# Robotics I

**September 21, 2017**

## Exercise 1

Consider the rigid body in Fig. 1, a thin rod of length $L$. The rod will be rotated by an angle $\alpha$ around the $\boldsymbol{z}$ axis, then by an angle $\beta$ around the resulting $\boldsymbol{x}$ axis, and finally by an angle $\gamma$ around the resulting $\boldsymbol{y}$ axis.

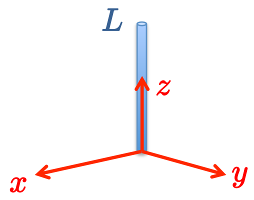


Figure 1: Initial placement of a rigid thin rod of length $L$ in an absolute reference frame.

- Provide the final orientation of the rod, as expressed by a rotation matrix, and the symbolic expression of its angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ in terms of the angles $(\alpha, \beta, \gamma)$ and their time derivatives $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$.
- Assuming the numerical values $L = 0.4$ [m] for the rod, $\alpha = 30°$, $\beta = -30°$, and $\gamma = 60°$ for the angles, and $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 45°/\text{s}$ for their time derivatives, determine:
  - the components of the final unit vector $\boldsymbol{y}$ attached to the rod after the three rotations;
  - the absolute coordinates of the position $\boldsymbol{p}$ of the end-point of the rod after the three rotations;
  - the angular velocity $\boldsymbol{\omega}$ of the rod, expressed both in the initial (absolute) reference frame and in the final rotated frame;
  - the absolute velocity $\dot{\boldsymbol{p}}$ of the end-point of the rod.

## Exercise 2

Using standard DH angles, a planar 2R robot with links of lengths $\ell_1 = \ell_2 = 0.5$ [m] is in the initial static configuration $\boldsymbol{q}_0 = (45°, -90°)$. Plan a rest-to-rest cubic polynomial trajectory in the joint space so as to reach the final end-effector position $\boldsymbol{p}_{goal} = (0.5, 0.866)$ [m] in minimum time under the following joint velocity limits: $|\dot{q}_1| \leq 30°/\text{s}$, $|\dot{q}_2| \leq 90°/\text{s}$.

- Provide the minimum time $T^*$ and the numerical values of the coefficients of the optimal cubic polynomials.
- If the robot performs the above minimum-time task with a *coordinated* motion in the joint space, what is the maximum velocity reached by each joint during motion?

**Exercise 3**

Consider the transmission/reduction assembly of a robot joint driving a single robot link, as sketched in Fig. 2. The actuator is an electrical motor with maximum rotation speed $\dot{\theta}_m$ of its output shaft equal to 2000 RPM. The transmission assembly consists of an Harmonic Drive (HD), with its wave generator mounted on the motor shaft ($A \leftrightarrow A'$) and whose circular spline has $n_{CS} = 64$ teeth, followed by a two-wheel toothed gear, with a smaller wheel of radius $r_1 = 0.9$ [cm] mounted on the HD output axis ($B \leftrightarrow B'$) and a larger wheel of radius $r_2 = 1.7$ [cm]. The link rotates with the larger wheel at an angular speed $\dot{\theta}_\ell$ and has a length $L = 0.68$ [m].

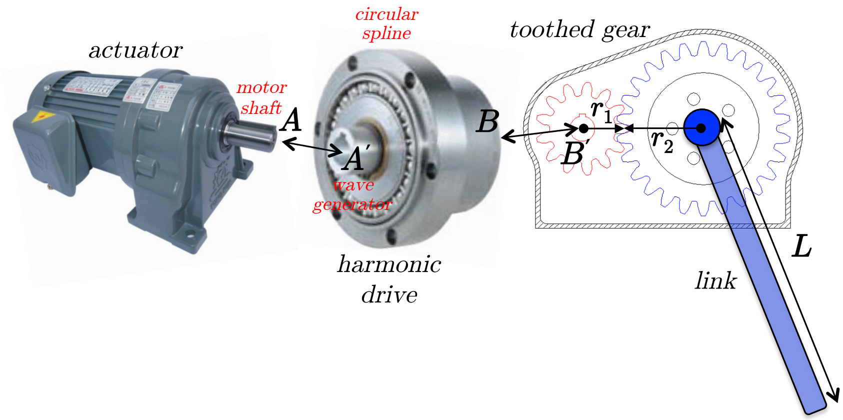


Figure 2: Transmission/reduction assembly of a robot joint.

- What is the reduction ratio $N = |\dot{\theta}_m/\dot{\theta}_\ell|$ of the complete transmission?
- Does the link rotate in the same or in the opposite direction of the motor shaft?
- What is the maximum velocity achievable by the end-point of the link?

**[180 minutes, open books but no computer or smartphone]**

# Solution

September 21, 2017

**Exercise 1**

The sequence of rotations about the first (fixed) axis $Z$ and the following (moving) axes $X'$ and $Y''$ provides a $ZXY$ Euler representation of orientation. Therefore, the final orientation of the rod is given by

$$
\begin{aligned}
\boldsymbol{R}_{ZXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Z(\alpha)\,\boldsymbol{R}_{X'}(\beta)\,\boldsymbol{R}_{Y''}(\gamma)\\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\beta & -\sin\beta\\ 0 & \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix} \cos\gamma & 0 & \sin\gamma\\ 0 & 1 & 0\\ -\sin\gamma & 0 & \cos\gamma\end{pmatrix}\\
&= \begin{pmatrix} \cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma+\sin\alpha\sin\beta\cos\gamma\\ \sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma-\cos\alpha\sin\beta\cos\gamma\\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\sin\gamma\end{pmatrix}\\
&= \begin{pmatrix} \boldsymbol{x}'''(\alpha,\beta,\gamma) & \boldsymbol{y}'''(\alpha,\beta) & \boldsymbol{z}'''(\alpha,\beta,\gamma)\end{pmatrix}
\end{aligned}
$$

In particular, the position of the end-point of the rod after the three rotations is $\boldsymbol{p} = L\,\boldsymbol{z}'''$.

Using the numerical data, the final rotation matrix is

$$
\boldsymbol{R}_{ZXY}\left(\frac{\pi}{6},-\frac{\pi}{6},\frac{\pi}{3}\right) = \begin{pmatrix} \frac{3\sqrt{3}}{8} & -\frac{\sqrt{3}}{4} & \frac{5}{8}\\ -\frac{1}{8} & \frac{3}{4} & \frac{3\sqrt{3}}{8}\\ -\frac{3}{4} & -\frac{1}{2} & \frac{\sqrt{3}}{4}\end{pmatrix} = \begin{pmatrix} 0.6495 & -0.4330 & 0.6250\\ -0.1250 & 0.7500 & 0.6495\\ -0.7500 & -0.5000 & 0.4330\end{pmatrix},
$$

while the unit vector $\boldsymbol{y}''' = \boldsymbol{y}''$ and the position vector $\boldsymbol{p}$ of the end-point of the rod after the rotations are, respectively,

$$
\boldsymbol{y}''\left(\frac{\pi}{6},-\frac{\pi}{6}\right) = \begin{pmatrix} -0.4330\\ 0.7500\\ -0.5000\end{pmatrix}, \qquad \boldsymbol{p} = 0.4\,\boldsymbol{z}'''\left(\frac{\pi}{6},-\frac{\pi}{6},\frac{\pi}{3}\right) = \begin{pmatrix} \frac{1}{4}\\ \frac{3\sqrt{3}}{20}\\ \frac{\sqrt{3}}{10}\end{pmatrix} = \begin{pmatrix} 0.2500\\ 0.2598\\ 0.1732\end{pmatrix}\ [\mathrm{m}].
$$

The angular velocity $\boldsymbol{\omega}$ of the rod associated to the time derivatives $(\dot\alpha,\dot\beta,\dot\gamma)$ in the configuration specified by the angles $(\alpha,\beta,\gamma)$ can be computed either from the relation

$$
\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_{ZXY}(\alpha,\beta,\gamma,\dot\alpha,\dot\beta,\dot\gamma)\,\boldsymbol{R}^T_{ZXY}(\alpha,\beta,\gamma) = \begin{pmatrix} 0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\omega}
$$

or, more explicitly, as

$$
\boldsymbol{\omega} = \boldsymbol{z}\,\dot\alpha + \boldsymbol{x}'\,\dot\beta + \boldsymbol{y}''\,\dot\gamma = \begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}\dot\alpha + \boldsymbol{R}_Z(\alpha)\begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}\dot\beta + \boldsymbol{R}_Z(\alpha)\,\boldsymbol{R}_{X'}(\beta)\begin{pmatrix} 0\\ 1\\ 0\end{pmatrix}\dot\gamma = \begin{pmatrix} \omega_x\\ \omega_y\\ \omega_z\end{pmatrix}.
$$

Performing computations, we obtain

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \\ 1 & 0 & \sin\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} \cos\alpha\,\dot{\beta} - \sin\alpha\cos\beta\,\dot{\gamma} \\ \sin\alpha\,\dot{\beta} + \cos\alpha\cos\beta\,\dot{\gamma} \\ \dot{\alpha} + \sin\beta\,\dot{\gamma} \end{pmatrix}.$$

This expression is referenced to the initial frame. In the rotated frame, one has

$$^{R}\boldsymbol{\omega} = \boldsymbol{R}_{ZXY}^{T}\,\boldsymbol{\omega} = \begin{pmatrix} -\cos\beta\sin\gamma\,\dot{\alpha} + \cos\gamma\,\dot{\beta} \\ \sin\beta\,\dot{\alpha} + \dot{\gamma} \\ \cos\beta\cos\gamma\,\dot{\alpha} + \sin\gamma\,\dot{\beta} \end{pmatrix}.$$

Finally, the linear velocity $\dot{\boldsymbol{p}}$ of the end-point of the rod is given by

$$\dot{\boldsymbol{p}} = \boldsymbol{\omega} \times \boldsymbol{p} = \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{p} = L \left[ \begin{pmatrix} \cos\alpha\sin\beta\cos\gamma - \sin\alpha\sin\gamma \\ \sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma \\ 0 \end{pmatrix} \dot{\alpha} \right.$$
$$\left. + \begin{pmatrix} \sin\alpha\cos\beta\cos\gamma \\ -\cos\alpha\cos\beta\cos\gamma \\ -\sin\beta\cos\gamma \end{pmatrix} \dot{\beta} + \begin{pmatrix} -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma \\ \cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma \\ -\cos\beta\sin\gamma \end{pmatrix} \dot{\gamma} \right].$$

Using the numerical data, we obtain

$$\boldsymbol{\omega} = \frac{\pi}{4} \begin{pmatrix} \frac{\sqrt{3}}{4} \\ \frac{5}{4} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.3401 \\ 0.9817 \\ 0.3927 \end{pmatrix} \text{[rad/s]}, \qquad {}^{R}\boldsymbol{\omega} = \frac{\pi}{4} \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ \frac{3\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} -0.1963 \\ 0.3927 \\ 1.0203 \end{pmatrix} \text{[rad/s]},$$

and

$$\dot{\boldsymbol{p}} = \begin{pmatrix} 0.0680 \\ 0.0393 \\ -0.1571 \end{pmatrix} \text{[m/s]}.$$

**Exercise 2**

We first solve the inverse kinematics for the final position $\boldsymbol{p}_{goal} = (p_{x,goal}, p_{y,goal}) = (0.5, \sqrt{3}/2)$. Noting that $\|\boldsymbol{p}_{goal}\| = 1 = \ell_1 + \ell_2$, this point is on the external boundary of the robot workspace. Thus, there is a single configuration with stretched arm as solution, given by

$$\boldsymbol{q}_{goal} = \begin{pmatrix} q_{1,goal} \\ q_{2,goal} \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\{p_{y,goal}, p_{x,goal}\} \\ 0 \end{pmatrix} = \begin{pmatrix} \pi/3 \\ 0 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 60^\circ \\ 0^\circ \end{pmatrix}.$$

As a result the joint displacement is

$$\boldsymbol{\Delta q} = \boldsymbol{q}_{goal} - \boldsymbol{q}_0 = \begin{pmatrix} 15^\circ \\ 90^\circ \end{pmatrix} = \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix}$$

and the associated rest-to-rest cubic trajectories for the two joints, each in time $T_i > 0$, $i = 1, 2$, will take the form

$$q_i(t) = q_{0,i} + \Delta q_i \left( 3 \left( \frac{t}{T_i} \right)^2 - 2 \left( \frac{t}{T_i} \right)^3 \right), \tag{1}$$

with velocities and accelerations

$$\dot{q}_i(t) = \frac{6\Delta q_i}{T_i}\left(\left(\frac{t}{T_i}\right) - \left(\frac{t}{T_i}\right)^2\right), \qquad \ddot{q}_i(t) = \frac{6\Delta q_i}{T_i^2}\left(1 - 2\left(\frac{t}{T_i}\right)\right), \qquad \text{for } i = 1, 2.$$

For each joint, the maximum velocity (in absolute value) will occur at the trajectory midpoint, where the acceleration is zero, i.e.,

$$\ddot{q}_i(t_i^*) = 0 \quad \Rightarrow \quad t_i^* = \frac{T_i}{2} \quad \Rightarrow \quad |\dot{q}_i(T_i/2)| = \frac{3\,|\Delta q_i|}{2T_i}.$$

Since both displacements $\Delta q_i$ are positive, we will discard from now on the absolute value. Imposing that the joint velocities both reach their admissible limits, respectively, $V_1 = 30°/\text{s}$ and $V_2 = 90°/\text{s}$, we have

$$|\dot{q}_i(T_i/2)| = V_i \quad \Rightarrow \quad T_i^* = \frac{1.5\,\Delta q_i}{V_i} = \begin{cases} 0.75, & \text{for } i = 1, \\ 1.5, & \text{for } i = 2. \end{cases}$$

Therefore, the minimum motion time will be

$$T^* = \max\{T_1^*, T_2^*\} = T_2^* = 1.5\,\text{s}.$$

The associated numerical coefficients of the optimal cubic polynomials in (1) are

$$\begin{aligned} q_1(t) &= 45 + 80\,t^2 - 71.1111\,t^3, & t \in [0, 0.75], \\ q_2(t) &= -90 + 120\,t^2 - 53.3333\,t^3, & t \in [0, 1.5]. \end{aligned}$$

In order to perform a coordinated motion in the joint space, the fastest joint, namely the first one, should uniformly slow down its motion so that the its final time equals $T^*$. Since $T^* = 2T_1^*$, the needed scaling is by a factor $k_1 = 2$ and, accordingly, the new maximum velocity of joint 1, reached again at the midpoint of the coordinated trajectory, will be reduced to $V_1/k_1 = 15°/\text{s}$. Indeed, no changes occur in the velocity profile of the second joint.

**Exercise 3**

The considered harmonic drive has $n_{CS} = 64$ teeth on the (internal side of the) circular spline and thus $n_{FS} = n_{CS} - 2 = 62$ teeth on the external side of the flexspline. Its reduction ratio is thus

$$N_{HD} = \frac{n_{FS}}{n_{CS} - n_{FS}} = \frac{n_{FS}}{2} = \frac{62}{2} = 31.$$

Since the toothed gear has reduction ratio

$$N_{TG} = \frac{r_2}{r_1} = \frac{1.7}{0.9} = 1.8889,$$

the reduction ratio of the transmission is

$$N = \left|\frac{\dot{\theta}_m}{\dot{\theta}_\ell}\right| = N_{HD} \cdot N_{TG} = 58.5556.$$

There is a double inversion of rotations in the overall transmission. Thus, the link will rotate in the same direction (clockwise or counterclockwise) of the motor shaft, i.e., $\text{sign}\{\dot{\theta}_m\} = \text{sign}\{\dot{\theta}_\ell\}$. Finally, the maximum velocity $v$ achievable by the end-point of the link is

$$v = L\,\max\{\dot{\theta}_\ell\} = L\,\frac{\max\{\dot{\theta}_m\}}{N} = 0.68\,[\text{m}]\,\frac{2000\,[\text{RPM}] \cdot (2\pi/60)[\text{rad/s}]}{58.5556} = 2.4322\,[\text{m/s}].$$

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