Robotics I

September 11, 2019

Exercise 1

The kinematics of a 3R robot is defined by the following Denavit-Hartenberg table (units in [m] or [rad]):

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 5$	q_1
2	0	$a_2 = 4$	0	q_2
3	0	$a_3 = 3$	0	q_3

Determine the 3×3 linear part of the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot. When the robot is in the configuration $\boldsymbol{q}_0 = (\pi/2, \pi/4, \pi/2)$ [rad] and has a joint velocity $\dot{\boldsymbol{q}}_0 = (1, 2, -2)$ [rad/s], determine, if possible, a joint acceleration $\ddot{\boldsymbol{q}}$ that realizes a zero end-effector acceleration, i.e., $\ddot{\boldsymbol{p}} = \mathbf{0}$. *[Bonus: What if the second link parameter is changed to $a_2 = 3$?]*

Exercise 2

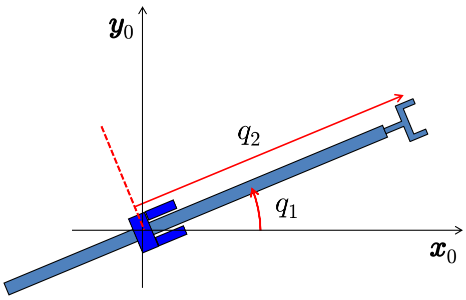

Figure 1: A RP planar robot, with the definition of the joint variables.

The RP robot shown in Fig. 1 starts from rest at time $t = 0$ in the configuration $\boldsymbol{q}(0) = (0, 1)$ [rad; m] and moves under the action of the following discontinuous joint acceleration commands for a time $T = 2$ [s]:

$$\ddot{q}_1(t) = \begin{cases} A_1 = 2 \text{ [rad/s}^2\text{]}, & t \in [0, T/4], \\ 0, & t \in [T/4, 3T/4], \\ -A_1 = -2 \text{ [rad/s}^2\text{]}, & t \in [3T/4, T]; \end{cases} \qquad \ddot{q}_2(t) = \begin{cases} -A_2 = -0.5 \text{ [m/s}^2\text{]}, & t \in [0, T/2], \\ A_2 = 0.5 \text{ [m/s}^2\text{]}, & t \in [T/2, T]. \end{cases}$$

a. Plot the time profiles of $q_i(t)$, $\dot{q}_i(t)$ and $\ddot{q}_i(t)$, for $i = 1, 2$.

b. Does the robot cross a singularity during this motion?

c. Compute the mid time configuration $\boldsymbol{q}(T/2)$ and the final configuration $\boldsymbol{q}(T)$ reached in this motion. Sketch the robot in these two configurations, as well as in the initial one.

d. Provide the analytic expressions of the end-effector velocity and acceleration norms, i.e., $\|\dot{\boldsymbol{p}}\|$ and $\|\ddot{\boldsymbol{p}}\|$.

e. Draw the end-effector velocity and acceleration vectors $\dot{\boldsymbol{p}}(T/2)$, $\ddot{\boldsymbol{p}}((T/2)^-)$ and $\ddot{\boldsymbol{p}}((T/2)^+)$ on the mid time configuration of the robot sketched at item c. Compute the numerical values of $\|\dot{\boldsymbol{p}}(T/2)\|$, $\|\ddot{\boldsymbol{p}}((T/2)^-)\|$ and $\|\ddot{\boldsymbol{p}}((T/2)^+)\|$.

Exercise 3

A link of length $L = 1.5$ m is rotated by a DC motor mounted at the base through a gear with reduction ratio $N_r = 4$. The motor has a quadrature incremental encoder with $N_p = 250$ pulses/turn and a digital counter of $n = 10$ bits. The link carries a laser scanner at its other end. The laser measures distances from the link tip to obstacles (in the same plane of link motion) up to a maximum distance of $d = 5$ m. The laser has a depth resolution $\Delta_\rho = 12$ mm and an angular resolution $\delta_s = 0.2°$ in the range $\alpha = \pm 90°$ (the zero is when the scanning ray is aligned with the link). Sketch the setup and analyze the resolution of this system for measuring the position of objects in the environment. In the worst-case condition, which is the largest possible Cartesian displacement Δ of an object that provides no change in the output readings?

[180 minutes; open books]

Solution

September 11, 2019

Exercise 1

The 3×3 Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that relates the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the linear velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^3$ of the end-effector of this robot can be equivalently computed either by differentiation of the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ or geometrically. Both methods require the information provided by the DH table. We follow here the first method. Note that the given DH table refers to a standard 3R elbow-type manipulator without offsets: this is helpful to know, but not really needed in the computations.

The end-effector position is obtained from the homogenous transformation matrices:

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)$$

$$\Longrightarrow \qquad \boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) \\ \sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) \\ d_1 + a_2\sin q_2 + a_3\sin(q_2+q_3) \end{pmatrix}. \tag{1}$$

Therefore, using the usual compact notation for trigonometric functions, we have

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1\,(a_2c_2 + a_3c_{23}) & -c_1\,(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1\,(a_2c_2 + a_3c_{23}) & -s_1\,(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}. \tag{2}$$

The Jacobian is singular when[1]

$$\det \boldsymbol{J}(\boldsymbol{q}) = -a_2a_3\,s_3\,(a_2c_2 + a_3c_{23}) = 0. \tag{3}$$

The end-effector acceleration $\ddot{\boldsymbol{p}}$ is computed as

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{4}$$

Thus, in order to realize a zero end-effector acceleration, we need to set $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ in (4) and solve for $\ddot{\boldsymbol{q}}$, or

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{5}$$

Indeed, this solution is valid as long as the robot is out of singularities. To evaluate (5), we need first to derive the time derivative of the robot Jacobian. Let $\boldsymbol{J}_i(\boldsymbol{q})$ be the ith column of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$, for $i = 1,2,3$. We compute [2]

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \frac{d\boldsymbol{J}(\boldsymbol{q})}{dt} \quad \left(= \sum_{i=1}^{3}\left(\frac{\partial \boldsymbol{J}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}}\right)\boldsymbol{e}_i^T = \sum_{j=1}^{3}\frac{\partial \boldsymbol{J}(\boldsymbol{q})}{\partial q_j}\,\dot{q}_j \right)$$

$$= \left(\begin{matrix} -c_1\dot{q}_1\,(a_2c_2 + a_3c_{23}) + s_1\,(a_2s_2\dot{q}_2 + a_3s_{23}(\dot{q}_2+\dot{q}_3)) \\ -s_1\dot{q}_1\,(a_2c_2 + a_3c_{23}) - c_1\,(a_2s_2\dot{q}_2 + a_3s_{23}(\dot{q}_2+\dot{q}_3)) \\ 0 \end{matrix}\right. \tag{6}$$

$$\left.\begin{matrix} s_1\dot{q}_1\,(a_2s_2 + a_3s_{23}) - c_1\,(a_2c_2\dot{q}_2 + a_3c_{23}(\dot{q}_2+\dot{q}_3)) & a_3s_1\dot{q}_1s_{23} - a_3c_1c_{23}(\dot{q}_2+\dot{q}_3) \\ c_1\dot{q}_1\,(a_2s_2 + a_3s_{23}) - s_1\,(a_2c_2\dot{q}_2 + a_3c_{23}(\dot{q}_2+\dot{q}_3)) & -a_3c_1\dot{q}_1s_{23} - a_3s_1c_{23}(\dot{q}_2+\dot{q}_3) \\ -\,(a_2s_2\dot{q}_2 - a_3s_{23}(\dot{q}_2+\dot{q}_3)) & -a_3s_{23}(\dot{q}_2+\dot{q}_3) \end{matrix}\right).$$

When the robot is in the configuration $\boldsymbol{q}_0 = (\pi/2, \pi/4, \pi/2)$ [rad] and has a joint velocity $\dot{\boldsymbol{q}}_0 = (1, 2, -2)$ [rad/s], evaluation of (2) and (6) gives

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -0.7071 & 0 & 0 \\ 0 & -4.9497 & -2.1213 \\ 0 & 0.7071 & -2.1213 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_0 = \dot{\boldsymbol{J}}(\boldsymbol{q}_0)|_{\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}_0} = \begin{pmatrix} 5.6569 & 4.9497 & 2.1213 \\ -0.7071 & -5.6569 & 0 \\ 0 & -5.6569 & 0 \end{pmatrix}.$$

Since $\det \boldsymbol{J}_0 = -8.4853 \neq 0$,we can use eq. (5) for computing the joint acceleration $\ddot{\boldsymbol{q}}$ that realizes a zero end-effector acceleration:

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}_0^{-1}\left(\dot{\boldsymbol{J}}_0\dot{\boldsymbol{q}}_0\right) = -\begin{pmatrix} -1.4142 & 0 & 0 \\ 0 & -0.1768 & 0.1768 \\ 0 & -0.0589 & -0.4125 \end{pmatrix}\begin{pmatrix} 11.3137 \\ -12.0208 \\ -11.3137 \end{pmatrix} = \begin{pmatrix} 16 \\ -0.1250 \\ -5.3750 \end{pmatrix} \text{ [rad/s}^2\text{]}. \tag{7}$$

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[1] This computation is made easier when expressing the Jacobian in frame 1, i.e., using ${}^1\boldsymbol{J}(\boldsymbol{q}) = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}(\boldsymbol{q})$.

[2] The first expression in the large parenthesis uses the dyadic expansion of a matrix: $\boldsymbol{J}(\boldsymbol{q}) = \sum_{i=1}^{3}\boldsymbol{J}_i(\boldsymbol{q})\,\boldsymbol{e}_i^T$, where $\boldsymbol{e}_i^T$ is the i-th row of the identity matrix.

Bonus part. If we change the second link parameter from $a_2 = 4$ to $a_2 = 3$, a singular configuration will be encountered. This can be recognized already from the direct klnematics (1); in fact, we have in this case

$$\boldsymbol{p}' = \boldsymbol{p}_{|a_2=3} = \begin{pmatrix} 0 & 0 & 9.2426 \end{pmatrix}^T,$$

namely the end-effector is placed on the axis of joint 1: any rotation $\dot{q}_1$ will not move the end-effector —a situation of singularity. Re-evaluating then the Jacobian (2) yields

$$\boldsymbol{J}_0' = \boldsymbol{J}(\boldsymbol{q}_0)_{|a_2=3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4.2426 & -2.1213 \\ 0 & 0 & -2.1213 \end{pmatrix} \quad \Rightarrow \quad \det \boldsymbol{J}_0' = 0, \quad \mathcal{R}\left\{\boldsymbol{J}_0'\right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In these cases, compensating with a suitable joint acceleration a drift of the end-effector acceleration due to the current joint velocity is still possible, provided that the Cartesian drift is in the span of the Jacobian, Moreover, the inversion in (5) should be replaced by a pseudoinversion of the Jacobian matrix, namely

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{8}$$

To check if this is the case, we re-evaluate with (6) the time derivative of the Jacobian, and then the drift term:

$$\dot{\boldsymbol{J}}_0' = \dot{\boldsymbol{J}}_{0|a_2=3} = \begin{pmatrix} 4.2426 & 4.2426 & 2.1213 \\ 0 & -4.2426 & 0 \\ 0 & -4.2426 & 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{J}}_0'\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{J}}_0' \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 8.4853 \\ -8.4853 \\ -8.4853 \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{J}_0'\right\}.$$

Therefore, even with the use of a pseudoinverse, we will not be able to impose the desired (zero) end-effector acceleration: an error (of minimum possible norm) will result. Computing the pseudoinverse and evaluating (8) gives

$$\boldsymbol{J}_0^{\#'} = \boldsymbol{J}^{\#}_{|a_2=3}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.2357 & 0.2357 \\ 0 & 0 & -0.4714 \end{pmatrix} \quad \Longrightarrow \quad \ddot{\boldsymbol{q}} = -\boldsymbol{J}_0^{\#'}\dot{\boldsymbol{J}}_0'\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}.$$

Checking the end-effector acceleration obtained with this joint solution,

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}_0'\dot{\boldsymbol{J}}_0'\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 8.4853 & 0 & 0 \end{pmatrix}^T \neq \boldsymbol{0}^T,$$

confirms that the component that is outside the range of the Jacobian (i.e., in the $\boldsymbol{x}$ direction) is not canceled, while the task is achieved for the remaining part.

Exercise 2

We proceed by integrating twice the joint acceleration commands, taking into account the initial state of the robot at time $t = 0$ ($\boldsymbol{q}(0) = (0, 1)$ [rad; m], $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$) and the total motion time $T = 2$ [s]. For the joint velocities, we have obtain

$$\dot{q}_1(t) = \begin{cases} A_1 t = 2\,t \text{ [rad/s]}, & t \in [0, 0.5], \\ V_1 = 1 \text{ [rad/s]}, & t \in [0.5, 1.5], \\ V_1 - A_1(t - 1.5) = 1 - 2(t - 1.5) = 4 - 2\,t \text{ [rad/s]}, & t \in [1.5, 2] \end{cases}$$

and

$$\dot{q}_2(t) = \begin{cases} -A_2 t = -0.5\,t \text{ [m/s]}, & t \in [0, 1], \\ -V_2 + A_2(t - 1) = -0.5 + 0.5(t - 1) = 0.5\,t - 1 \text{ [m/s]}, & t \in [1, 2]. \end{cases}$$

For the joint positions, we obtain

$$q_1(t) = \begin{cases} q_1(0) + \frac{1}{2}A_1 t^2 = t^2 \text{ [rad]}, & t \in [0, 0.5], \\ q_1(0.5) + V_1(t - 0.5) = 0.25 + (t - 0.5) = t - 0.25 \text{ [rad]}, & t \in [0.5, 1.5], \\ q_1(1.5) + V_1(t - 1.5) - \frac{1}{2}A_1(t - 1.5)^2 = 1.25 + (t - 1.5) - (t - 1.5)^2 \text{ [rad]}, & t \in [1.5, 2] \end{cases}$$

and

$$q_2(t) = \begin{cases} q_2(0) - \frac{1}{2}A_2 t^2 = 1 - 0.25\,t^2 \text{[m]}, & t \in [0, 1], \\ q_2(1) - V_2(t - 1) + \frac{1}{2}A_2(t - 1)^2 = 0.75 - 0.5(t - 1) + 0.25(t - 1)^2 \text{ [m]}, & t \in [1, 2]. \end{cases}$$

The qualitative time profiles of $q_i(t)$, $\dot{q}_i(t)$ and $\ddot{q}_i(t)$, for $i = 1, 2$, are shown in Fig. 2, with joint variations $\Delta q_1 = 1.5$ [rad] and $\Delta q_2 = -0.5$ [m]. The robot never crosses its kinematic singularities, i.e., any

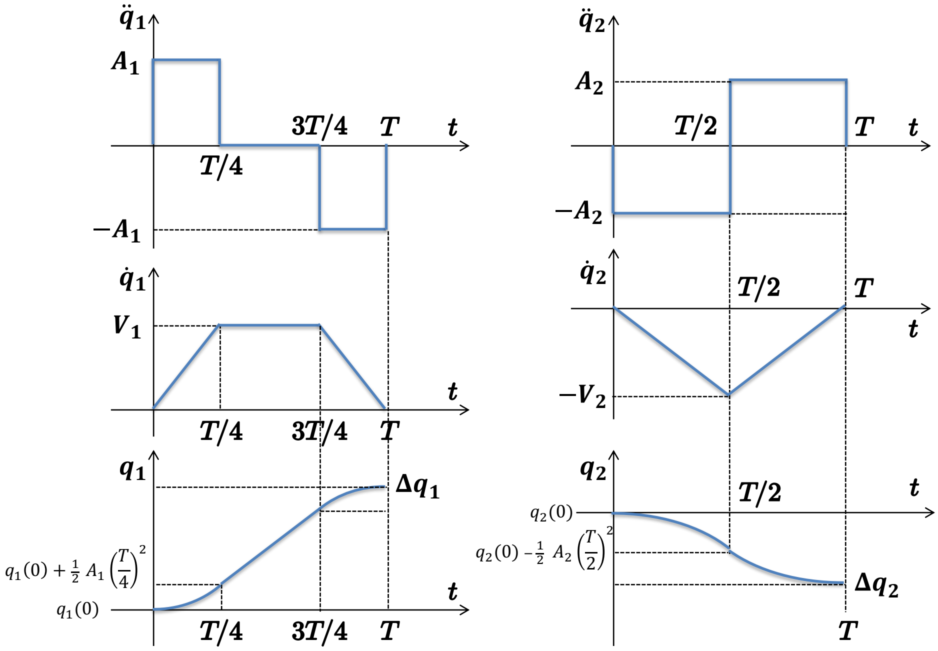

Figure 2: Qualitative plots of $\ddot{\boldsymbol{q}}(t)$, $\dot{\boldsymbol{q}}(t)$ and $\boldsymbol{q}(t)$ for the RP robot, as specified in the text.

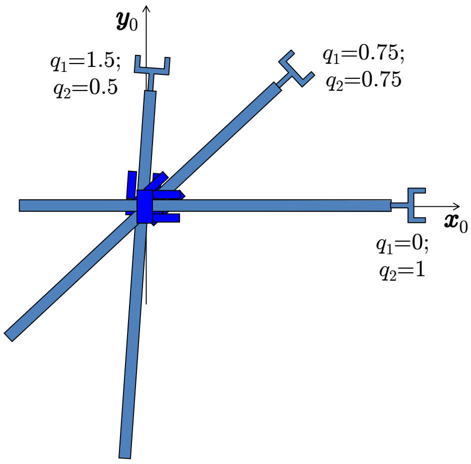

Figure 3: The RP robot in the initial, mid and final time configurations (values in [rad;m])

configuration $\boldsymbol{q}_s = (*, 0)$ with the second link fully retracted. The initial, mid time and final configurations are

$$\boldsymbol{q}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{q}(1) = \begin{pmatrix} 0.75 \\ 0.75 \end{pmatrix}, \qquad \boldsymbol{q}(2) = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix} \qquad \text{[rad; m]}.$$

Fig. 3 sketches the RP robot in these configurations.

The end-effector velocity and acceleration of the RP robot are computed by differentiation of its direct kinematics

$$\boldsymbol{p} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}.$$

We have

$$\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{q}_2 \cos q_1 - \dot{q}_1 q_2 \sin q_1 \\ \dot{q}_2 \sin q_1 + \dot{q}_1 q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix}, \tag{9}$$

where a planar 2×2 rotation matrix $\boldsymbol{R}$ by an angle q_1 has been put in evidence. The last vector in (9) is the end-effector velocity expressed in the frame rotated by the angle q_1, i.e., ${}^1\dot{\boldsymbol{p}}$. The norm of vector $\dot{\boldsymbol{p}}$ is computed as follows:

$$\|\dot{\boldsymbol{p}}\|^2 = \dot{\boldsymbol{p}}^T \dot{\boldsymbol{p}} = \begin{pmatrix} \dot{q}_2 & q_2 \dot{q}_1 \end{pmatrix} \boldsymbol{R}^T(q_1) \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix} = \left\| \begin{pmatrix} \dot{q}_2 \\ q_2 \dot{q}_1 \end{pmatrix} \right\|^2 = q_2^2\, \dot{q}_1^2 + \dot{q}_2^2 \quad \Rightarrow \quad \|\dot{\boldsymbol{p}}\| = \sqrt{q_2^2\, \dot{q}_1^2 + \dot{q}_2^2}. \tag{10}$$

For computing the acceleration $\ddot{\boldsymbol{p}}$, note first that

$$\dot{\boldsymbol{R}}(q_1) = \begin{pmatrix} -\sin q_1 & -\cos q_1 \\ \cos q_1 & -\sin q_1 \end{pmatrix} \dot{q}_1 = \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{q}_1 = \boldsymbol{R}_1(q_1) \begin{pmatrix} 0 & -\dot{q}_1 \\ \dot{q}_1 & 0 \end{pmatrix}.$$

Differentiation of eq. (9) provides

$$\begin{aligned} \ddot{\boldsymbol{p}} &= \boldsymbol{R}(q_1) \begin{pmatrix} \ddot{q}_2 \\ \dot{q}_1\dot{q}_2 + q_2\ddot{q}_1 \end{pmatrix} + \dot{\boldsymbol{R}}(q_1) \begin{pmatrix} \dot{q}_2 \\ q_2\dot{q}_1 \end{pmatrix} \\ &= \boldsymbol{R}(q_1) \left(\begin{pmatrix} \ddot{q}_2 \\ \dot{q}_1\dot{q}_2 + q_2\ddot{q}_1 \end{pmatrix} + \begin{pmatrix} 0 & -\dot{q}_1 \\ \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ q_2\dot{q}_1 \end{pmatrix} \right) = \boldsymbol{R}(q_1) \begin{pmatrix} \ddot{q}_2 - q_2\,\dot{q}_1^2 \\ q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2 \end{pmatrix} = \boldsymbol{R}(q_1)\,{}^1\ddot{\boldsymbol{p}}. \end{aligned} \tag{11}$$

Moreover, the norm of this vector is

$$\|\ddot{\boldsymbol{p}}\| = \|{}^1\ddot{\boldsymbol{p}}\| = \sqrt{(\ddot{q}_2 - q_2\,\dot{q}_1^2)^2 + (q_2\ddot{q}_1 + 2\dot{q}_1\dot{q}_2)^2}. \tag{12}$$

The evaluation of (9) and (10) at $t = T/2 = 1$ s yields

$$\dot{\boldsymbol{p}}(1) = \boldsymbol{R}(q_1(1)) \begin{pmatrix} \dot{q}_2(1) \\ q_2(1)\dot{q}_1(1) \end{pmatrix} = \boldsymbol{R}(q_1(1)) \begin{pmatrix} -0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} -0.8771 \\ 0.2079 \end{pmatrix}$$

and

$$\|\dot{\boldsymbol{p}}(1)\| = \sqrt{\dot{q}_2^2(1) + q_2^2(1)\dot{q}_1^2(1)} = 0.9014.$$

The evaluation of (11) at $t = T/2 = 1$ s should take into account the discontinuity of the acceleration of the second joint at the mid time of motion. Therefore, we should consider the two values just before ($^-$) and just after ($^+$) the mid time instant:

$$\ddot{\boldsymbol{p}}(1^-) = \boldsymbol{R}(q_1(1)) \begin{pmatrix} \ddot{q}_2(1^-) - q_2(1)\dot{q}_1^2(1) \\ q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1) \end{pmatrix} = \boldsymbol{R}(q_1(1)) \begin{pmatrix} -1.25 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.2330 \\ -1.5837 \end{pmatrix}$$

and

$$\ddot{\boldsymbol{p}}(1^+) = \boldsymbol{R}(q_1(1)) \begin{pmatrix} \ddot{q}_2(1^+) - q_2(1)\dot{q}_1^2(1) \\ q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1) \end{pmatrix} = \boldsymbol{R}(q_1(1)) \begin{pmatrix} -0.25 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.4987 \\ -0.9021 \end{pmatrix}.$$

Similarly, for the norm (12) we have

$$\left\|\ddot{\boldsymbol{p}}(1^-)\right\| = \sqrt{\left(\ddot{q}_2(1^-) - q_2(1)\dot{q}_1^2(1)\right)^2 + \left(q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1)\right)^2} = 1.6008$$

and

$$\left\|\ddot{\boldsymbol{p}}(1^+)\right\| = \sqrt{\left(\ddot{q}_2(1^+) - q_2(1)\dot{q}_1^2(1)\right)^2 + \left(q_2(1)\ddot{q}_1(1) + 2\dot{q}_1(1)\dot{q}_2(1)\right)^2} = 1.0308.$$

Finally, Fig. 4 shows the vectors $\dot{\boldsymbol{p}}(1)$, $\ddot{\boldsymbol{p}}(1^-)$ and $\ddot{\boldsymbol{p}}(1^+)$ on the RP robot in the mid time configuration. For this picture, it is more convenient to use the vectors expressed in the rotated frame, i.e., to draw for instance ${}^1\dot{\boldsymbol{p}}(1)$ on the rotated second link (rather than attempting directly to draw $\dot{\boldsymbol{p}}(1)$). Note that the relative scales of these vectors are somewhat arbitrary.

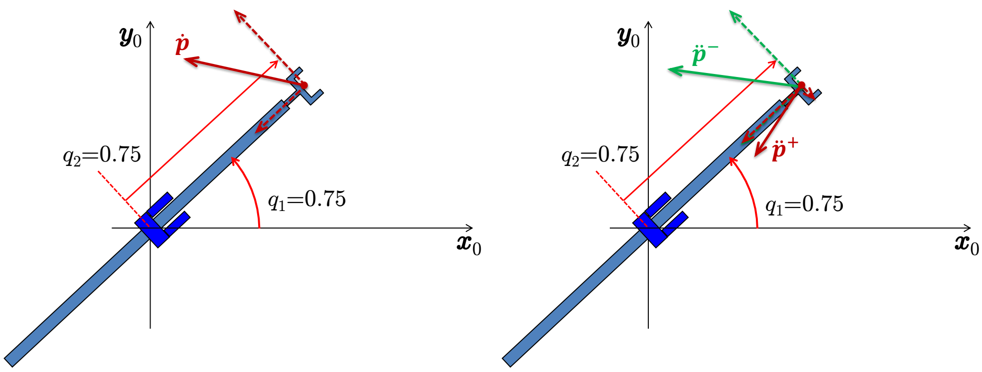

Figure 4: The mid time robot configuration with: $\dot{\boldsymbol{p}}(1)$ [left]; $\ddot{\boldsymbol{p}}(1^-)$ and $\ddot{\boldsymbol{p}}(1^+)$ [right].

Exercise 3

The measurement system is composed by two parts, the laser scanner and the rotating link carrying it, see Fig. 5. The rotation added by the actuated link is useful because it enlarges the angular range of the sensor. On the other hand, uncertainty is added to the laser measurement, due to the angular resolution of the encoder which translates into an uncertain localization of the base of the sensor. To analyze the overall behavior, we consider first the two systems separately.

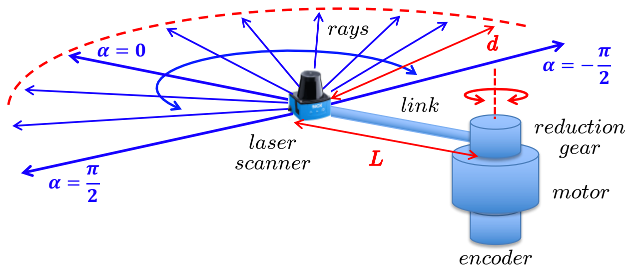

Figure 5: The measurement system made by a rotating link that carries a laser scanner.

For the motor-link assembly, the angular resolution δ_m of the encoder mounted on the motor shaft (after electronic multiplication by 4) and the lateral uncertainty Δ_L at the link end are computed as

$$\delta_m = \frac{360°}{4 \times N_p}\left(\frac{\pi}{180°}\right) = \frac{2\pi}{4 \times 250} = 0.00628\,[\text{rad}], \qquad \Delta_L = \frac{\delta_m}{N_r}\,L = \frac{0.00628}{4}\,1.5 = 0.0024\,[\text{m}] = 2.4\,[\text{mm}].$$

Note that the $n = 10$ bits of the digital counter in the encoder are sufficient to represent the full rotation, since $2^n = 2^{10} = 1024 > 1000$ (the number of electrical pulses per turn).

For the laser scanner, the angular resolution δ_s corresponds to an uncertainty in the lateral positioning (w.r.t. the pointing ray) of a sensed object. The worst-case situation is when the object is placed at the maximum sensing distance d from the laser source. The (Cartesian) width resolution Δ_ϕ in this case is

$$\Delta_\phi = \delta_s\, d = 0.2°\left(\frac{\pi}{180°}\right) 5 = 0.00349 \times 5 = 0.0175\,[\text{m}] = 17.5\,[\text{mm}].$$

Instead, the depth resolution Δ_ρ is rather independent from the distance. Thus, the region of uncertainty in the scanning process, when the base of the laser sensor is in a fixed, known position, can be approximated by a rectangle of size $\Delta_\rho \times \Delta_\phi = 12 \times 17.5$ [mm×mm]. A displacement of an object within this small area will not generate any change in the sensor reading. In particular, if it crosses this area in diagonal, we get

$$\Delta \simeq \sqrt{\Delta_\rho^2 + \Delta_\phi^2} = \sqrt{12^2 + 17.5^2} = 21.2\ [\text{mm}]. \qquad (13)$$

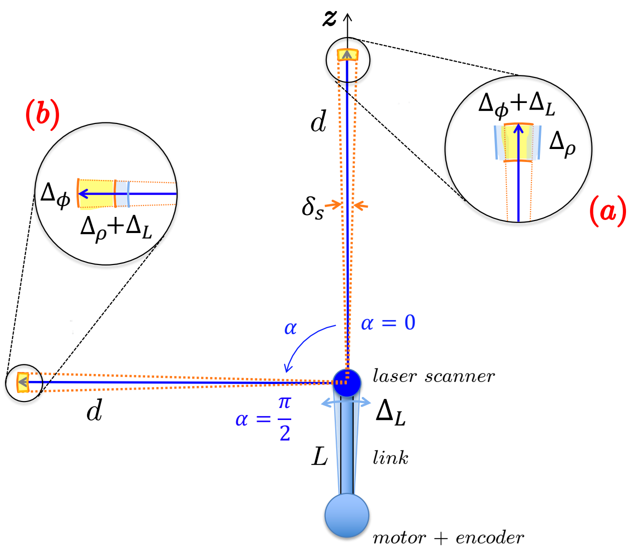

Figure 6: Two (expanded) Cartesian regions of measurement uncertainty: $(a)\ \alpha = 0$; $(b)\ \alpha = \pi/2$.

When combining the scanning process with the variable orientation of the link, the measurement uncertainty on the object position will increase, due to the uncertainty Δ_L on the lateral positioning at the link end where the sensor is placed. However, the outcome of this combination will depend on the relative angle between the direction of the laser ray and the direction of the link that carries the sensor. With reference to Fig. 6, we consider two limit cases: when the ray is aligned with the link, at $\alpha = 0$ [case (a)], and when the ray is at the boundary of its angular sensing range, e.g., at $\alpha = \pi/2$ [case (b)]. In the first case, Δ_L adds to Δ_ϕ while the depth resolution Δ_ρ remains unaffected. In the second case, Δ_L adds to Δ_ρ while the width resolution Δ_ϕ is unaffected. All other feasible values of α lead to intermediate situations. The largest Cartesian displacement of an object that would provide no change in the output reading is again on the diagonal of the rectangle of uncertainly. We have:

$$\Delta_a \simeq \sqrt{\Delta_\rho^2 + (\Delta_\phi + \Delta_L)^2}, \qquad \Delta_b \simeq \sqrt{(\Delta_\rho + \Delta_L)^2 + \Delta_\phi^2}.$$

Since for the given data $\Delta_\phi > \Delta_\rho$, it follows that $\Delta_a > \Delta_b$. Thus, the worst increase in uncertainty will happen in case (a):

$$\Delta = \Delta_a \simeq \sqrt{12^2 + (17.5 + 2.4)^2} = 23.2 \text{ [mm]}.$$

The resolution of the measurement system (or, equivalently, the largest positional uncertainty of the sensed object) in the mobile case has worsened by about 2 mm with respect to the fixed case.

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