Robotics I

February 4, 2016

Exercise 1

We are given an incomplete time-varying rotation matrix from frame 0 to frame 1:

$$ {}^0\boldsymbol{R}_1(t) = \begin{pmatrix} \cos t & a(t) & b(t) \\ \sin t & \dfrac{k(t)}{\sqrt{2}}\cos t & c(t) \\ 0 & -\dfrac{k(t)}{\sqrt{2}}\sin t & d(t) \end{pmatrix}. $$

Determine the expressions of $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $k(t)$ in a consistent way.

Exercise 2

The table of Denavit-Hartenberg parameters of a 2-dof robot is:

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	0	0	q_2	0

The two joints have a range limitation: $|q_1| \leq 120°$ and $|q_2| \leq 2$ [m]. Determine all feasible inverse kinematics solutions, if any, when the origin of frame 2 needs to be placed at ${}^0\boldsymbol{p} = (-1, 1)$ [m].

Exercise 3

Consider a planar 4R robot with links of lengths $\ell_i = 0.25$ [m], $i = 1, \ldots, 4$. The robot performs simultaneously two tasks: moving the end-effector at a desired velocity $\boldsymbol{v}_E$ and moving a midpoint in the structure, i.e., the end of link 2, at another desired velocity $\boldsymbol{v}_M$, as in Fig. 1. Formalize the problem and investigate the conditions for its solvability. When the robot is in the configuration $\boldsymbol{q} = (\pi/3, \pi/6, 0, -\pi/2)$ [rad], determine if there exists a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ realizing the two Cartesian velocities $\boldsymbol{v}_M = (-0.2, 0.1)$ [m/s] and $\boldsymbol{v}_E = (0.2, 0)$ [m/s]. If so, compute a solution. Is it unique?

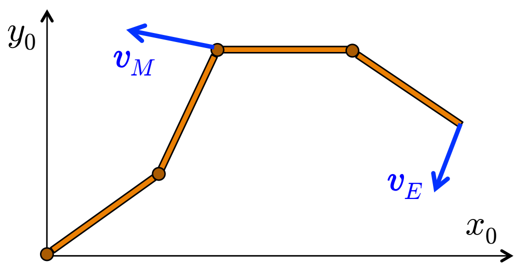

Figure 1: A 4R planar robot with a double motion task

[turn the sheet for next exercise]

Exercise 4

The end-effector of a planar robot moves in a cycle along the rectangular path $ABCD$, having short side M and long side L, placed as in Fig. 2. The robot end-effector should pass through the corner points. The Cartesian speed of the end-effector is limited above by $V_{max} > 0$, while the Cartesian acceleration is bounded in norm as $\|\ddot{\boldsymbol{p}}\| \leq A_{max} > 0$. The trajectory should start at rest from point A and return at rest to the same point at the end. The Cartesian velocity $\dot{\boldsymbol{p}}(t)$ should be continuous everywhere.

a) Determine the minimum feasible motion time T in a parametric way, sketching the speed profile along the entire path.

b) Provide the numerical value of T using the following data:

$$M = 0.4 \text{ [m]}, \qquad L = 1.6 \text{ [m]}, \qquad V_{max} = 1 \text{ [m/s]}, \qquad A_{max} = 2 \text{ [m/s}^2\text{]}.$$

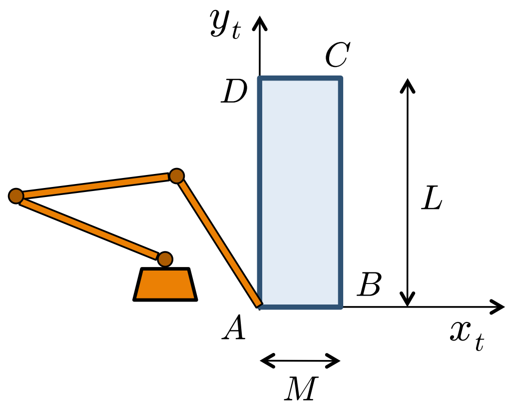

Figure 2: The cyclic rectangular path $A \to B \to C \to D \to A$

[210 minutes; open books]

Solution

February 4, 2016

Exercise 1

We need to impose orthonormality conditions to the columns of ${}^0\boldsymbol{R}_1(t)$ and check finally that $\det {}^0\boldsymbol{R}_1(t) = +1$, for all times t. The first column $\boldsymbol{r}_1$ is already of unitary norm. For the second column $\boldsymbol{r}_2$, we need to impose the unit norm condition

$$\|\boldsymbol{r}_2\|^2 = a^2(t) + \frac{k^2(t)\cos^2 t}{2} + \frac{k^2(t)\sin^2 t}{2} = a^2(t) + \frac{k^2(t)}{2} = 1 \tag{1}$$

and the condition of orthogonality $\boldsymbol{r}_2 \perp \boldsymbol{r}_1$

$$a(t)\cos t + \frac{k(t)\cos t}{\sqrt{2}}\sin t = 0.$$

The latter provides $a(t) = -k(t)\sin t/\sqrt{2}$. Substituting in (1) yields

$$\frac{k^2(t)\sin^2 t}{2} + \frac{k^2(t)}{2} = 1 \qquad \Rightarrow \qquad k(t) = \pm\sqrt{\frac{2}{1+\sin^2 t}}. \tag{2}$$

Therefore, the second column of ${}^0\boldsymbol{R}_1(t)$ is

$$\boldsymbol{r}_2 = \begin{pmatrix} \dfrac{\mp\sin t}{\sqrt{1+\sin^2 t}} & \dfrac{\pm\cos t}{\sqrt{1+\sin^2 t}} & \dfrac{\mp\sin t}{\sqrt{1+\sin^2 t}} \end{pmatrix}^T. \tag{3}$$

Similarly, for the third column $\boldsymbol{r}_3$, we impose first the orthogonality $\boldsymbol{r}_3 \perp \boldsymbol{r}_1$

$$b(t)\cos t + c(t)\sin t = 0 \qquad \Rightarrow \qquad b(t) = \alpha(t)\sin t, \quad c(t) = -\alpha(t)\cos t. \tag{4}$$

Using (3) and (4), we impose next the orthogonality $\boldsymbol{r}_3 \perp \boldsymbol{r}_2$ as[1]

$$\alpha(t)\,\frac{\sin^2 t}{\sqrt{1+\sin^2 t}} + \alpha(t)\,\frac{\cos^2 t}{\sqrt{1+\sin^2 t}} + d(t)\,\frac{\sin t}{\sqrt{1+\sin^2 t}} = 0 \quad \Rightarrow \quad \alpha(t) = -d(t)\sin t.$$

Finally, the unit norm condition provides

$$\|\boldsymbol{r}_3\|^2 = 1 \quad \Rightarrow \quad d^2(t)\left(\sin^4 t + \sin^2 t\cos^2 t + 1\right) = 1 \quad \Rightarrow \quad d(t) = \frac{\pm 1}{\sqrt{1+\sin^2 t}}. \tag{5}$$

The uncertainty left in the signs of $k(t)$ and $d(t)$, respectively in eq. (2) and eq. (5), is eliminated by imposing the determinant of ${}^0\boldsymbol{R}_1(t)$ to be equal to $+1$. This holds true when choosing either both positive signs for $k(t)$ and $d(t)$, or both negative. The first solution is

$${}^0\boldsymbol{R}_1(t) = \begin{pmatrix} \cos t & -\dfrac{\sin t}{\sqrt{1+\sin^2 t}} & -\dfrac{\sin^2 t}{\sqrt{1+\sin^2 t}} \\ \sin t & \dfrac{\cos t}{\sqrt{1+\sin^2 t}} & \dfrac{\sin t\cos t}{\sqrt{1+\sin^2 t}} \\ 0 & -\dfrac{\sin t}{\sqrt{1+\sin^2 t}} & \dfrac{1}{\sqrt{1+\sin^2 t}} \end{pmatrix}, \tag{6}$$

and corresponds to the case when ${}^0\boldsymbol{R}_1(0) = \boldsymbol{I}$. The second solution is as in (6), but with each element of the second and third column having the opposite sign.

[1] The same $\mp$ sign is factored out in all three terms, and thus eliminated as irrelevant in a homogenous equation.

Exercise 2

The given table of parameters refers to the planar RP robot in Fig. 3, where the associated Denavit-Hartenberg frames are also shown. Please note the definition of the first joint angle q_1, which differs from what one may expect (there is an additional $\pi/2$ with respect to the second link orientation).

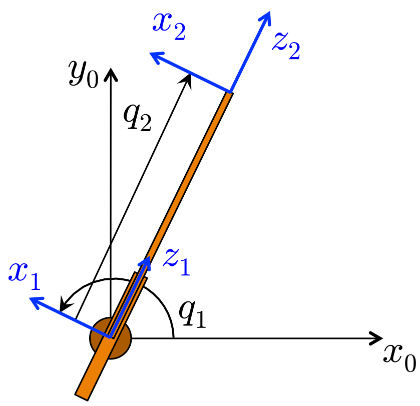

Figure 3: The RP robot, with its Denavit-Hartenberg frames and joint coordinates

The direct kinematics for the position $\boldsymbol{p}$ of the origin of frame 2 is then

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \sin q_1 \\ -q_2 \cos q_1 \end{pmatrix}.$$

Out of the singularity ($q_2 \neq 0 \Leftrightarrow \boldsymbol{p} \neq 0$), the two solutions of the inverse kinematics are analytically found as

$$q_2 = \pm \|\boldsymbol{p}\| = \pm \sqrt{p_x^2 + p_y^2}, \qquad q_1 = \text{ATAN2}\left\{\frac{p_x}{q_2}, -\frac{p_y}{q_2}\right\}. \tag{7}$$

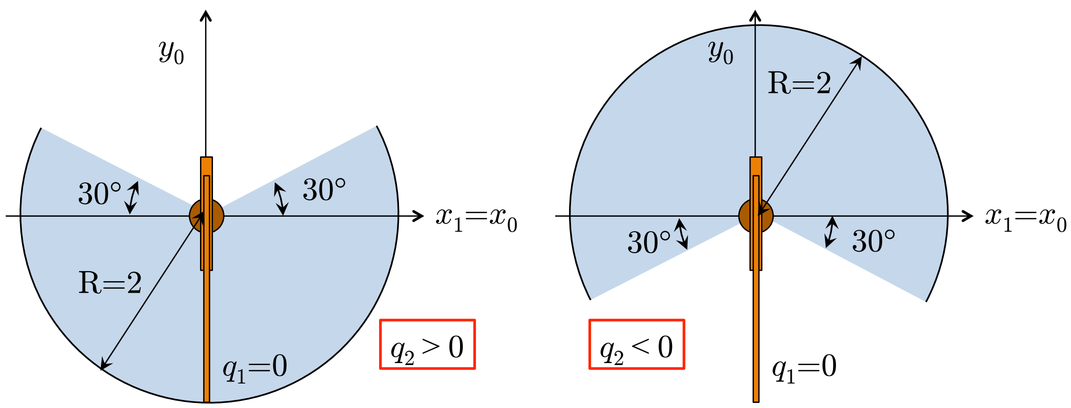

Figure 4: Robot workspace for $|q_1| \leq 120°$, $|q_2| \leq 2$, shown when $q_2 > 0$ (left) and $q_2 < 0$ (right)

For the desired position $\boldsymbol{p} = (-1, 1)$, we obtain

$$\boldsymbol{q}' = \begin{pmatrix} -\dfrac{3\pi}{4} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} -135^\circ \\ \sqrt{2} \end{pmatrix}, \qquad \boldsymbol{q}'' = \begin{pmatrix} \dfrac{\pi}{4} \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 45^\circ \\ -\sqrt{2} \end{pmatrix}.$$

Thus, only the solution $\boldsymbol{q}''$ is within the joint range. This end-effector position belongs to the robot workspace shown on the right in Fig. 4.

As a further numerical example, let the desired end-effector position be $\boldsymbol{p} = (0.25, 0.5)$ (a point in the first quadrant). From eqs. (7), we have

$$\boldsymbol{q}' = \begin{pmatrix} 150^\circ \\ \dfrac{\sqrt{3}}{2} \end{pmatrix}, \qquad \boldsymbol{q}'' = \begin{pmatrix} -30^\circ \\ -\dfrac{\sqrt{3}}{2} \end{pmatrix},$$

and the solution $\boldsymbol{q}''$ is again the only feasible one. Indeed, for any $\boldsymbol{p} \in \mathbb{R}^2$ belonging to the intersection of the two 'half' workspaces in Fig. 4 (two cones of 60° around the positive and negative $\boldsymbol{x}_0$ axis), there will be two feasible solutions to the inverse kinematics.

Exercise 3

Consider the position $\boldsymbol{p}_M$ of the midpoint along the robot structure and the position $\boldsymbol{p}_E$ of the end-effector. Use the DH joint angles and partition the four-dimensional joint configuration $\boldsymbol{q}$ into $\boldsymbol{q}_M = (q_1, q_2)$ and $\boldsymbol{q}_E = (q_3, q_4)$. The two relevant direct kinematics maps are

$$\boldsymbol{p}_M = \boldsymbol{f}_M(\boldsymbol{q}_M) = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} \\ \ell_1 s_1 + \ell_2 s_{12} \end{pmatrix} \tag{8}$$

and

$$\boldsymbol{p}_E = \boldsymbol{f}_E(\boldsymbol{q}_M, \boldsymbol{q}_E) = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} + \ell_4 c_{1234} \\ \ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + \ell_4 s_{1234} \end{pmatrix} = \boldsymbol{p}_M + \boldsymbol{p}_{ME}, \tag{9}$$

with

$$\boldsymbol{p}_{ME} = \boldsymbol{f}_{ME}(\boldsymbol{q}_M, \boldsymbol{q}_E) = \begin{pmatrix} \ell_3 c_{123} + \ell_4 c_{1234} \\ \ell_3 s_{123} + \ell_4 s_{1234} \end{pmatrix}, \tag{10}$$

and where the usual shorthand notation for trigonometric quantities (e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$) has been used.

Differentiating w.r.t. time eq. (8) and (9) yields

$$\boldsymbol{v}_M = \dot{\boldsymbol{p}}_M = \frac{\partial \boldsymbol{f}_M(\boldsymbol{q}_M)}{\partial \boldsymbol{q}_M} \dot{\boldsymbol{q}}_M = \begin{pmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}_{MM}(\boldsymbol{q}_M) \dot{\boldsymbol{q}}_M \tag{11}$$

and

$$
\begin{aligned}
\boldsymbol{v}_E = \dot{\boldsymbol{p}}_E &= \frac{\partial \boldsymbol{f}_E(\boldsymbol{q}_M, \boldsymbol{q}_E)}{\partial \boldsymbol{q}_M}\,\dot{\boldsymbol{q}}_M + \frac{\partial \boldsymbol{f}_E(\boldsymbol{q}_M, \boldsymbol{q}_E)}{\partial \boldsymbol{q}_E}\,\dot{\boldsymbol{q}}_E \\
&= \begin{pmatrix} -\left(\ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + \ell_4 s_{1234}\right) & -\left(\ell_2 s_{12} + \ell_3 s_{123} + \ell_4 s_{1234}\right) \\ \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} + \ell_4 c_{1234} & \ell_2 c_{12} + \ell_3 c_{123} + \ell_4 c_{1234} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} -\ell_3 s_{123} - \ell_4 s_{1234} & -\ell_4 s_{1234} \\ \ell_3 c_{123} + \ell_4 c_{1234} & \ell_4 c_{1234} \end{pmatrix} \begin{pmatrix} \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} \\
&= \boldsymbol{J}_{EM}(\boldsymbol{q}_M, \boldsymbol{q}_E)\dot{\boldsymbol{q}}_M + \boldsymbol{J}_{EE}(\boldsymbol{q}_M, \boldsymbol{q}_E)\dot{\boldsymbol{q}}_E.
\end{aligned}
\tag{12}
$$

Note also that, from (9) and (10),

$$
\boldsymbol{J}_{EE}(\boldsymbol{q}_M, \boldsymbol{q}_E) = \frac{\partial \boldsymbol{f}_E(\boldsymbol{q}_M, \boldsymbol{q}_E)}{\partial \boldsymbol{q}_E} = \frac{\partial \boldsymbol{f}_{ME}(\boldsymbol{q}_M, \boldsymbol{q}_E)}{\partial \boldsymbol{q}_E}.
$$

The simultaneous execution of the double task is represented by the 4×4 composite Jacobian $\boldsymbol{J}(\boldsymbol{q})$ as

$$
\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_M \\ \boldsymbol{v}_E \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{MM}(\boldsymbol{q}_M) & \boldsymbol{O} \\ \boldsymbol{J}_{EM}(\boldsymbol{q}_M, \boldsymbol{q}_E) & \boldsymbol{J}_{EE}(\boldsymbol{q}_M, \boldsymbol{q}_E) \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}}_M \\ \dot{\boldsymbol{q}}_E \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{13}
$$

The block triangular structure of $\boldsymbol{J}$ indicates that the problem is solvable for any pair of generic desired velocities $\boldsymbol{v}_E \in \mathbb{R}^2$ and $\boldsymbol{v}_M \in \mathbb{R}^2$ if and only if the two diagonal blocks $\boldsymbol{J}_{MM}$ and $\boldsymbol{J}_{EE}$ are both nonsingular. It is easy to see that $\boldsymbol{J}_{MM}$ is the Jacobian of the 2R robot sub-structure made by the first two links. Thus

$$
\det \boldsymbol{J}_{MM}(\boldsymbol{q}_M) = 0 \qquad \Longleftrightarrow \qquad q_2 = 0 \text{ (stretched) or } \pi \text{ (folded)}. \tag{14}
$$

On the other hand, the block $\boldsymbol{J}_{EE}$ can be expressed in the DH frame 2, i.e., premultiplied by the transpose of the 2×2 (planar) rotation matrix ${}^0\boldsymbol{R}_2(\boldsymbol{q}_M)$, resulting in

$$
\begin{aligned}
{}^0\boldsymbol{R}_2^T(\boldsymbol{q}_M)\boldsymbol{J}_{EE}(\boldsymbol{q}_M, \boldsymbol{q}_E) &= \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} -\ell_3 s_{123} - \ell_4 s_{1234} & -\ell_4 s_{1234} \\ \ell_3 c_{123} + \ell_4 c_{1234} & \ell_4 c_{1234} \end{pmatrix} \\
&= \begin{pmatrix} -\ell_3 s_3 - \ell_4 s_{34} & -\ell_4 s_{34} \\ \ell_3 c_3 + \ell_4 c_{34} & \ell_4 c_{34} \end{pmatrix}.
\end{aligned}
$$

Therefore, we recognize that the singularities of $\boldsymbol{J}_{EE}$ are those of the Jacobian of the 2R robot sub-structure made by the last two links, or

$$
\det \boldsymbol{J}_{EE}(\boldsymbol{q}) = 0 \qquad \Longleftrightarrow \qquad q_4 = 0 \text{ (stretched) or } \pi \text{ (folded)}. \tag{15}
$$

When none of the singularity conditions (14) and (15) holds, the solution to (13) is given by blockwise inversion of matrix $\boldsymbol{J}$

$$
\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v} = \begin{pmatrix} \boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M) & \boldsymbol{O} \\ -\boldsymbol{J}_{EE}^{-1}(\boldsymbol{q})\boldsymbol{J}_{EM}(\boldsymbol{q})\boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M) & \boldsymbol{J}_{EE}^{-1}(\boldsymbol{q}) \end{pmatrix} \boldsymbol{v} \tag{16}
$$

or

$$
\dot{\boldsymbol{q}}_M = \boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M)\boldsymbol{v}_M, \qquad \dot{\boldsymbol{q}}_E = \boldsymbol{J}_{EE}^{-1}(\boldsymbol{q})\left(\boldsymbol{v}_E - \boldsymbol{J}_{EM}(\boldsymbol{q})\dot{\boldsymbol{q}}_M\right). \tag{17}
$$

Note that the term in the last parentheses in (17) represents the part of the desired end-effector velocity that is still missing, once the contribution given by the velocity $\dot{\boldsymbol{q}}_M$ of the first two joints has been taken into account.

Turning now to the numerical evaluation, the configuration $\boldsymbol{q} = (\pi/3, \pi/6, 0, -\pi/2)$ is shown in Fig. 5 and is clearly nonsingular.

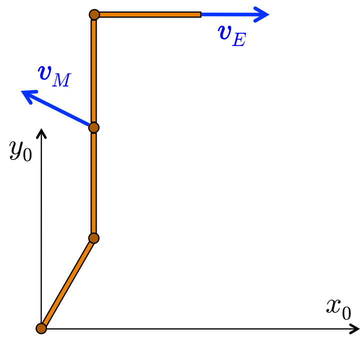

Figure 5: The 4R planar robot in the configuration $\boldsymbol{q} = (\pi/3, \pi/6, 0, -\pi/2)$ with the prescribed double motion task $\boldsymbol{v}_M = (-0.2, 0.1)$ and $\boldsymbol{v}_E = (0.2, 0)$

Using $\ell_i = 0.25$, $i = 1, \dots, 4$, the blocks of the complete Jacobian are

$$\boldsymbol{J}_{MM} = \begin{pmatrix} -0.4665 & -0.25 \\ 0.125 & 0 \end{pmatrix}, \quad \boldsymbol{J}_{EM} = \begin{pmatrix} -0.7165 & -0.5 \\ 0.375 & 0.25 \end{pmatrix}, \quad \boldsymbol{J}_{EE} = \begin{pmatrix} -0.25 & 0 \\ 0.25 & 0.25 \end{pmatrix}.$$

The joint velocity $\dot{\boldsymbol{q}}$ realizing the two Cartesian velocities $\boldsymbol{v}_M = (-0.2, 0.1)$ and $\boldsymbol{v}_E = (0.2, 0)$ are computed as in (17), yielding

$$\dot{\boldsymbol{q}}_M = \begin{pmatrix} 0.8 \\ -0.6928 \end{pmatrix} \text{[rad/s]}, \qquad \dot{\boldsymbol{q}}_E = \begin{pmatrix} -1.7072 \\ 1.2 \end{pmatrix} \text{[rad/s]}, \qquad \dot{\boldsymbol{q}} = \begin{pmatrix} \dot{\boldsymbol{q}}_M \\ \dot{\boldsymbol{q}}_E \end{pmatrix} \in \mathbb{R}^4. \tag{18}$$

This solution is indeed unique.

Final note. A more complex approach to determine the solution would have been the following. Let the solution to the first task be $\dot{\boldsymbol{q}}_M = \boldsymbol{J}_{MM}^{-1}(\boldsymbol{q}_M)\boldsymbol{v}_M$ and consider the second (redundant) task

$$\boldsymbol{J}_E(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{J}_{EM}(\boldsymbol{q}) & \boldsymbol{J}_{EE}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}}_M \\ \dot{\boldsymbol{q}}_E \end{pmatrix} = \boldsymbol{v}_E, \tag{19}$$

where the Jacobian $\boldsymbol{J}_E(\boldsymbol{q})$ is a 2×4 matrix. All solutions to (19) can be written as

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{\boldsymbol{q}}_M^* \\ \dot{\boldsymbol{q}}_E^* \end{pmatrix} = \boldsymbol{J}_E^{\#}(\boldsymbol{q})\boldsymbol{v}_E + \left(\boldsymbol{I} - \boldsymbol{J}_E^{\#}(\boldsymbol{q})\boldsymbol{J}_E(\boldsymbol{q})\right)\dot{\boldsymbol{q}}_0, \qquad \text{with arbitrary } \dot{\boldsymbol{q}}_0 \in \mathbb{R}^4. \tag{20}$$

The first term in (20) is the minimum norm joint velocity solution given by the pseudoinverse of the Jacobian $\boldsymbol{J}_E$. The second term is a joint velocity vector belonging to the null space $\mathcal{N}\{\boldsymbol{J}_E\}$ of $\boldsymbol{J}_E$, thanks to the presence of the projection matrix $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}_E^{\#}\boldsymbol{J}_E$. The null space is explored by changing the generic joint velocity $\dot{\boldsymbol{q}}_0$. For $\dot{\boldsymbol{q}}_0 = \boldsymbol{0}$, the upper part $\dot{\boldsymbol{q}}_M^*$ of the minimum norm solution obtained will differ in general from the solution found for the first task, $\dot{\boldsymbol{q}}_M^* \neq \boldsymbol{J}_{MM}^{-1}\boldsymbol{v}_M$, showing an incompatibility at the level of the velocities of the first two joints. This is what happens in fact with the given numerical data:

$$\dot{\boldsymbol{q}}^* = \boldsymbol{J}_E^{\#}(\boldsymbol{q})\boldsymbol{v}_E = \begin{pmatrix} -0.2037 & -0.1591 & 0.1018 & 0.3627 \end{pmatrix}^T,$$

which differs in the first two components from (18). However, there exists indeed a choice of $\dot{\boldsymbol{q}}_0$ in (20) that will provide a fully consistent solution. This is guaranteed by the fact that we found already the solution (18) to our simultaneous double velocity task problem. For the case study, setting for instance

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 1.0037 & -0.5337 & -1.8090 & 0.8373 \end{pmatrix}^T$$

in (20) will provide back the solution (18). We note also that $\dot{\boldsymbol{q}}_0 \in \mathcal{N}\{\boldsymbol{J}_E\}$, and thus $\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}_0$.

Exercise 4

The problem addressed in the Cartesian space. To guarantee continuity of the end-effector velocity $\boldsymbol{p}(t)$ during the entire motion, it is necessary to stop at each of the path corners B, C, and D (because the tangent to the path is discontinuous there). Therefore, we can treat separately each side of the rectangle. The minimum time motion along a side will have either a trapezoidal speed profile or a (degenerate) bang-bang acceleration profile. The type of profile will be identical on two opposite sides, since it depends only on the length of the segment (M or L), once V_{max} and A_{max} are assigned. In order for a 'coast' phase to exist (i.e., the maximum admissible speed is reached, at least for one instant) on each of the four sides, it is necessary and sufficient that

$$\textbf{Case I:} \quad M \geq \frac{V_{max}^2}{A_{max}} \text{ (on the short sides)} \quad \Rightarrow \quad L \geq M \geq \frac{V_{max}^2}{A_{max}} \text{ (\textbf{also} on the long sides).}$$

Conversely, the profiles on all sides will be of the bang-bang acceleration type if and only if

$$\textbf{Case II:} \quad L \leq \frac{V_{max}^2}{A_{max}} \text{ (on the long sides)} \quad \Rightarrow \quad M \leq L \leq \frac{V_{max}^2}{A_{max}} \text{ (\textbf{also} on the short sides).}$$

Indeed, a mixed situation occurs when

$$\textbf{Case III:} \quad M \leq \frac{V_{max}^2}{A_{max}} \leq L \quad \text{(bang-bang on short sides, trapezoidal speed on long sides).}$$

From the known expression of the minimum time needed for a rest-to-rest motion along a straight path of length δ with a trapezoidal speed profile

$$T_\delta = \frac{\delta A_{max} + V_{max}^2}{A_{max}V_{max}}, \qquad \text{for } \delta = \{M, L\},$$

the motion time in **Case I** will be:

$$T = 2\left(\frac{MA_{max} + V_{max}^2}{A_{max}V_{max}} + \frac{LA_{max} + V_{max}^2}{A_{max}V_{max}}\right) = \frac{2(M+L)A_{max} + 4V_{max}^2}{A_{max}V_{max}}. \tag{21}$$

For **Case II**, the velocity profile on each side will be triangular, with maximum acceleration and deceleration phases. Let T_Δ be the travel time on one of the sides. At the mid time $t = T_\Delta/2$, the peak speed $A_{max}(T_\Delta/2)$ is reached. The displacement will be equal to $\frac{1}{2}A_{max}(T/2)^2$, where half of the length of the side has been traced. Therefore,

$$\frac{1}{2}A_{max}(T_\Delta/2)^2 = \frac{\Delta}{2} \qquad \Rightarrow \qquad T_\Delta = 2\sqrt{\frac{\Delta}{A_{max}}}, \qquad \text{for } \ \Delta = \{M, L\},$$

and the total motion time will be

$$T = 2\left(2\sqrt{\frac{M}{A_{max}}} + 2\sqrt{\frac{L}{A_{max}}}\right) = 4\,\frac{\sqrt{M}+\sqrt{L}}{\sqrt{A_{max}}}. \tag{22}$$

Finally, **Case III** will be a combination of the two formulas (21) and (22). Thus,

$$T = 2\,\frac{LA_{max} + V_{max}^2}{A_{max}V_{max}} + 4\sqrt{\frac{M}{A_{max}}}. \tag{23}$$

Using the numerical data, we see that **Case III** applies since

$$M = 0.4 < \left(\frac{V_{max}^2}{A_{max}} = \frac{1}{2} = \right) 0.5 < 1.6 = L.$$

From (23), the total travel time is then $T = 5.989$ s.

Note that the total length of the rectangular path is $2(M+L) = 4$ [m]; if we could trace it always at maximum speed $V_{max} = 1$ m/s from the beginning to its end, this would take $T_{ideal} = 4$ s. Because of the limited acceleration and of the required continuity of velocity, motion lasts about 50% longer than in the ideal (but not realizable) limit.

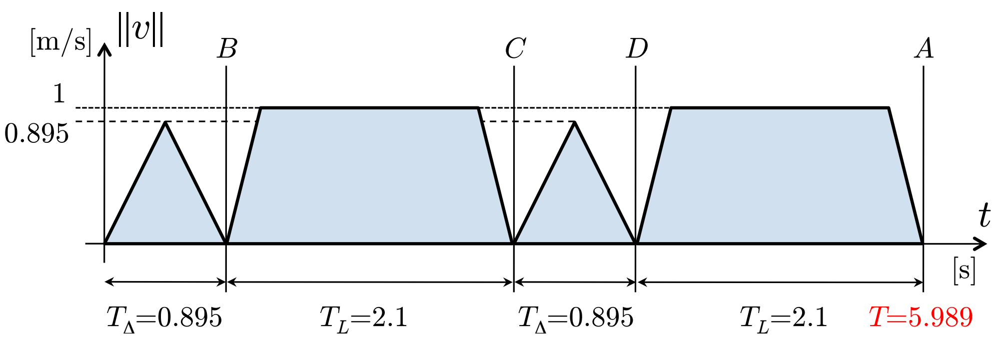

Figure 6: Time profile of the scalar speed along the rectangular path

Figure 6 gives the profile of the (scalar) speed along the entire rectangular path. Note that this speed is always non-negative. Figure 7 reports the associated profiles of the v_x and v_y components of the Cartesian velocity $\boldsymbol{v} = \dot{\boldsymbol{p}}$. Indeed, continuity is enforced at all times.

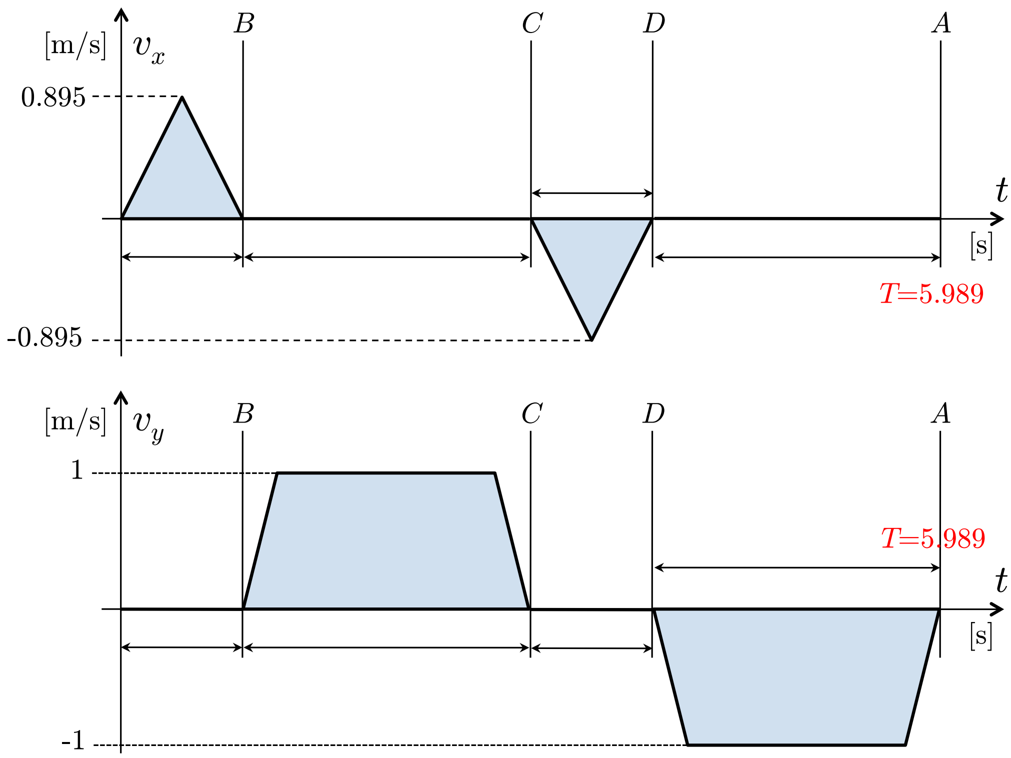

Figure 7: Time profiles of the components of the Cartesian velocity $\boldsymbol{v}$ along the rectangular path of Fig. 2: v_x (top) and v_y (bottom)

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