the RP robot at $\boldsymbol{q}^{\{0\}}$, we have

$$\boldsymbol{J}(\boldsymbol{q}^{\{0\}}) = \left.\begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{q}^{\{0\}}} = \begin{pmatrix} -\varepsilon\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \\ \varepsilon\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix}.$$

It is $\det \boldsymbol{J}(\boldsymbol{q}^{\{0\}}) = -\varepsilon$, and thus a singularity is approached when $\varepsilon \to 0$. For the desired end-effector position of case I, with the Newton method we have at the first iteration

$$\begin{aligned}
\boldsymbol{q}^{\{1\}}_{Newton,\mathrm{I}} &= \boldsymbol{q}^{\{0\}} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{\{0\}})\left(\boldsymbol{p}_{d,\mathrm{I}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\
&= \begin{pmatrix} \dfrac{\pi}{4} \\ \varepsilon \end{pmatrix} + \begin{pmatrix} -\dfrac{\sqrt{2}}{2\varepsilon} & \dfrac{\sqrt{2}}{2\varepsilon} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon\dfrac{\sqrt{2}}{2} \\ \varepsilon\dfrac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \dfrac{\pi}{4} + \dfrac{\sqrt{2}}{\varepsilon} \\ 0 \end{pmatrix},
\end{aligned}$$

whereas with the Gradient method (for a generic step size $\alpha > 0$) it is

$$\begin{aligned}
\boldsymbol{q}^{\{1\}}_{Gradient,\mathrm{I}} &= \boldsymbol{q}^{\{0\}} + \alpha\, \boldsymbol{J}^{T}(\boldsymbol{q}^{\{0\}})\left(\boldsymbol{p}_{d,\mathrm{I}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\
&= \begin{pmatrix} \dfrac{\pi}{4} \\ \varepsilon \end{pmatrix} + \alpha \begin{pmatrix} -\varepsilon\dfrac{\sqrt{2}}{2} & \varepsilon\dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon\dfrac{\sqrt{2}}{2} \\ \varepsilon\dfrac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \dfrac{\pi}{4} + \alpha\sqrt{2}\,\varepsilon \\ \varepsilon\,(1-\alpha) \end{pmatrix}.
\end{aligned}$$

For $\varepsilon \to 0$, it is easy to see that

$$\boldsymbol{q}^{\{1\}}_{Newton,\mathrm{I}} \to \begin{pmatrix} \infty \\ 0 \end{pmatrix}, \qquad \boldsymbol{q}^{\{1\}}_{Gradient,\mathrm{I}} \to \begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix} = \boldsymbol{q}^{\{0\}},$$

illustrating how the Newton method diverges while approaching a singularity, while the Gradient method simply stops. In fact, when $\varepsilon = 0$, the position error $\boldsymbol{e}^{\{0\}} = \boldsymbol{p}_{d,\mathrm{I}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}}) = \boldsymbol{p}_{d,\mathrm{I}} = (-1, 1)$ belongs to the null space of $\boldsymbol{J}^T(\boldsymbol{q}^{\{0\}})|_{\varepsilon=0}$. For case II, with the Newton method one has

$$\begin{aligned}\boldsymbol{q}^{\{1\}}_{Newton,\mathrm{II}} &= \boldsymbol{q}^{\{0\}} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{\{0\}})\left(\boldsymbol{p}_{d,\mathrm{II}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\ &= \begin{pmatrix} \frac{\pi}{4} \\ \varepsilon \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{2}}{2\varepsilon} & \frac{\sqrt{2}}{2\varepsilon} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon\frac{\sqrt{2}}{2} \\ \varepsilon\frac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \frac{\pi}{4} \\ \sqrt{2} \end{pmatrix} = \boldsymbol{q}^*.\end{aligned}$$

Being $\boldsymbol{f}(\boldsymbol{q}^*) = \boldsymbol{p}_{d,\mathrm{II}}$, we have found a solution $\boldsymbol{q}^*$ to the inverse kinematics problem in just one iteration (thanks to the simple structure of this specific problem). Moreover, this holds true independently from the value of ε, which in fact cancels out. On the other hand, with the Gradient method, and for a generic step size $\alpha > 0$, it is

$$\begin{aligned}\boldsymbol{q}^{\{1\}}_{Gradient,\mathrm{II}} &= \boldsymbol{q}^{\{0\}} + \alpha\, \boldsymbol{J}^T(\boldsymbol{q}^{\{0\}})\left(\boldsymbol{p}_{d,\mathrm{II}} - \boldsymbol{f}(\boldsymbol{q}^{\{0\}})\right) \\ &= \begin{pmatrix} \frac{\pi}{4} \\ \varepsilon \end{pmatrix} + \alpha \begin{pmatrix} -\varepsilon\frac{\sqrt{2}}{2} & \varepsilon\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \varepsilon\frac{\sqrt{2}}{2} \\ \varepsilon\frac{\sqrt{2}}{2} \end{pmatrix} \right) = \begin{pmatrix} \frac{\pi}{4} \\ \alpha\sqrt{2} + \varepsilon\,(1-\alpha) \end{pmatrix}.\end{aligned}$$

When $\varepsilon \to 0$, we have

$$\boldsymbol{q}^{\{1\}}_{Gradient.\mathrm{II}} \to \begin{pmatrix} \frac{\pi}{4} \\ \alpha\sqrt{2} \end{pmatrix},$$

so that the Gradient method will escape the singularity (here, $\boldsymbol{e}^{\{0\}} = \boldsymbol{p}_{d,\mathrm{II}} \notin \mathcal{N}\left\{\boldsymbol{J}^T(\boldsymbol{q}^{\{0\}})|_{\varepsilon=0}\right\}$). One can easily see that also the Gradient method may find the solution $\boldsymbol{q}^*$ in just one iteration, but only if the step size is chosen as $\alpha = 1$ (and then, once again, this would then occur independently from the actual value of ε). However, α is usually chosen smaller than unitary in the final iterations in order to avoid missing a close solution. Therefore, the Gradient method will typically approach the solution $\boldsymbol{q}^*$ at a slower rate than the Newton method. ■

Question #6 [all students]

The 3R robotic device in Fig. 3 has joint axes that intersect two by two. The second joint axis is inclined by an angle $\delta \approx 20°$. This structure is mainly intended for pointing the final axis $\boldsymbol{n}$ at a moving target in 3D. Provide the explicit expression of the square angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of this robot. Find the singularities, if any, of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$. Compute the relation between $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and the time derivative $\dot{\boldsymbol{n}}$ of the pointing axis.

Reply #6

The angular part of the geometric Jacobian for this 3R robotic pointing device is the 3×3 matrix

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{z}_2 \end{pmatrix}$$

where $\boldsymbol{z}_{i-1}$ is a unit vector along the axis of joint i, for $i = 1, 2, 3$, expressed by default in the robot base frame RF_0, and ${}^j\boldsymbol{z}_j = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for any index j. In order to provide the explicit expression of the elements in this matrix, it is convenient to define a set of frames according to the DH convention, e.g., as in Fig. 7. From the DH table reported therein[2], one has

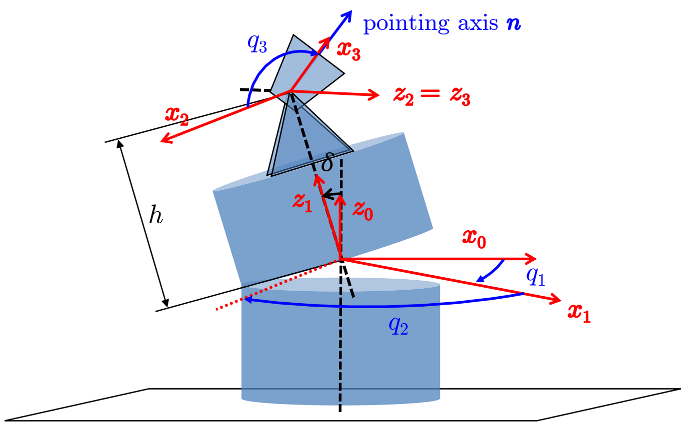
$$
{}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & -\cos\delta \sin q_1 & \sin\delta \sin q_1 \\ \sin q_1 & \cos\delta \cos q_1 & -\sin\delta \cos q_1 \\ 0 & \sin\delta & \cos\delta \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & -\sin q_2 \\ \sin q_2 & 0 & \cos q_2 \\ 0 & -1 & 0 \end{pmatrix},
$$

and so

$$
{}^0\boldsymbol{R}_2(q_1, q_2) = \begin{pmatrix} \cos q_1 \cos q_2 - \cos\delta \sin q_1 \sin q_2 & -\sin\delta \sin q_1 & -\cos q_1 \sin q_2 - \cos\delta \sin q_1 \cos q_2 \\ \sin q_1 \cos q_2 + \cos\delta \cos q_1 \sin q_2 & \sin\delta \cos q_1 & \cos\delta \cos q_1 \cos q_2 - \sin q_1 \sin q_2 \\ \sin\delta \sin q_2 & -\cos\delta & \sin\delta \cos q_2 \end{pmatrix}.
$$

As a result,

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & \sin\delta \sin q_1 & -\cos q_1 \sin q_2 - \cos\delta \sin q_1 \cos q_2 \\ 0 & -\sin\delta \cos q_1 & \cos\delta \cos q_1 \cos q_2 - \sin q_1 \sin q_2 \\ 1 & \cos\delta & \sin\delta \cos q_2 \end{pmatrix}.
$$



i	α_i	a_i	d_i	θ_i
1	δ	0	0	q_1
2	$-\pi/2$	0	h	q_2
3	0	0	0	q_3

Figure 7: DH frames and table used for defining the Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ of the pointing device.

The singularities of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ occur when

$$
\det \boldsymbol{J}_A(\boldsymbol{q}) = -\sin\delta \sin q_2 = 0 \qquad \Longleftrightarrow \qquad q_2 = \{0, \pi\},
$$

[2]Since $\boldsymbol{z}_3$ does not enter in the Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$, the choice of a particular final twist α_3 is irrelevant here.

namely when the axis $\boldsymbol{x}_1$ and the projection of the axis $\boldsymbol{x}_2$ on the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ are aligned. In this situation, the following instantaneous joint velocities

$$\dot{\boldsymbol{q}} = \lambda \begin{pmatrix} -1 \\ \cos\delta \\ \sin\delta \end{pmatrix}, \text{ for } q_2 = 0 \qquad \text{or} \qquad \dot{\boldsymbol{q}} = \lambda \begin{pmatrix} -1 \\ \cos\delta \\ -\sin\delta \end{pmatrix}, \text{ for } q_2 = \pi,$$

lie in the null space of $\boldsymbol{J}$ and will produce thus $\boldsymbol{\omega} = \boldsymbol{0}$, $\forall\lambda$. Finally, the pointing axis $\boldsymbol{n}$ is given by

$$\begin{aligned} \boldsymbol{n} &= {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (\cos q_1 \cos q_2 - \cos\delta \sin q_1 \sin q_2)\cos q_3 - \sin\delta \sin q_1 \sin q_3 \\ (\sin q_1 \cos q_2 + \cos\delta \cos q_1 \sin q_2)\cos q_3 + \sin\delta \cos q_1 \sin q_3 \\ \sin\delta \sin q_2 \cos q_3 - \cos\delta \sin q_3 \end{pmatrix} = \begin{pmatrix} n_1(\boldsymbol{q}) \\ n_2(\boldsymbol{q}) \\ n_3(\boldsymbol{q}) \end{pmatrix}. \end{aligned}$$

Let $\boldsymbol{R}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} \end{pmatrix}$. The time derivative of the unit vector $\boldsymbol{n}$ is computed then as

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R} \quad \Rightarrow \quad \dot{\boldsymbol{n}} = \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{n} = \boldsymbol{\omega} \times \boldsymbol{n} = (\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}) \times \boldsymbol{n} = (\boldsymbol{z}_0 \times \boldsymbol{n})\,\dot{q}_1 + (\boldsymbol{z}_1 \times \boldsymbol{n})\,\dot{q}_2 + (\boldsymbol{z}_3 \times \boldsymbol{n})\,\dot{q}_3.$$

The following computations can be conveniently performed with a symbolic code in Matlab:

$$\boldsymbol{z}_0 \times \boldsymbol{n} = \begin{pmatrix} -n_2(\boldsymbol{q}) \\ n_1(\boldsymbol{q}) \\ 0 \end{pmatrix},$$

$$\boldsymbol{z}_1 \times \boldsymbol{n} = \left({}^0\boldsymbol{R}_1\,{}^1\boldsymbol{z}_1\right) \times \boldsymbol{n} = \begin{pmatrix} \sin\delta \sin q_1 \\ -\sin\delta \cos q_1 \\ \cos\delta \end{pmatrix} \times \boldsymbol{n} = \begin{pmatrix} -(\cos q_1 \sin q_2 + \cos\delta \sin q_1 \cos q_2)\cos q_3 \\ -(\sin q_1 \sin q_2 - \cos\delta \cos q_1 \cos q_2)\cos q_3 \\ \sin\delta \cos q_2 \cos q_3 \end{pmatrix},$$

and

$$\begin{aligned} \boldsymbol{z}_2 \times \boldsymbol{n} &= \left({}^0\boldsymbol{R}_2\,{}^2\boldsymbol{z}_2\right) \times \boldsymbol{n} = \begin{pmatrix} \cos q_1 \sin q_2 - \cos\delta \sin q_1 \cos q_2 \\ \cos\delta \cos q_1 \cos q_2 - \sin q_1 \sin q_2 \\ \sin\delta \cos q_2 \end{pmatrix} \times \boldsymbol{n} \\ &= \begin{pmatrix} \cos\delta \sin q_1 \sin q_2 \sin q_3 - \sin\delta \sin q_1 \cos q_3 - \cos q_1 \cos q_2 \sin q_3 \\ \sin\delta \cos q_1 \cos q_3 - \sin q_1 \cos q_2 \sin q_3 - \cos\delta \cos q_1 \sin q_2 \sin q_3 \\ -\cos\delta \cos q_3 - \sin\delta \sin q_2 \sin q_3 \end{pmatrix}. \end{aligned}$$

■

Question #7 [all students]

The desired initial and final orientation of the end effector of a certain robot are specified at time $t = 0$ and $t = T$, respectively by

$$\boldsymbol{R}(0) = \boldsymbol{R}_{in} = \begin{pmatrix} 0.5 & 0 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & -0.5 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{R}(T) = \boldsymbol{R}_{fin} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ -0.5 & -0.5 & -\sqrt{2}/2 \\ 0.5 & 0.5 & -\sqrt{2}/2 \end{pmatrix}.$$

The end effector should start with zero angular velocity and acceleration ($\boldsymbol{\omega}_{in} = \dot{\boldsymbol{\omega}}_{in} = \mathbf{0}$, at $t = 0$) and reach the final orientation with angular velocity and acceleration given by

$$\boldsymbol{\omega}(T) = \boldsymbol{\omega}_{fin} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \text{[rad/s]}, \qquad \dot{\boldsymbol{\omega}}(T) = \dot{\boldsymbol{\omega}}_{fin} = \mathbf{0}.$$

Plan a smooth and coordinated trajectory for the end-effector orientation that satisfies all the given boundary conditions for a generic motion time $T > 0$. Setting next $T = 1$, compute at the mid-time instant $t = T/2$ the numerical values of the resulting orientation $\boldsymbol{R}(T/2)$ and angular velocity $\boldsymbol{\omega}(T/2)$ of the robot end effector.

Reply #7

This trajectory planning problem for the end-effector orientation has the special feature of requiring a desired non-zero angular velocity at the final instant of motion. So, it is very unlikely that addressing the reorientation using the axis-angle method will work (in fact, this has zero probability to occur!). The reason is that, if we extract a unit axis $\boldsymbol{r}$ and an angle θ_{if} from the relative rotation $\boldsymbol{R}_{if} = \boldsymbol{R}_{in}^T \boldsymbol{R}_{fin}$ and then perform the rotation around $\boldsymbol{r}$ with any possible timing $\theta(t)$, the associated angular velocity $\boldsymbol{\omega}_r(t) = \boldsymbol{r}\dot{\theta}(t)$ will always be aligned with $\boldsymbol{r}$. In particular, at $t = T$, it will be $\boldsymbol{\omega}_r(T) \neq \boldsymbol{\omega}_{fin}$, and we cannot satisfy such equality for an arbitrary $\boldsymbol{\omega}_{fin}$ (one can check that this in fact the case here too). Therefore, we pursue a solution by planning the motion for the three angles of a minimal representation of orientation, and imposing suitable boundary conditions. Indeed, many choices are possible and the only precaution is to avoid singularities of the representation during the entire reorientation. In the following, we shall use the sequence of XYZ Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$.

For this Euler representation of orientation, we have the rotation matrix

$$\begin{aligned} \boldsymbol{R}_{E,XYZ}(\boldsymbol{\phi}) &= \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_Z(\gamma) \\ &= \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix}. \end{aligned} \tag{4}$$

At the differential level, we have also the relationship between $\dot{\boldsymbol{\phi}}$ and the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$:

$$\begin{aligned} \boldsymbol{\omega} &= \begin{pmatrix} {}^0\boldsymbol{x}_0 & \boldsymbol{R}_X(\alpha)\,{}^1\boldsymbol{y}_1 & \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\,{}^2\boldsymbol{z}_2 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} \dot{\alpha} & & \\ 0 & \boldsymbol{R}_X(\alpha)\begin{pmatrix} 0 \\ \dot{\beta} \\ 0 \end{pmatrix} & \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} \\ 0 & & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\,\dot{\boldsymbol{\phi}}. \end{aligned} \tag{5}$$

Matrix $\boldsymbol{T}$ has $\det \boldsymbol{T}(\boldsymbol{\phi}) = \cos\beta$. This is indeed the singularity of the chosen Euler representation. One should make sure that the condition $\beta = \pm\pi/2$ is never crossed in the solution of the problem (otherwise, we should change representation and restart from scratch). With the expression (4) at hand, one can convert the given initial and final rotation matrices $\boldsymbol{R}_{in}$ and $\boldsymbol{R}_{fin}$ into XYZ Euler angles by using general inversion formulas that hold in the non-singular case only, i.e., when

$$\cos\beta = \pm\sqrt{R_{11}^2 + R_{12}^2} \neq 0.$$

This happens to be the case for both rotation matrices. Then, from

$$\alpha = \text{ATAN2}\left\{\frac{-R_{23}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \qquad \beta = \text{ATAN2}\left\{R_{13}, \cos\beta\right\}, \qquad \gamma = \text{ATAN2}\left\{\frac{-R_{12}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}, \quad (6)$$

we obtain the following pairs of solutions (depending on the sign chosen in the expression of $\cos\beta$)

$$\boldsymbol{R}_{in} \quad \Rightarrow \quad \cos\beta = \pm 0.5 \qquad \Rightarrow \qquad \boldsymbol{\phi}_{in} = (\alpha_{in}, \beta_{in}, \gamma_{in}) = \begin{cases} \boldsymbol{\phi}_{in}^{I} = (\frac{\pi}{2}, -\frac{\pi}{3}, 0) \\ \boldsymbol{\phi}_{in}^{II} = (-\frac{\pi}{2}, -\frac{2\pi}{3}, \pi) \end{cases}$$

$$\boldsymbol{R}_{fin} \quad \Rightarrow \quad \cos\beta = \pm 1 \qquad \Rightarrow \qquad \boldsymbol{\phi}_{fin} = (\alpha_{fin}, \beta_{fin}, \gamma_{fin}) = \begin{cases} \boldsymbol{\phi}_{fin}^{I} = (\frac{3\pi}{4}, 0, \frac{\pi}{4}) \\ \boldsymbol{\phi}_{fin}^{II} = (-\frac{\pi}{4}, \pi, -\frac{3\pi}{4}). \end{cases}$$

Keeping into account the need to avoid the crossing of a value $\beta = \pm\pi/2$, we choose the combination (out of four possible) $\boldsymbol{\phi}_{in}^{I}$ (with $\beta_{in} = -\pi/3$) and $\boldsymbol{\phi}_{fin}^{I}$ (with $\beta_{fin} = 0$) as boundary conditions for the Euler angles trajectories[3]. In this way, a smooth interpolation should also guarantee that $\beta(t)$ remains in the singularity-free interval $[-\pi/3, 0]$ for all $t \in [0, T]$. We use finally eq. (5) to convert the given angular velocity $\boldsymbol{\omega}_{fin}$ into a boundary condition for the first derivative of the Euler angles at the final time $t = T$. In view of the non-singularity of $\boldsymbol{T}(\boldsymbol{\phi}_{fin})$, we obtain

$$\dot{\boldsymbol{\phi}}_{fin} = \boldsymbol{T}^{-1}(\boldsymbol{\phi}_{fin})\,\boldsymbol{\omega}_{fin} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2.1213 \\ \sqrt{2}/2 \end{pmatrix} \text{ [rad/s]}.$$

The interpolation problem for the Euler angles requires generating a $\boldsymbol{\phi}(t)$ for $t \in [0, T]$ such that

$$\boldsymbol{\phi}(0) = \boldsymbol{\phi}_{in}, \quad \dot{\boldsymbol{\phi}}(0) = \mathbf{0}, \quad \ddot{\boldsymbol{\phi}}(0) = \mathbf{0}, \qquad \boldsymbol{\phi}(T) = \boldsymbol{\phi}_{fin}, \quad \dot{\boldsymbol{\phi}}(T) = \dot{\boldsymbol{\phi}}_{fin}, \quad \ddot{\boldsymbol{\phi}}(T) = \mathbf{0}.$$

We use then quintic polynomials in normalized time of the form

$$\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}_{in} + \boldsymbol{a}_3\tau^3 + \boldsymbol{a}_4\tau^4 + \boldsymbol{a}_5\tau^5, \qquad \tau = \frac{t}{T} \in [0, 1], \quad (7)$$

where the vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5 \in \mathbb{R}^3$ contain the non-vanishing coefficients of the three quintic polynomials for $\alpha(\tau)$, $\beta(\tau)$, and $\gamma(\tau)$. The solution is illustrated next for a generic scalar component $\phi_i(\tau)$ of $\boldsymbol{\phi}(\tau)$, with $i = 1, 2, 3$, dropping also the index. The chosen structure of the polynomial in (7) already satisfies the three boundary conditions at $\tau = 0$. For the remaining three conditions at $\tau = 1$, we have:

$$\begin{aligned} \phi(1) &= \phi_{in} + a_3 + a_4 + a_5 = \phi_{fin} \\ \dot{\phi}(1) &= \left.\frac{d\phi}{d\tau}\right|_{\tau=1} \cdot \frac{d\tau}{dt} = (3a_3 + 4a_4 + 5a_5)\,\frac{1}{T} = \dot{\phi}_{fin} \\ \ddot{\phi}(1) &= \left.\frac{d^2\phi}{d\tau^2}\right|_{\tau=1} \cdot \left(\frac{d\tau}{dt}\right)^2 = (6a_3 + 12a_4 + 20a_5)\,\frac{1}{T^2} = 0, \end{aligned}$$

[3] In the following, we shall drop for conciseness the index I from $\boldsymbol{\phi}_{in}$ and $\boldsymbol{\phi}_{fin}$.

or

$$\boldsymbol{M}\boldsymbol{a} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} \phi_{fin} - \phi_{in} \\ \dot{\phi}_{fin} T \\ 0 \end{pmatrix} = \boldsymbol{b},$$

with $\det \boldsymbol{M} = 2$. Solving this linear system yields

$$\boldsymbol{a} = \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{b} = \begin{pmatrix} 10 & -4 & 0.5 \\ -15 & 7 & -1 \\ 6 & -3 & 0.5 \end{pmatrix} \begin{pmatrix} \phi_{fin} - \phi_{in} \\ \dot{\phi}_{fin} T \\ 0 \end{pmatrix},$$

or

$$a_3 = 10(\phi_{fin} - \phi_{in}) - 4\dot{\phi}_{fin}T, \quad a_4 = -15(\phi_{fin} - \phi_{in}) + 7\dot{\phi}_{fin}T, \quad a_5 = 6(\phi_{fin} - \phi_{in}) - 3\dot{\phi}_{fin}T.$$

In vector form, the solution can be rewritten as

$$\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}_{in} + \left(\boldsymbol{\phi}_{fin} - \boldsymbol{\phi}_{in}\right)\left(10\tau^3 - 15\tau^4 + 6\tau^5\right) + \dot{\boldsymbol{\phi}}_{fin} T \left(-4\tau^3 + 7\tau^4 - 3\tau^5\right), \quad \tau \in [0, 1].$$

Moreover, the first and second time derivatives are

$$\dot{\boldsymbol{\phi}}(\tau) = \frac{\boldsymbol{\phi}_{fin} - \boldsymbol{\phi}_{in}}{T}\left(30\tau^2 - 60\tau^3 + 30\tau^4\right) + \dot{\boldsymbol{\phi}}_{fin}\left(-12\tau^2 + 28\tau^3 - 15\tau^4\right),$$

$$\ddot{\boldsymbol{\phi}}(\tau) = \frac{\boldsymbol{\phi}_{fin} - \boldsymbol{\phi}_{in}}{T^2}\left(60\tau - 180\tau^2 + 120\tau^3\right) + \frac{\dot{\boldsymbol{\phi}}_{fin}}{T}\left(-24\tau + 84\tau^2 - 60\tau^3\right).$$

By setting now $T = 1$ [s] as motion time, we can fully evaluate the coefficients of the three quintic polynomials. The result is

$$\boldsymbol{\phi}(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix} = \begin{pmatrix} 1.5708 - 4.1460\,t^3 + 9.2190\,t^4 - 4.2876\,t^5 \\ -1.0472 + 1.9867\,t^3 - 0.8587\,t^4 - 0.0808\,t^5 \\ 5.0256\,t^3 - 6.8312\,t^4 + 2.5911\,t^5 \end{pmatrix}, \quad t \in [0, 1].$$

Figure 8 shows the trajectories of these interpolating XYZ Euler angles, together with the evolutions of their first and second time derivatives. At the motion mid-time $t = 0.5$ [s], we have

$$\boldsymbol{\phi}_m = \boldsymbol{\phi}(0.5) = \begin{pmatrix} 1.4947 \\ -0.8551 \\ 0.2822 \end{pmatrix} \text{[rad]} \quad \Rightarrow \quad \boldsymbol{R}_{E,XYZ}(\boldsymbol{\phi}_m) = \begin{pmatrix} 0.6302 & -0.1827 & -0.7546 \\ -0.7015 & 0.2825 & -0.6543 \\ 0.3327 & 0.9417 & 0.04985 \end{pmatrix}.$$

Moreover, from the first derivative of the Euler angles trajectory evaluated at the motion mid-time $t = 0.5$ [s],

$$\dot{\boldsymbol{\phi}}_m = \dot{\boldsymbol{\phi}}(0.5) = \begin{pmatrix} 0.3202 \\ 2.0708 \\ 2.3265 \end{pmatrix} \text{[rad/s]},$$

we also obtain

$$\boldsymbol{\omega}_m = \boldsymbol{\omega}(0.5) = \boldsymbol{T}(\boldsymbol{\phi}_m)\dot{\boldsymbol{\phi}}_m = \begin{pmatrix} 1 & 0 & -0.7546 \\ 0 & 0.0760 & -0.6543 \\ 0 & 0.9971 & 0.04985 \end{pmatrix} \dot{\boldsymbol{\phi}}_m = \begin{pmatrix} -1.8511 \\ 2.4771 \\ -2.8774 \end{pmatrix} \text{[rad/s]}. \qquad \blacksquare$$

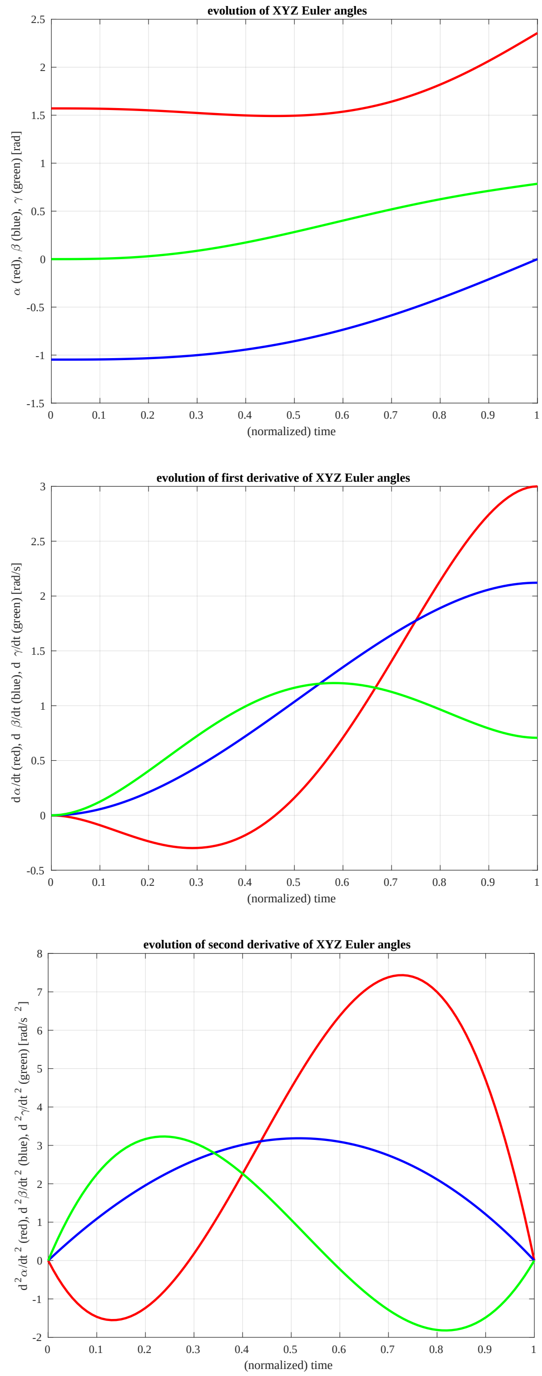


Figure 8: Trajectories of the interpolating XYZ Euler angles (top), with their velocities (center) and accelerations (bottom).

Question #8 [all students]

The planar 3R robot with unitary link lengths shown in Fig. 4 is initially in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\ 11\pi/18,\ -\pi/4)$. *Commanded by a joint velocity* $\dot{\boldsymbol{q}}(t)$ *that uses feedback from the current* $\boldsymbol{q}(t)$, *the robot should perform a self-motion so as to reach asymptotically the final value* $q_{3,fin} = -\pi/2$ *for the third joint, while keeping the position of its end-effector always at the same initial point* P_{in}. *Verify first that such task is feasible. Design then a control scheme that completes the task in a robust way, i.e., by rejecting also possible transient errors and without encountering any singular situation in which the control law is ill conditioned.* Hint: Use an approach based on joint space decomposition.

Reply #8

The planar robot has $n = 3$ joints and is kinematically redundant for the positioning of its end effector in the plane ($m = 2$). The requested task requires the use of this redundancy, exploring the null space motions so as to get to the desired joint configuration while keeping the end-effector at rest in the initial position. Before proposing a control solution, we have to verify whether or not the end effector of the 3R robot with the third link at $q_3 = -\pi/2$ can still reach the point P_{in}. We shall call the 3R robot with the third joint at this angle, a 'reduced 2R' robot having its second equivalent link of length $\sqrt{l_2 + l_3} = \sqrt{2}$. Using the initial configuration $\boldsymbol{q}_{in}$, we compute first the position of point P_{in} as

$$\boldsymbol{p}_{in} = \boldsymbol{f}(\boldsymbol{q}_{in}) = \begin{pmatrix} \cos q_1 + \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ \sin q_1 + \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_{in}} = \begin{pmatrix} 1.6468 \\ 1.3651 \end{pmatrix},$$

obtaining $\|\boldsymbol{p}_{in}\| = 2.1390$ [m] as distance from the origin. The workspace of the reduced 2R robot is a circular annulus with inner and outer radius given by $R_{min} = \left|1 - \sqrt{2}\right| = 0.4142$ and $R_{max} = 1 + \sqrt{2} = 2.4142$, respectively. Therefore, the point P_{in} will be inside the workspace of the reduced 2R robot when at destination. In principle, we should be able to keep the end effector at P_{in} during the entire self-motion task —see also Fig. 9.

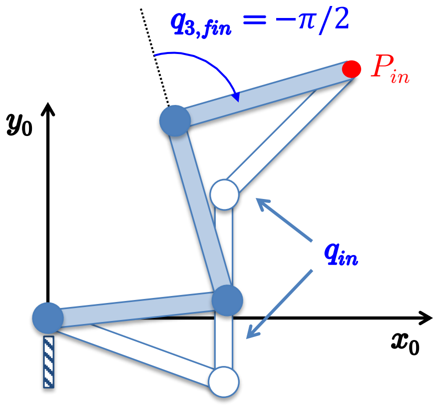


Figure 9: Configuration reached by the 3R robot at the end of the controlled self-motion task.

A possible approach would be to define the joint velocity $\dot{\boldsymbol{q}}$ as the projection of a suitable command $\boldsymbol{\xi} \in \mathbb{R}^3$ in the null space of the 2×3 task Jacobian $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})$, namely[4]

$$\dot{\boldsymbol{q}} = \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\boldsymbol{\xi}.$$

[4]See also slide #19 in the block 12_InverseDiffKinStatics.pdf (later revised as slide #22 in the lecture block 12a_InverseDifferentialKinematics.pdf).

However, in this way we would have no control on which configuration would be eventually reached by the robot. Therefore, we opt for a more tailored solution that focuses on the third joint, the only one that has a special target, leaving to the other two joints the task of keeping the end effector at the desired position P_{in}. This is also called a joint space decomposition approach. Indeed, it could be used in a planning fashion, but here a feedback solution is required. With this in mind, we set

$$\dot{q}_3 = k_3 \left(q_{3,fin} - q_3 \right), \qquad k_3 > 0. \tag{8}$$

This guarantees that q_3 will converge exponentially from any position to the desired $q_{3,fin}$. Moreover, the natural motion of q_3 will always remain in the interval $[q_{3,fin}, q_{3,in}] = [-\pi/2, -\pi/4]$. Next, decompose the differential kinematics as follows:

$$\begin{aligned} \dot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_{12}(\boldsymbol{q})\dot{\boldsymbol{q}}_{12} + \boldsymbol{J}_3(\boldsymbol{q})\,\dot{q}_3 \\ &= \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) - \sin(q_1+q_2+q_3) & -\sin(q_1+q_2) - \sin(q_1+q_2+q_3) \\ \cos q_1 + \cos(q_1+q_2) + \cos(q_1+q_2+q_3) & \cos(q_1+q_2) + \cos(q_1+q_2+q_3) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} -\sin(q_1+q_2+q_3) \\ \cos(q_1+q_2+q_3) \end{pmatrix} \dot{q}_3. \end{aligned} \tag{9}$$

The square sub-Jacobian made by the first two columns of $\boldsymbol{J}$ has $\det \boldsymbol{J}_{12}(\boldsymbol{q}) = \sin q_2 + \sin(q_2+q_3)$. As long as this determinant is different from zero, we can set $\dot{\boldsymbol{p}} = \boldsymbol{0}$ in (9) and solve for $\dot{\boldsymbol{q}}_{12}$ so as to realize our self-motion task by

$$\dot{\boldsymbol{q}}_{12} = -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\boldsymbol{J}_3(\boldsymbol{q})\,\dot{q}_3, \tag{10}$$

for any motion $\dot{q}_3$, in particular that given by (8). To introduce more robustness in the task of keeping the end-effector position at $\boldsymbol{p}_{in}$, we replace

$$\dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_{in} = \boldsymbol{0} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_{in} + \boldsymbol{K}_P \left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q}) \right) = \boldsymbol{K}_P \left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q}) \right), \tag{11}$$

with a (diagonal) 2×2 gain matrix $\boldsymbol{K}_P > 0$ weighting the position error. The final control solution is obtained by using (8) and (11) in eq. (9) and solving again for $\dot{\boldsymbol{q}}_{12}$:

$$\dot{\boldsymbol{q}}_{12} = \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) \left(\boldsymbol{K}_P \left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q}) \right) - k_3\, \boldsymbol{J}_3(\boldsymbol{q}) \left(q_{3,fin} - q_3 \right) \right). \tag{12}$$

We can also combine (8) and (12) in a single formula as

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{\boldsymbol{q}}_{12} \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) & -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\,\boldsymbol{J}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_P \left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q}) \right) \\ k_3 \left(q_{3,fin} - q_3 \right) \end{pmatrix}. \tag{13}$$

The last thing to check is the absence of singularities for $\boldsymbol{J}_{12}(\boldsymbol{q})$ during the self-motion under the feedback control law (13). It can be shown that $\det \boldsymbol{J}_{12}(\boldsymbol{q}) = \sin q_2 + \sin(q_2+q_3) = 0$ if and only if the end-effector of the 3R robot finds itself aligned with the first link of the structure. From the illustration in Fig. 9, it is rather evident that such condition is not encountered in this task. ■

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