Robotics I

February 5, 2018

Exercise 1

The Italian robot manufacturer Comau has recently put on the market two educational manipulators of small size and weight called *e.Do*. The version with four actuated revolute joints is shown in Fig. 1.

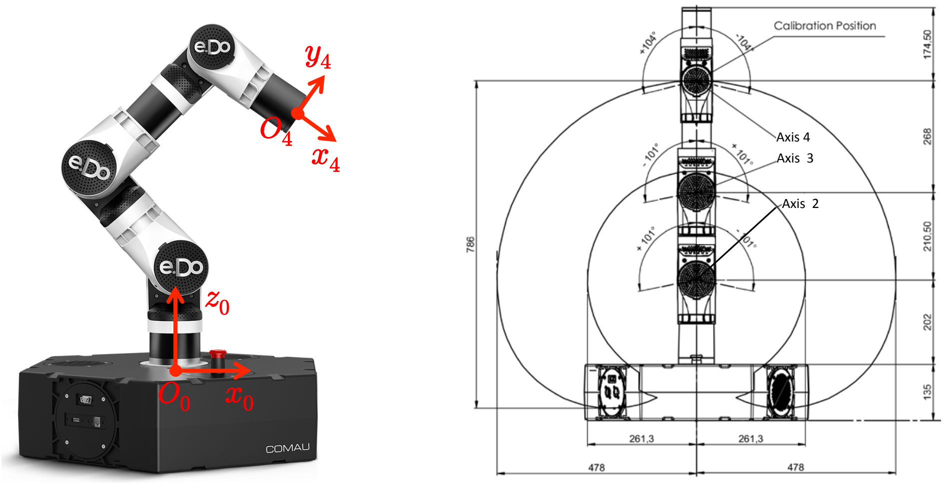


Figure 1: The 4R *e.Do* manipulator by Comau with base and end-effector frames [left] and its relevant dimensions [right].

Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all constant parameters are *non-negative*. Specify also their numerical values. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 and 4 are already assigned and should not be modified. Finally, write the DH homogeneous transformation matrices. [Please, make clean drawings and return the completed sheet with your name written on it.]

Exercise 2

A number of statements are reported on the extra sheet #2, regarding sensor devices for fixed-base manipulators and related measurement issues. Check if each statement is **True** or **False**, providing also a *very short* motivation/explanation for your answer. [Return the completed sheet with your name on it.]

Exercise 3

Determine the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for the robot in Fig. 1 (do not enter numerical values). Partition this matrix in blocks as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{1}$$

- Find all configurations $\boldsymbol{q}_L^*$, if any, where $\boldsymbol{J}_L(\boldsymbol{q})$ loses rank.
- Determine the range space of all feasible angular velocities $\boldsymbol{\omega} \in \mathbb{R}^3$.
- Find all singular configurations $\boldsymbol{q}^*$ of $\boldsymbol{J}(\boldsymbol{q})$, if any.

Choose next a configuration $\boldsymbol{q}_0$ where $\boldsymbol{J}_L$ is full rank, and substitute all the available numerical data in this matrix. Sketch this configuration and compute then a non-zero joint velocity $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^4$ such that the resulting linear velocity $\boldsymbol{v}$ of the robot end-effector at $\boldsymbol{q}_0$ is identically zero.

[turn for next exercise]

Exercise 4

Plan a cubic spline trajectory $q(t)$ that interpolates the following data at given time instants

$$t_1 = 1,\ q(t_1) = 45^\circ, \quad t_2 = 2,\ q(t_2) = 90^\circ, \quad t_3 = 2.5,\ q(t_3) = -45^\circ, \quad t_4 = 4,\ q(t_4) = 45^\circ, \tag{2}$$

starting with $\dot{q}(t_1) = 0$ and arriving with $\dot{q}(t_4) = 0$.

- Give an expression and the associated numerical values of the coefficients of each cubic polynomial.
- Find the maximum (absolute) values attained by the velocity $\dot{q}(t)$ and the acceleration $\ddot{q}(t)$ over the whole motion interval $[t_1, t_4]$, as well as the time instants at which these occur.
- Check if the following bounds are satisfied throughout the motion,

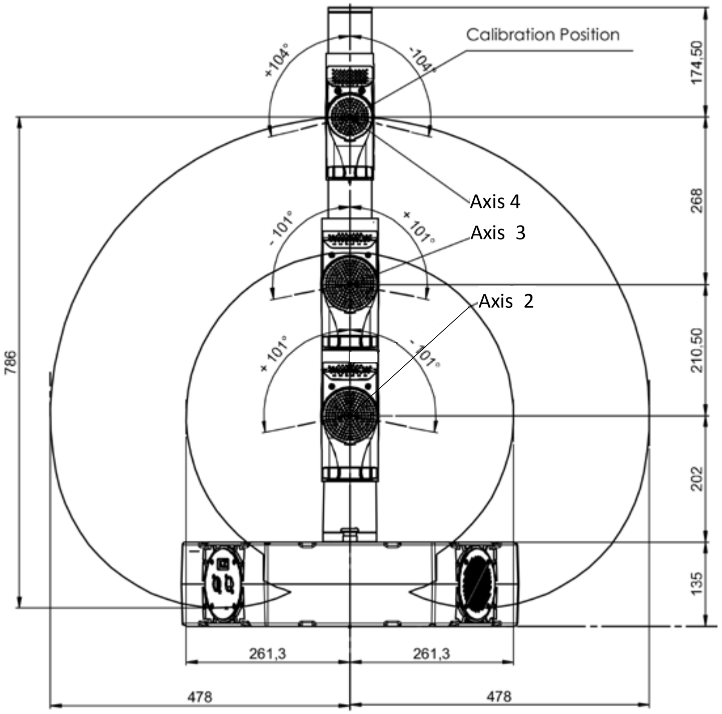
$$|\dot{q}(t)| \le V_{\max} = 250^\circ/\mathrm{s}, \qquad |\ddot{q}(t)| \le A_{\max} = 1000^\circ/\mathrm{s}^2, \tag{3}$$

  and, if needed, determine the minimum uniform scaling factor for the trajectory so that feasibility is recovered.
- Provide the total motion time of the feasible trajectory and sketch as accurately as possible the profiles of the resulting velocity and acceleration.

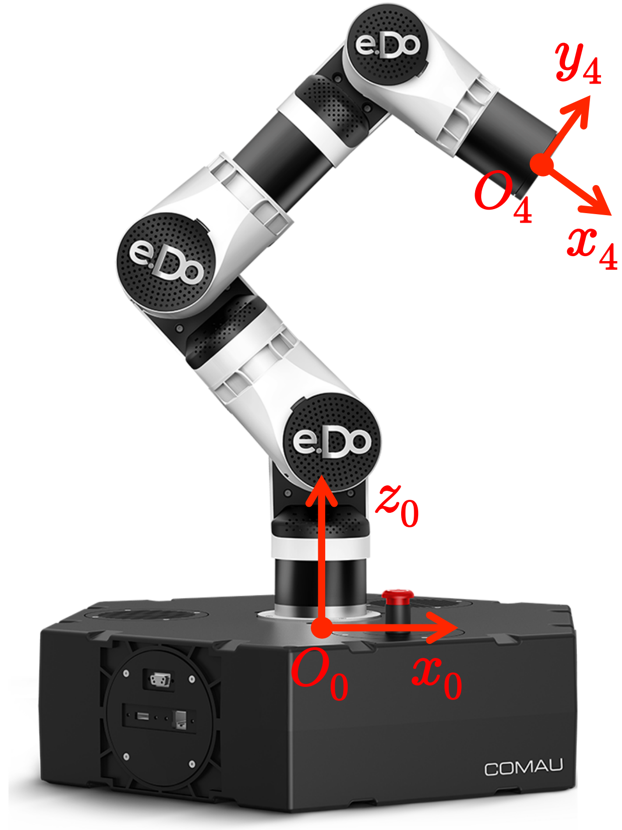
[210 minutes, open books but no computer or smartphone]

4R e.Do robot by Comau – DH frames assignment and table

Name: ____



i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				



all constant DH parameters should be ≥ 0

Robotics I – Sheet for Exercise 1, February 5, 2018

Robotics I - Sheet for Exercise 2

February 5, 2018

Name: ____

Consider motion sensing devices available for fixed-base robot manipulators and related issues in the measurement process. Check if each of the following statements is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. Encoders of the absolute type cannot be used for estimating joint velocity.

   True ☐ **False** ☐

   ____

2. Encoders should never be mounted beyond the reduction element in motor-link transmission systems.

   True ☐ **False** ☐

   ____

3. Dynamic repeatability of a robot improves when the robot is moving at slow speed.

   True ☐ **False** ☐

   ____

4. Absolute encoders need no calibration before being operative.

   True ☐ **False** ☐

   ____

5. For estimating velocity, integration of accelerometer data outperforms differentiation of encoder data.

   True ☐ **False** ☐

   ____

6. Vision systems are preferred when a direct measure of the robot end-effector position is needed.

   True ☐ **False** ☐

   ____

7. An incremental encoder with 6000 ppt has a better resolution than an absolute encoder with 15 tracks.

   True ☐ **False** ☐

   ____

8. With a sensor mounted on the motor, the larger is the reduction ratio N of the transmission, the better the resolution of the link position estimate is.

   True ☐ **False** ☐

   ____

9. In general, repeatability of a sensor can be improved by calibration, whereas accuracy cannot.

   True ☐ **False** ☐

   ____

10. Sensor devices should be used only in their domain of linearity (within $2 \div 3\%$ of deviation).

    True ☐ **False** ☐

    ____

Solution

February 5, 2018

Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1, respectively, together with the numerical values of the constant parameters (all non-negative, as requested).

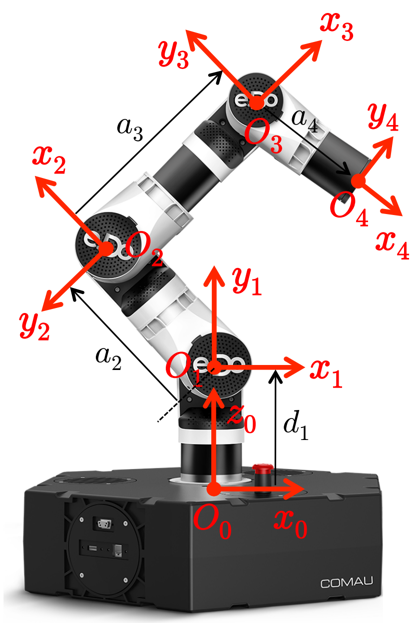


Figure 2: A DH frame assignment for the Comau *e.Do* robot, with associated length parameters.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 = 202$	q_1
2	0	$a_2 = 210.5$	0	q_2
3	0	$a_3 = 268$	0	q_3
4	0	$a_4 = 174.5$	0	q_4

Table 1: Parameters associated to the DH frames in Fig. 2. Lengths are in [mm].

The robot is equivalent to a planar 3R structure in a vertical plane, mounted on a rotating first axis. Based on Tab. 1, the four DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{i-1}\boldsymbol{A}_i(q_i) = \begin{pmatrix} \cos q_i & -\sin q_i & 0 & a_i \cos q_i \\ \sin q_i & \cos q_i & 0 & a_i \sin q_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad i = 2,3,4. \tag{4}
$$

Exercise 2

1. Encoders of the absolute type cannot be used for estimating joint velocity.

   False. Both types can be used, although incremental encoders output directly Δq (and $\Delta q/\Delta t \simeq \dot{q}$).

2. Encoders should never be mounted beyond the reduction element in motor-link transmission systems.

   False. An encoder on the link side provides a better measure of its position (e.g., for flexible shafts).

3. Dynamic repeatability of a robot improves when the robot is moving at slow speed.

   True. Position errors in the execution of reference trajectories usually increase with larger speeds.

4. Absolute encoders need no calibration before being operative.

   False. A 'homing' at start is not needed, but calibration will recover an erroneous/rotated mounting.

5. For estimating velocity, integration of accelerometer data outperforms differentiation of encoder data.

   False. All the rest being equal, integration of signals is usually subject to drifts over time.

6. Vision systems are preferred when a direct measure of the robot end-effector position is needed.

   True. An external camera senses directly the position, by-passing inaccuracies of robot kinematics.

7. An incremental encoder with 6000 ppt has a better resolution than an absolute encoder with 15 tracks.

   False. $\Delta\theta_{\text{inc}} = 360°/6000 = 0.06°$ ($0.015°$ with quadrature). $\Delta\theta_{\text{abs}} = 360°/2^{15} = 0.011°$ is better.

8. With a sensor mounted on the motor, the larger is the reduction ratio N of the transmission, the better the resolution of the link position estimate is.

   True. Given a resolution $\Delta\theta_m$ on the motor side, the resolution on the link side is $\Delta\theta_\ell = \Delta\theta_m/N$.

9. In general, repeatability of a sensor can be improved by calibration, whereas accuracy cannot.

   False. Calibration affects accuracy reducing systematic errors. Repeatability relies on sensor quality.

10. Sensor devices should be used only in their domain of linearity (within $2 \div 3\%$ of deviation).

    True. Superposition of physical effects should hold on measurements, possibly after their equalization.

Exercise 3

Using (4), we compute first the direct kinematics of the end-effector position as

$$\boldsymbol{p}_{\text{hom}} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left[{}^1\boldsymbol{A}_2(q_2)\left[{}^2\boldsymbol{A}_3(q_3)\left[{}^3\boldsymbol{A}_4(q_4)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]\right]\right] = \begin{pmatrix} c_1\,(a_2c_2 + a_3c_{23} + a_4c_{234}) \\ s_1\,(a_2c_2 + a_3c_{23} + a_4c_{234}) \\ a_2s_2 + a_3s_{23} + a_4s_{234} \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \tag{5}$$

with the usual shorthand notation for trigonometric terms (e.g., $s_{234} = \sin(q_2 + q_3 + q_4)$). The linear part of the geometric Jacobian is easily obtained by time differentiation of $\boldsymbol{p}$ in (5) as

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{6}$$

yielding

$$\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1\,(a_2c_2 + a_3c_{23} + a_4c_{234}) & -c_1\,(a_2s_2 + a_3s_{23} + a_4s_{234}) & -c_1\,(a_3s_{23} + a_4s_{234}) & -a_4c_1s_{234} \\ c_1\,(a_2c_2 + a_3c_{23} + a_4c_{234}) & -s_1\,(a_2s_2 + a_3s_{23} + a_4s_{234}) & -s_1\,(a_3s_{23} + a_4s_{234}) & -a_4s_1s_{234} \\ 0 & a_2c_2 + a_3c_{23} + a_4c_{234} & a_3c_{23} + a_4c_{234} & a_4c_{234} \end{pmatrix}. \tag{7}$$

For analysis purposes, the structure of matrix $\boldsymbol{J}_L$ can be manipulated by (invertible) transformations on the rows and on the columns. It is easy to see that simplifications are obtained by writing the Jacobian in the rotated DH frame 1 (i.e., expressing linear velocity as ${}^1\boldsymbol{v} = {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{v}$) and by factoring out recursive expressions in (7). Using the rotational part of matrix ${}^0\boldsymbol{A}_1(q_1)$ in (4), the first step leads in fact to

$$
\begin{aligned}
{}^1\boldsymbol{J}_L(\boldsymbol{q}) &= {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) = \\
&= \begin{pmatrix} 0 & -\left(a_2 s_2 + a_3 s_{23} + a_4 s_{234}\right) & -\left(a_3 s_{23} + a_4 s_{234}\right) & -a_4 s_{234} \\ 0 & a_2 c_2 + a_3 c_{23} + a_4 c_{234} & a_3 c_{23} + a_4 c_{234} & a_4 c_{234} \\ -\left(a_2 c_2 + a_3 c_{23} + a_4 c_{234}\right) & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{8}
$$

The second step requires post-multiplication of matrix ${}^1\boldsymbol{J}_L$ by a non-singular constant matrix $\boldsymbol{H}$ as follows:

$$
\begin{aligned}
{}^1\boldsymbol{J}_{L,\mathrm{abs}}(\boldsymbol{q}) &= {}^1\boldsymbol{J}_L(\boldsymbol{q})\boldsymbol{H} = {}^1\boldsymbol{J}_L(\boldsymbol{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \\
&= \begin{pmatrix} 0 & -a_2 s_2 & -a_3 s_{23} & -a_4 s_{234} \\ 0 & a_2 c_2 & a_3 c_{23} & a_4 c_{234} \\ -\left(a_2 c_2 + a_3 c_{23} + a_4 c_{234}\right) & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{9}
$$

The subscript 'abs' is there to remind that the upper two rows and last three columns of the obtained matrix have the same structure of the Jacobian that we would have when considering a planar 3R robot and using absolute coordinates w.r.t. to a horizontal axis.

Analyzing the rank of ${}^1\boldsymbol{J}_{L,\mathrm{abs}}(\boldsymbol{q})$ is easy. The last row will become dependent from the other two (and actually vanish) if and only if

$$
\boldsymbol{q}_L^*: \quad a_2 c_2 + a_3 c_{23} + a_4 c_{234} = \sqrt{p_x^2 + p_y^2} = 0, \tag{10}
$$

namely, if the robot end-effector lies on the axis $\boldsymbol{z}_0$ of joint 1. Moreover, the first two rows (deleting their useless, zero first column) will be linearly dependent if and only if the three 2×2 minors that can be extracted are all equal to zero, or

$$
s_3 = 0, \quad s_4 = 0, \quad s_{34} = 0 \quad \Longleftrightarrow \quad \boldsymbol{q}_L^*: \quad q_3 = \{0, \pi\}, \quad q_4 = \{0, \pi\}, \tag{11}
$$

namely when the third and fourth link are aligned with the second link, either stretched or folded. Each of the two singularity types (10) and (11) reduces by one the rank of ${}^1\boldsymbol{J}_{L,\mathrm{abs}}(\boldsymbol{q})$, which is clearly equal to the rank of $\boldsymbol{J}_L(\boldsymbol{q})$. When the arm is fully aligned with the axis of joint 1, either in a stretched or in a folded configuration, then rank $\boldsymbol{J}_L(\boldsymbol{q}) = 1$ (i.e., it drops by two at the intersection of the singularities).

Since all joints are revolute, the angular part of the geometric Jacobian is obtained from the general expression

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix}, \tag{12}
$$

where $\boldsymbol{z}_{i-1}$ is the unit vector aligned with the ith joint axis, and expressed in the base frame (of index 0). Using the DH rotation matrices, we have

$$
\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0, \quad \boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0, \quad \boldsymbol{z}_3 = {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2){}^2\boldsymbol{R}_3(q_3)\boldsymbol{z}_0, \tag{13}
$$

and performing easy computations

$$z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad z_3 = z_2 = z_1 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}. \tag{14}$$

Therefore, we obtain

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & s_1 & s_1 & s_1 \\ 0 & -c_1 & -c_1 & -c_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

This matrix has constant rank equal to two. All angular velocities that can be generated as $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ take the form

$$\boldsymbol{\omega} = \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \qquad \text{for any } \alpha,\, \beta. \tag{16}$$

Vice versa, angular velocities that cannot be generated by this robot take the form

$$\boldsymbol{\omega} = \gamma \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, \qquad \text{for any } \gamma \neq 0. \tag{17}$$

For further analysis, matrix $\boldsymbol{J}_A(\boldsymbol{q})$ can be written in the rotated DH frame 1, similarly to (8). We obtain the constant matrix

$${}^1\boldsymbol{J}_A = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{18}$$

Observing that

$$\text{rank } \boldsymbol{J}(\boldsymbol{q}) = \text{rank } \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \text{rank } \begin{pmatrix} {}^1\boldsymbol{J}_{L,\text{abs}}(\boldsymbol{q}) \\ {}^1\boldsymbol{J}_A \end{pmatrix}, \tag{19}$$

we can stack the two matrices in eq. (9) and (18) and analyze the rank of this 6×4 matrix. Deleting the third row (as linearly dependent on the fifth row) and the fourth (null) row, we are left with the 4×4 matrix

$$\boldsymbol{J}_{\text{red}}(\boldsymbol{q}) = \begin{pmatrix} 0 & -a_2 s_2 & -a_3 s_{23} & -a_4 s_{234} \\ 0 & a_2 c_2 & a_3 c_{23} & a_4 c_{234} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

Thus, a singularity of $\boldsymbol{J}(\boldsymbol{q})$ (rank $\boldsymbol{J}(\boldsymbol{q}^*) < 4$) will occur if and only if the determinant of $\boldsymbol{J}_{\text{red}}(\boldsymbol{q})$ vanishes, or

$$\det \boldsymbol{J}_{\text{red}}(\boldsymbol{q}) = 0 \quad \Longleftrightarrow \quad \boldsymbol{q}^* : \quad a_2 a_3 s_3 + a_3 a_4 s_4 - a_2 a_4 s_{34} = 0. \tag{21}$$

We note that when the robot is in a stretched of folded configuration and $\boldsymbol{J}_L(\boldsymbol{q})$ loses rank, as expressed by (11), then condition (21) is also satisfied and $\boldsymbol{J}(\boldsymbol{q})$ is necessarily singular too (i.e., $\boldsymbol{J}_A(\boldsymbol{q})$ does not help in recovering full rank).

Finally, we choose a simple configuration that is nonsingular for $\boldsymbol{J}_L(\boldsymbol{q})$ in (7), say $\boldsymbol{q}_0 = \begin{pmatrix} 0 & 0 & \pi/2 & 0 \end{pmatrix}^T$ (the robot has the second link horizontal and the third and fourth links vertical, pointing upwards: a sketch is left to the reader), and plug in the numerical values for $a_2 = 210.5$, $a_3 = 268$, and $a_4 = 174.5$ [mm], as found in Exercise #1. We obtain the numerical matrix (with rank equal to 3)

$$\boldsymbol{J}_L(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -442.5 & -442.5 & -174.5 \\ 210.5 & 0 & 0 & 0 \\ 0 & 210.5 & 0 & 0 \end{pmatrix}. \tag{22}$$

A velocity vector $\dot{\boldsymbol{q}}_0$ in the null space of $\boldsymbol{J}_L(\boldsymbol{q}_0)$ is

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 & 0 & -0.3669 & 0.9303 \end{pmatrix}^T \text{ [rad/s]},$$

and it is easy to see that $\boldsymbol{J}_L(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$.

Exercise 4

Using time normalization, the three cubic tracts of the interpolating spline are conveniently defined as

$$q_A(\tau_A) = q_1 + a_1\tau_A + a_2\tau_A^2 + a_3\tau_A^3, \qquad \tau_A = \frac{t-t_1}{t_2-t_1} \in [0,1], \quad t \in [t_1,t_2] \tag{23}$$

$$q_B(\tau_B) = q_2 + b_1\tau_B + b_2\tau_B^2 + b_3\tau_B^3, \qquad \tau_B = \frac{t-t_2}{t_3-t_2} \in [0,1], \quad t \in [t_2,t_3] \tag{24}$$

$$q_C(\tau_C) = q_3 + c_1\tau_C + c_2\tau_C^2 + c_3\tau_C^3, \qquad \tau_C = \frac{t-t_3}{t_4-t_3} \in [0,1], \quad t \in [t_3,t_4], \tag{25}$$

with the nine coefficients $a_1,\ldots,c_3$ determined by satisfying the nine boundary conditions

$$\begin{array}{l} q_A(1) = q_2, \\ q_B(1) = q_3, \\ q_C(1) = q_4, \end{array} \quad \begin{array}{llll} \dot{q}_A(0) = 0, & \dot{q}_A(1) = \dot{q}_B(0)\,[=v_2\,], & \ddot{q}_A(1) = \ddot{q}_B(0), \\ \dot{q}_C(1) = 0, & \dot{q}_B(1) = \dot{q}_C(0)\,[=v_3\,], & \ddot{q}_B(1) = \ddot{q}_C(0). \end{array} \tag{26}$$

Assume, for the time being, that we know the value of the velocities v_2 and v_3 in the two intermediate knots (at $t=t_2$ and $t=t_3$, respectively). The coefficients of each of the three cubic polynomials would then be completely defined by the four local boundary conditions on position and velocity at the two extremes of their interval of definition. Performing computations for the cubic A yields the coefficients

$$a_1 = 0, \qquad a_2 = 3(q_2-q_1) - v_2(t_2-t_1), \qquad a_3 = v_2(t_2-t_1) - 2(q_2-q_1), \tag{27}$$

and thus

$$\ddot{q}_A(1) = \frac{2a_2+6a_3}{(t_2-t_1)^2} = \frac{4v_2}{t_2-t_1} - \frac{6(q_2-q_1)}{(t_2-t_1)^2}. \tag{28}$$

Similarly, for the cubic B

$$b_1 = v_2(t_3-t_2), \quad b_2 = 3(q_3-q_2) - (2v_2+v_3)(t_3-t_2), \quad b_3 = -2(q_3-q_2) + (v_2+v_3)(t_3-t_2), \tag{29}$$

and thus

$$\ddot{q}_B(0) = \frac{2b_2}{(t_3-t_2)^2} = \frac{6(q_3-q_2)}{(t_3-t_2)^2} - \frac{4v_2+2v_3}{t_3-t_2} \tag{30}$$

and

$$\ddot{q}_B(1) = \frac{2b_2+6b_3}{(t_3-t_2)^2} = \frac{2v_2+4v_3}{t_3-t_2} - \frac{6(q_3-q_2)}{(t_3-t_2)^2}. \tag{31}$$

Finally, for the cubic C

$$c_1 = v_3(t_4-t_3), \qquad c_2 = 3(q_4-q_3) - 2v_3(t_4-t_3), \qquad c_3 = v_3(t_4-t_3) - 2(q_4-q_3), \tag{32}$$

and thus

$$\ddot{q}_C(0) = \frac{2c_2}{(t_4-t_3)^2} = \frac{6(q_4-q_3)}{(t_4-t_3)^2} - \frac{4v_3}{t_4-t_3}. \tag{33}$$

Imposing continuity of the acceleration at the internal knots

$$\ddot{q}_A(1) = \ddot{q}_B(0), \qquad \ddot{q}_B(1) = \ddot{q}_C(0),$$

and using eqs. (28), (30–31) and (33), leads to the linear system of equations

$$\boldsymbol{A}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{b}, \tag{34}$$

with[1]

$$\boldsymbol{A} = \begin{pmatrix} 2(t_3 - t_1) & (t_2 - t_1) \\ (t_4 - t_3) & 2(t_4 - t_2) \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 3(q_3 - q_2)\,\dfrac{t_2 - t_1}{t_3 - t_2} + 3(q_2 - q_1)\,\dfrac{t_3 - t_2}{t_2 - t_1} \\ 3(q_4 - q_3)\,\dfrac{t_3 - t_2}{t_4 - t_3} + 3(q_3 - q_2)\,\dfrac{t_4 - t_3}{t_3 - t_2}. \end{pmatrix}.$$

Replacing the numerical data (degrees are used everywhere here), the system is solved as

$$\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{pmatrix} -175.7143 \\ -215.3571 \end{pmatrix} \; [°/\mathrm{s}],$$

and the coefficients (27), (29), and (32) of the three cubic polynomials take then the numerical values

$$\begin{gathered} a_0 = q_1 = 45, \quad a_1 = 0, \quad a_2 = 310.7143, \quad a_3 = -265.7143, \\ b_0 = q_2 = 90, \quad b_1 = -87.8571, \quad b_2 = -121.6071, \quad b_3 = 74.4643, \\ c_0 = q_3 = -45, \quad c_1 = -323.0357, \quad c_2 = 916.0714, \quad c_3 = -503.0357. \end{gathered}$$

The plots of the interpolating cubic spline $q(t)$, for $t \in [t_1, t_4] = [1, 4]$, and of its velocity and acceleration are shown in Fig. 3. We can see that velocity is peaking between knots 2 and 3, whereas the maximum (absolute) value of the acceleration is reached at knot 2. Apart from this visualization, we should work indeed analytically in order to check if and how the bounds (3) on the velocity and acceleration are satisfied.

Being piecewise linear, the spline acceleration can assume its maximum values only at the boundaries of each time sub-interval. Therefore, we evaluate the acceleration at the knots (expressed in [°/s^2]):

$$\begin{aligned} A_1 = \ddot{q}(t_1) &= \ddot{q}_A(0) = \frac{2a_2}{(t_2 - t_1)^2} = 621.4286, \\ A_2 = \ddot{q}(t_2) &= \ddot{q}_B(0) = \frac{2b_2}{(t_3 - t_2)^2} = -972.8571, \\ A_3 = \ddot{q}(t_3) &= \ddot{q}_C(0) = \frac{2c_2}{(t_4 - t_3)^2} = 814.2857, \\ A_4 = \ddot{q}(t_4) &= \ddot{q}_C(1) = \frac{2c_2 + 6c_3}{(t_4 - t_3)^2} = -527.1429. \end{aligned} \tag{35}$$

As a result, none of the (absolute) values exceeds the limit of $A_{\max} = 1000\,°/\mathrm{s}^2$.

On the other hand, being piecewise quadratic, the spline velocity can assume its maximum values only where the acceleration is zero, or at the boundaries of each time sub-interval. An instant with zero acceleration occurs inside a given sub-interval if and only if the acceleration changes sign between the boundary knots (i.e., $A_i\,A_{i+1} < 0$). Looking at (35), this happens in fact in all three intervals. We should test then the spline velocity also at the instants $t_{\mathrm{acc_0,i}}$ where acceleration vanishes, or

$$t_{\mathrm{acc_0,i}} = t_i + \frac{|A_i|}{|A_i - A_{i+1}|}(t_{i+1} - t_i), \qquad i = 1, 2, 3.$$

[1] The first equation has been multiplied conveniently by $(t_3 - t_2)(t_2 - t_1)/2$, the second by $(t_4 - t_3)(t_3 - t_2)/2$. This makes $\boldsymbol{A}$ and $\boldsymbol{b}$ in eq. (34) identical to those in the lecture slides, for $N = 4$ and with $v_1 = v_4 = 0$.

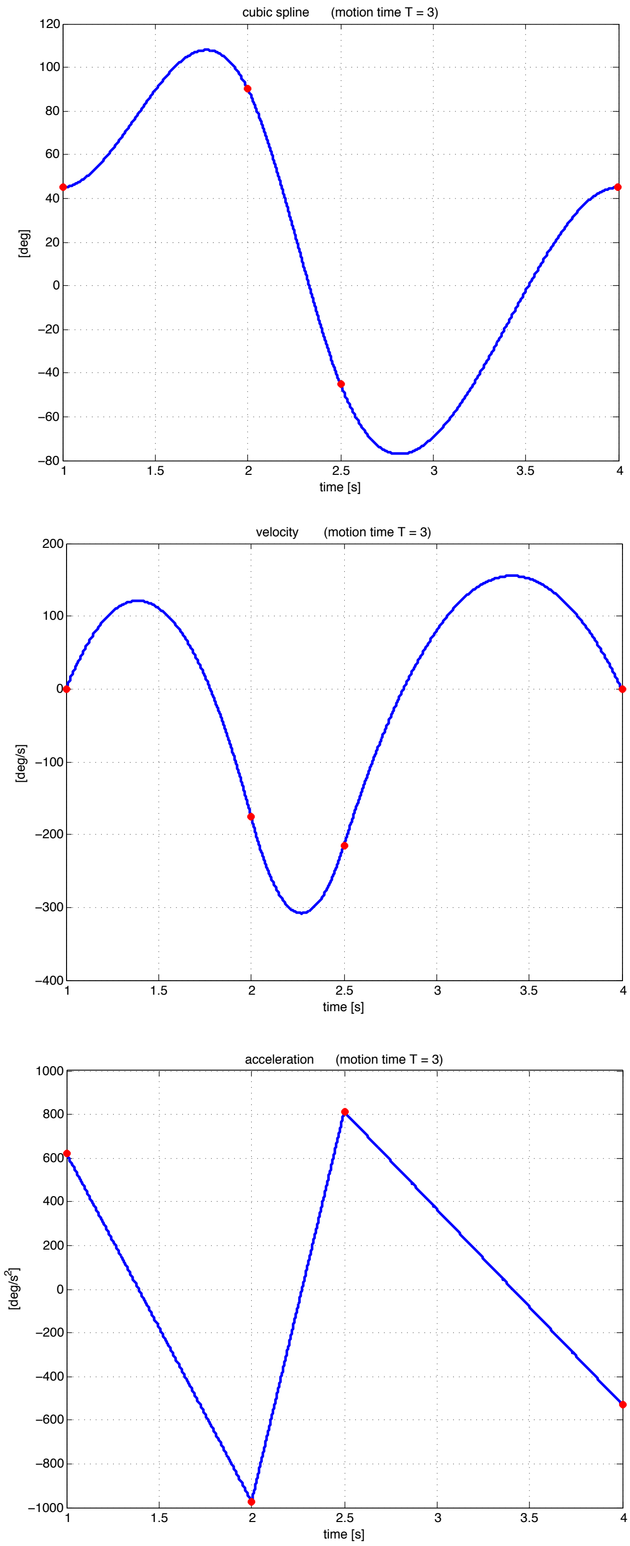


Figure 3: Planned spline trajectory $q(t)$, velocity, and acceleration. Total time $T = t_4 - t_1 = 3$ [s].

Therefore, we first evaluate the velocity at the knots (expressed in [°/s]):

$$\begin{aligned} v_1 &= \dot{q}(t_1) = 0, \\ v_2 &= \dot{q}(t_2) = -175.7143, \\ v_3 &= \dot{q}(t_3) = -215.3571, \\ v_4 &= \dot{q}(t_4) = 0. \end{aligned} \tag{36}$$

Next, having set[2]

$$\tau_{\text{acc}_0,\text{i}} = \frac{t_{\text{acc}_0,\text{i}} - t_i}{t_{i+1} - t_i} = \frac{|A_i|}{|A_i - A_{i+1}|} = \frac{|A_i|}{|A_i| + |A_{i+1}|} \in [0,1], \qquad i = 1,2,3, \tag{37}$$

we evaluate velocity also in the intermediate instants (expressed in [°/s]):

$$\begin{aligned} \dot{q}(t_{\text{acc}_0,1}) &= \dot{q}_A(\tau_{\text{acc}_0,1}) = \frac{a_1 + 2a_2\tau_{\text{acc}_0,1} + 3a_3\tau^2_{\text{acc}_0,1}}{t_2 - t_1} = 121.1118, \\ \dot{q}(t_{\text{acc}_0,2}) &= \dot{q}_B(\tau_{\text{acc}_0,2}) = \frac{b_1 + 2b_2\tau_{\text{acc}_0,2} + 3b_3\tau^2_{\text{acc}_0,2}}{t_3 - t_2} = -308.1115, \\ \dot{q}(t_{\text{acc}_0,3}) &= \dot{q}_C(\tau_{\text{acc}_0,3}) = \frac{c_1 + 2c_2\tau_{\text{acc}_0,3} + 3c_3\tau^2_{\text{acc}_0,3}}{t_4 - t_3} = 155.3640. \end{aligned} \tag{38}$$

As a result, the velocity at the time instant $t_{\text{acc}_0,2} = 2.2722$ [s] (in the second interval) is the only one that violates the bound specified in (3): $V_{\text{peak}} = |\dot{q}(t_{\text{acc}_0,2})| = 308.1115 > 250 = V_{\max}$.

In order to recover feasibility, we should then uniformly scale the total motion time $T = t_4 - t_1 = 3$ s by the factor

$$k = \frac{V_{\text{peak}}}{V_{\max}} = 1.2324 \quad \Rightarrow \quad T_{\text{scaled}} = kT = k(t_4 - t_1) = 3.6973. \tag{39}$$

The spline trajectory $q_{\text{scaled}}(t_{\text{scaled}})$, for $t_{\text{scaled}} \in [t_1, t_1 + T_{\text{scaled}}] = [1, 4.6973]$, is shown in Fig. 4, together with its scaled velocity and acceleration. Some caution should be used to handle a non-zero value for the initial time $t_1 = 1$ in the planned motion. We have $t_{\text{scaled}} = t_1 + k(t - t_1)$ in this case (rather than simply $t_{\text{scaled}} = kt$), and the interpolation of the original knots will be achieved at the new instants

$$t_1 \to t_{\text{scaled},1} = t_1 = 1 \text{ (unchanged)} \Rightarrow q_1, \qquad t_2 \to t_{\text{scaled},2} = t_1 + k(t_2 - t_1) = 2.2324 \Rightarrow q_2,$$

$$t_3 \to t_{\text{scaled},3} = t_1 + k(t_3 - t_1) = 2.8487 \Rightarrow q_3, \qquad t_4 \to t_{\text{scaled},4} = t_1 + k(t_4 - t_1) = 4.6973 \Rightarrow q_4.$$

Accordingly,

$$q_{\text{scaled}}(t_{\text{scaled}}) = q(t), \qquad \dot{q}_{\text{scaled}}(t_{\text{scaled}}) = \frac{\dot{q}(t)}{k}, \qquad \ddot{q}_{\text{scaled}}(t_{\text{scaled}}) = \frac{\dot{q}(t)}{k^2}.$$

The scaled velocity (in absolute value) reaches now its limit $V_{\text{scaled,peak}} = V_{\max} = 250°/\text{s}$ at the new time instant $t_{\text{scaled,peak}} = t_1 + k\,(t_{\text{acc}_0,2} - t_1) = 1 + 1.2324 \cdot (2.2722 - 1) = 2.5679$ s.

* * * * *

[2] The last equality in (37) holds because of the assumed condition $A_i\, A_{i+1} < 0$, under which the zero acceleration instant occurs inside $[t_i, t_{i+1}]$.

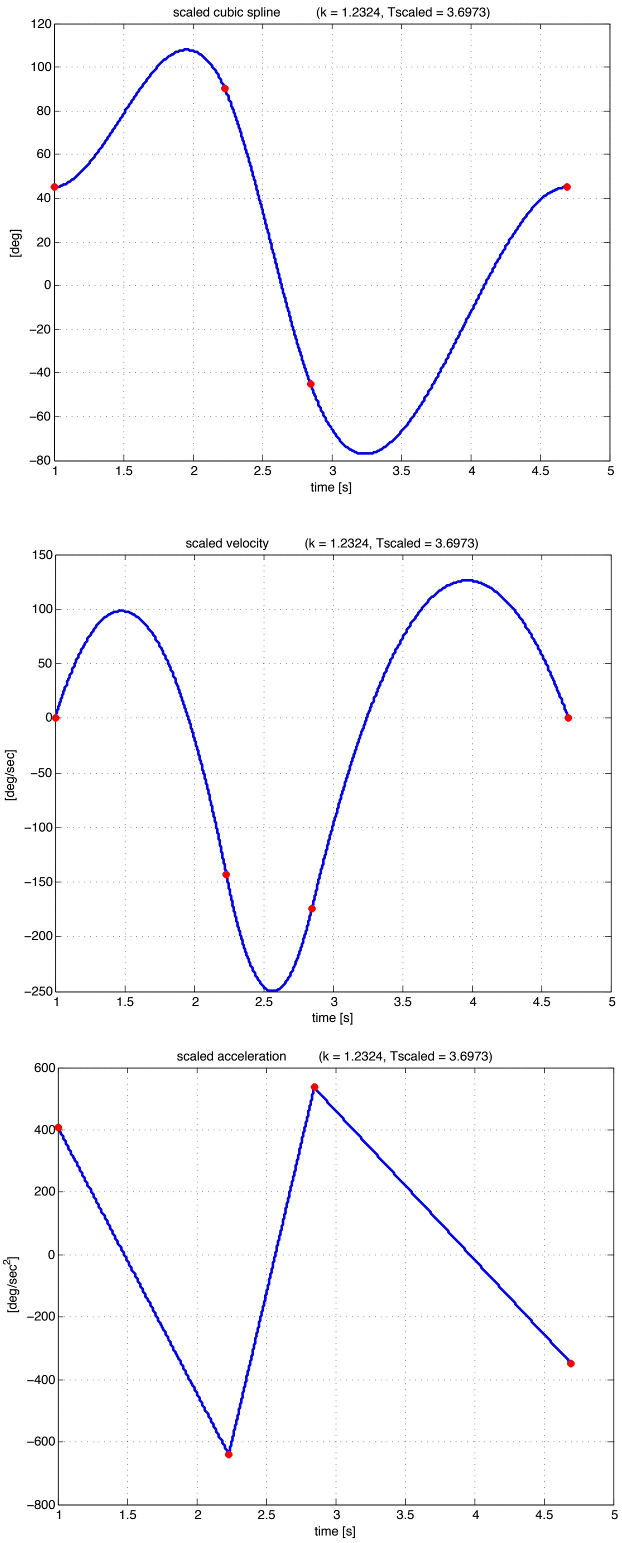


Figure 4: Scaled spline trajectory $q_{\text{scaled}}(t_{\text{scaled}})$, with velocity and acceleration. The scaled total time of motion is $T_{\text{scaled}} = 3.6973$ [s].