Robotics I

July 11, 2016

Exercise 1

A robot joint should be moved between an initial value q_0 at time $t = 0$ and a desired value $q_1 > q_0$ at the final time $t = T$, starting at rest and reaching a velocity $v_1 > 0$ at the final time.

- Provide the expression of a smooth trajectory $q_d(t)$ that solves this interpolation problem.
- For this trajectory, give the conditions on the problem data under which the velocity $\dot{q}_d(t)$ remains confined in the domain $[0, v_1]$ during the whole time interval $[0, T]$.
- Using the data $q_0 = 15°$, $q_1 = 45°$, and $v_1 = 30°/\text{s}$, provide the minimum motion time T^* that can be achieved with the chosen trajectory, such that the bound $|\dot{q}_d(t)| \le V_{max} = 90°/\text{s}$ holds for all $t \in [0, T^*]$. Sketch the obtained velocity profile.

Exercise 2

The kinematics of a 3R spatial robot is described by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	0	0	0	q_3

Table 1: Denavit-Hartenberg parameters of a 3R robot.

Provide the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ between the joint velocity $\dot{\boldsymbol{q}}$ and the angular velocity $\boldsymbol{\omega}$ of the third (last) robot reference frame and determine its singularities, if any.

Exercise 3

The end effector of a planar 2R robot having links of length $\ell_1 = 1.5$ and $\ell_2 = 2$ [m] should move from point $\boldsymbol{A} = (1, 1)$ [m] to point $\boldsymbol{B} = (0.5, 1.5)$ [m] along a straight line and at constant speed $V = 1$ m/s. Denote with $\boldsymbol{p}_d(t)$ this desired Cartesian trajectory and with $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ the direct kinematics of the robot for this task.

- Determine initial values $\boldsymbol{q}(0) = \boldsymbol{q}_A$ and $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_A$ such that the desired Cartesian trajectory $\boldsymbol{p}_d(t)$ can be traced exactly right from the initial time $t = 0$.
- Assume $\boldsymbol{q}(0) \neq \boldsymbol{q}_A$, so that an initial end-effector position error $\boldsymbol{e}_P(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0)) \neq \boldsymbol{0}$ results. Provide a kinematic control law at the joint velocity level such that the error $\boldsymbol{e}_P(t)$ is reduced in norm to less than 5% of its initial value before reaching halfway of the nominal Cartesian motion, and eventually vanishes. Disregard the possible presence of singularities.

[210 minutes; open books]

Solution

June 6, 2016

Exercise 1

Choosing the class of cubic polynomial trajectories and imposing the four boundary conditions

$$q_d(0) = q_0, \qquad q_d(0) = q_1, \qquad \dot{q}_d(T) = 0, \qquad \dot{q}_d(T) = v_1,$$

yields the unique trajectory

$$q_d(t) = q_0 + (q_1 - q_0)\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right) + v_1 T\left(\left(\frac{t}{T}\right)^3 - \left(\frac{t}{T}\right)^2\right), \qquad t \in [0, T]. \tag{1}$$

The first and second time derivatives of (1) are

$$\dot{q}_d(t) = \frac{6(q_1 - q_0)}{T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right) + v_1\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)\right),$$

and

$$\ddot{q}_d(t) = \frac{6(q_1 - q_0)}{T^2}\left(1 - 2\left(\frac{t}{T}\right)\right) + \frac{v_1}{T}\left(6\left(\frac{t}{T}\right) - 2\right).$$

In particular, the initial and final accelerations are

$$\ddot{q}_d(0) = \frac{1}{T^2}\left(6(q_1 - q_0) - 2v_1 T\right), \qquad \ddot{q}_d(T) = \frac{1}{T^2}\left(4v_1 T - 6(q_1 - q_0)\right).$$

In order to have $\dot{q}(t) \in [0, v_1]$, for all $t \in [0, T]$, the velocity (a quadratic polynomial) should always be increasing in the closed interval of motion, with its maximum reached at the final boundary instant. As a consequence, the (linear) acceleration profile should always remain positive in the (open) interval $(0, T)$. This occurs if and only if both $\ddot{q}_d(0)$ and $\ddot{q}_d(T)$ are non-negative, i.e.,

$$6(q_1 - q_0) - 2v_1 T \geq 0, \qquad 4v_1 T - 6(q_1 - q_0) \geq 0,$$

or

$$v_1 \leq 3\,\frac{q_1 - q_0}{T} \leq 2v_1, \tag{2}$$

which is the sought condition.

Substituting now the given data values yields the acceleration profile

$$\ddot{q}_d(t) = \frac{180°}{T^2}\left(1 - 2\left(\frac{t}{T}\right)\right) + \frac{30°}{T}\left(6\left(\frac{t}{T}\right) - 2\right) = \frac{180°}{T^2}\left((T-2)\left(\frac{t}{T}\right) + \left(1 - \frac{T}{3}\right)\right),$$

which crosses zero at the instant t^* (conveniently normalized as $\tau^* = t^*/T$)

$$\ddot{q}_d(t^*) = 0 \qquad \Rightarrow \qquad \tau^* = \frac{t^*}{T} = \frac{(T/3) - 1}{T - 2},$$

being $\tau^* \in (0, 1)$ only when $T \notin (1.5, 3)$. The maximum (absolute value) of the velocity in the closed time interval $[0, T]$ occurs either at one of the two boundaries or at t^*, but only when this instant is inside $[0, T]$. Since $\dot{q}_d(0) = 0$ and $\dot{q}_d(T) = v_1 = 30°/\text{s} < 90°/\text{s} = V_{max}$, only the last case is of interest for reaching the maximum velocity bound. From

$$\begin{aligned}\dot{q}_d(t) &= \frac{180°}{T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right) + 30°\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)\right) \\ &= \frac{180°}{T}\left(\frac{t}{T}\right)\left(\left(\frac{T}{2} - 1\right)\left(\frac{t}{T}\right) + \left(1 - \frac{T}{3}\right)\right),\end{aligned}$$

we have

$$\dot{q}_d(t^*) = \frac{180°}{T}\,\frac{(T/3) - 1}{T - 2}\left(\left(\frac{T}{2} - 1\right)\frac{(T/3) - 1}{T - 2} + \left(1 - \frac{T}{3}\right)\right) = \frac{90°\,(1 - (T/3))^2}{T(2 - T)}.$$

Setting $|\dot{q}_d(t^*)| = V_{max} = 90°/\text{s}$ leads to

$$\frac{(1 - (T/3))^2}{T\,|2 - T|} = 1.$$

When looking for the minimum feasible value T^* for T, we can assume for the time being that $T < 2$ and eliminate the need for the absolute value. Thus, we solve

$$(1 - (T/3))^2 = T(2 - T) \quad \Rightarrow \quad \frac{10}{9}T^2 - \frac{8}{3}T + 1 = 0 \quad \Rightarrow \quad T_{1,2} = \{0.4652, 1.9348\},$$

obtaining as the minimum feasible motion time

$$T^* = 0.4652 \text{ s}. \tag{3}$$

Indeed, the minimum time found satisfies $T^* \notin (1.5, 3)$, and is also consistent with the assumption made ($T^* < 2$). According to (3), the instant $t^* \in [0, T^*]$ when the maximum velocity $V_{max} = 90°/\text{s}$ is reached is

$$t^* = T^*\,\frac{(T^*/3) - 1}{T^* - 2} = 0.2561 \text{ s}.$$

Figure 1 shows the obtained position, velocity, and acceleration profiles.

Exercise 2

The requested mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ involves the angular part of the geometric Jacobian, usually expressed in the base frame 0. Since the robot has three revolute joints, it is

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{z}_2 \end{pmatrix},$$

with ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for $i = 0, 1, 2$. From Tab. 1, the symbolic expressions of the DH rotation matrices are

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & \sin q_2 \\ \sin q_2 & 0 & -\cos q_2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus

$${}^0\boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{z}_1 = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}, \quad {}^0\boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1)\left({}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{z}_2\right) = \begin{pmatrix} \cos q_1 \sin q_2 \\ \sin q_1 \sin q_2 \\ -\cos q_2 \end{pmatrix},$$

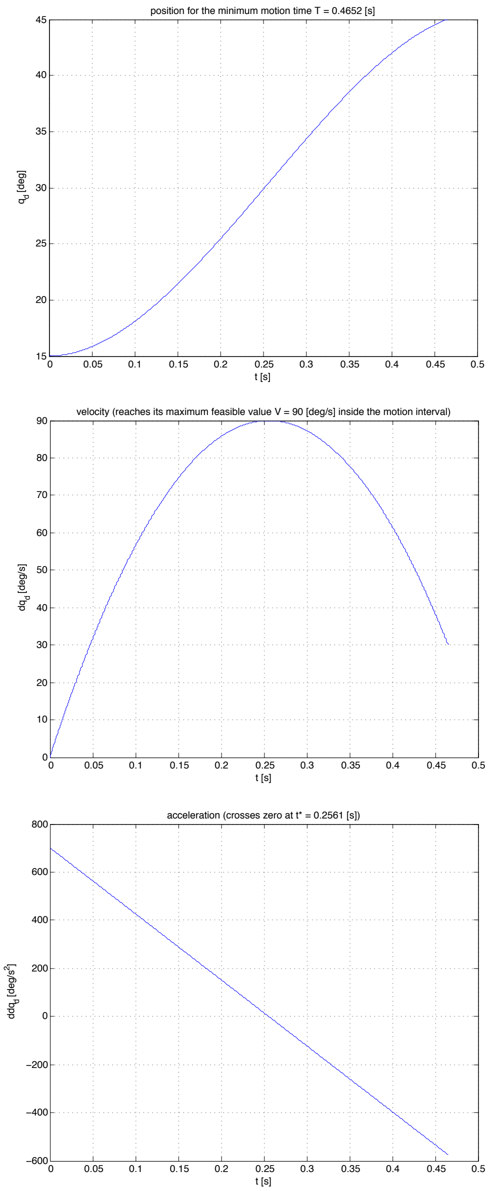

Figure 1: Minimum time position, velocity, and acceleration profiles for Exercise 1.

and the 3×3 Jacobian is

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & \sin q_1 & \cos q_1 \sin q_2 \\ 0 & -\cos q_1 & \sin q_1 \sin q_2 \\ 1 & 0 & -\cos q_2 \end{pmatrix},$$

which is singular when

$$\det \boldsymbol{J}_A(\boldsymbol{q}) = \sin q_2 = 0 \qquad \Rightarrow \qquad q_2 = \{0, \pm\pi\}.$$

Exercise 3

Both points $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to the (interior of the) robot workspace, being

$$\|\boldsymbol{A}\| = \sqrt{2}, \qquad \|\boldsymbol{B}\| = \sqrt{2.5}, \quad \text{and} \quad WS = \left\{\boldsymbol{p} \in \mathbb{R}^2 : 0.5 = |\ell_1 - \ell_2| \le \|\boldsymbol{p}\| \le \ell_1 + \ell_2 = 2.5\right\}.$$

It is easy to see that the whole linear path from $\boldsymbol{A}$ to $\boldsymbol{B}$ will also belong to the workspace interior. As a result, no singularities are encountered along the nominal Cartesian trajectory. Being $L = \|\boldsymbol{B} - \boldsymbol{A}\| = 1/sqrt2 = 0.7071$ [m] the length of the path from $\boldsymbol{A}$ to $\boldsymbol{B}$, a motion at the constant speed $V = 1$ [m/s] will reach halfway of this path at $t = T_{mid} = 0.5L/V = 0.3536$ [s]. Moreover, the desired Cartesian velocity at the initial point $\boldsymbol{A}$ is

$$\dot{\boldsymbol{p}}_d(0) = V\,\frac{\boldsymbol{B} - \boldsymbol{A}}{\|\boldsymbol{B} - \boldsymbol{A}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}.$$

The direct kinematics of the 2R planar robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2) \end{pmatrix}$$

and its Jacobian is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(\ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2)\right) & -\ell_2 \sin(q_1 + q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) & \ell_2 \cos(q_1 + q_2) \end{pmatrix}.$$

Solving the inverse kinematics problem —see the textbook— at the point $\boldsymbol{A}$ gives the two solutions (R = right arm solution, L = left arm solution)

$$\boldsymbol{q}_{A,R} = \begin{pmatrix} -0.7264 \\ 2.3579 \end{pmatrix}, \qquad \boldsymbol{q}_{A,L} = \begin{pmatrix} 2.2972 \\ -2.3579 \end{pmatrix}. \tag{4}$$

These are the two robot configurations $\boldsymbol{q}(0)$ at $t = 0$ that can be associated to the starting point of the desired Cartesian trajectory. In these two configurations, the robot Jacobian takes respectively the values

$$\boldsymbol{J}(\boldsymbol{q}_{A,R}) = \begin{pmatrix} -1 & -1.9963 \\ 1 & -0.1213 \end{pmatrix}, \qquad \boldsymbol{J}(\boldsymbol{q}_{A,L}) = \begin{pmatrix} -1 & 0.1213 \\ 1 & 1.9963 \end{pmatrix},$$

so that the initial joint velocity $\dot{\boldsymbol{q}}(0)$ at $t = 0$ needed for tracing exactly the desired Cartesian trajectory can be computed as

$$\dot{\boldsymbol{q}}_A = \boldsymbol{J}^{-1}(\boldsymbol{q}_{A,R})\,\dot{\boldsymbol{p}}_d(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_{A,L})\,\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} \dfrac{\sqrt{2}}{2} \\ 0 \end{pmatrix},$$

with a *single* common value in this very particular case[1].

[1] In general, two different joint velocities $\dot{\boldsymbol{q}}_{A.R}$ and $\dot{\boldsymbol{q}}_{A.L}$ will be found that execute the same Cartesian velocity $\dot{\boldsymbol{p}}_d(0)$ from two different solutions of the inverse kinematics problem. In this case, the uniqueness of the obtained joint velocity depends on the particular direction of the desired Cartesian velocity specified by the problem.

To recover an initial position error of the robot end-effector with respect to the desired trajectory, $\boldsymbol{e}_P(0) = \boldsymbol{p}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0)) \neq \boldsymbol{0}$, the following Cartesian kinematic control law can be used for the joint velocity commands

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_p\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right)\right), \qquad \boldsymbol{K}_p = k_p \boldsymbol{I}_{2\times 2} > 0.$$

This imposes the following dynamics to the Cartesian position error:

$$\dot{\boldsymbol{e}}_P = -k_p \boldsymbol{e}_P \quad \Rightarrow \quad \boldsymbol{e}_P(t) = \exp(-k_p t)\, \boldsymbol{e}_P(0) \quad \Rightarrow \quad \|\boldsymbol{e}_P(t)\| = \exp(-k_p t)\, \|\boldsymbol{e}_P(0)\|.$$

In order to have a reduction in norm of the position error to less than 5% of its initial value by the time $t = T_{mid} = 0.3536$ [s], we need to have

$$\exp(-0.3536\, k_p) \leq \exp(-3) \simeq 0.0498 < 0.05 \qquad \Rightarrow \qquad k_p \geq \frac{3}{0.3536} = 8.48. \tag{5}$$

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