Robotics I

October 27, 2015

Exercise 1

Consider a planar 2R robot with links of equal length $\ell_1 = \ell_2 = 0.5$ [m] and assume that the robot controller generates a *joint acceleration* command $\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 & \ddot{q}_2 \end{pmatrix}^T$. While in motion, at a given instant the robot end-effector P reaches the position $\boldsymbol{p} = \begin{pmatrix} 0.6 & 0.2 \end{pmatrix}^T$ [m] with a Cartesian velocity $\dot{\boldsymbol{p}} = \begin{pmatrix} -0.5 & 0.5 \end{pmatrix}^T$ [m/s]. In this state, the task requires a desired Cartesian acceleration $\ddot{\boldsymbol{p}}_d = \begin{pmatrix} -0.7 & 1 \end{pmatrix}^T$ [m/s^2].

- Describe in general the procedure for obtaining the command $\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_d$ that realizes the desired task acceleration $\ddot{\boldsymbol{p}}_d$ using only the above information. Is the solution unique? Does it always exist? Explain your answers.
- Compute numerically $\ddot{\boldsymbol{q}}_d$ with the given data.

Exercise 2

In the transmission gears shown in Fig. 1, the four toothed wheels W_i, each of radius $r_i > 0$ (for $i = 1, \dots, 4$), are perfectly engaged each to other.

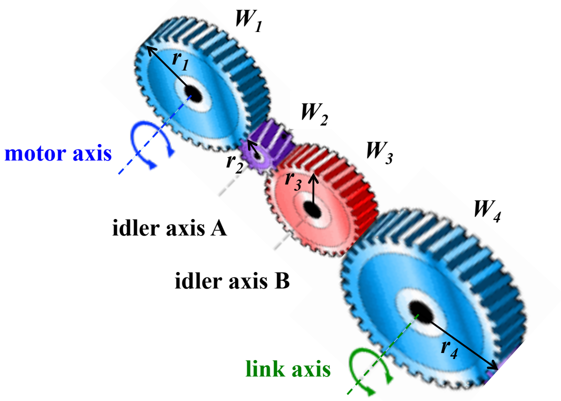

Figure 1: A sequence of four toothed wheels in a motor-to-link transmission

- Using the available information, determine the expression of the total transmission ratio N between the angular speed $\dot{\theta}_1$ of the motor axis and the angular speed $\dot{\theta}_4$ of the link axis. Is the link rotating in the same direction (CW or CCW) of the motor, or in the opposite one? Is there any restriction among the (positive) values of the radiuses r_i in order to have an increase of the torque produced when passing from the motor axis to the link axis?
- Compute the value of N when

$$r_1 = 0.025, \quad r_2 = 0.01, \quad r_3 = 0.02, \quad r_4 = 0.035 \qquad \text{[m]}.$$

What happens to N if we double the radius of both wheels W_2 and W_3 (possibly redesigning the teeth)? Provide any comments you may have on this issue.

[120 minutes; open books]

Solution

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Exercise 1

In the absence of redundancy and away from kinematic singularities, the solution is found through the following procedure.

1. Solve the inverse kinematics for the current Cartesian position $\boldsymbol{p}$. We shall have multiple (analytic or numerical) solutions, each in the form $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p})$.
2. For each actual joint configuration $\boldsymbol{q}$ found in this way, evaluate the (square and nonsingular) Jacobian $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q})$ and solve for the actual joint velocity $\dot{\boldsymbol{q}}$ from the current Cartesian position $\dot{\boldsymbol{p}}$ as $\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{p}}$.
3. The second-order differential kinematics is given by $\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Therefore, using the $\boldsymbol{q}$ and associated $\dot{\boldsymbol{q}}$ obtained so far, compute the vector $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ and solve for the joint acceleration command associated to the desired Cartesian acceleration $\ddot{\boldsymbol{p}}_d$ as $\ddot{\boldsymbol{q}}_d = \boldsymbol{J}^{-1}\left(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}\right)$.

The solution is not unique (because of the multiplicity of solutions in step 1). At least one solution exists, provided that $\boldsymbol{p}$ belongs to the robot workspace. When a singularity is encountered, the problem may degenerate at step 2 or at step 3. When the Cartesian velocity $\dot{\boldsymbol{p}}$ is not in the range of the Jacobian $\boldsymbol{J}$, i.e., $\dot{\boldsymbol{p}} \notin \mathcal{R}\{\boldsymbol{J}\}$, or, similarly, when the modified Cartesian acceleration vector $\ddot{\boldsymbol{p}}_d - \boldsymbol{n} \notin \mathcal{R}\{\boldsymbol{J}\}$, there will be no solution.

Applying the above procedure to the given planar 2R robot and data yields the following solution.

Step 1. From the direct kinematics

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2) \end{pmatrix}, \tag{1}$$

two inverse kinematic solutions, labeled $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$, are obtained from the analytic formulas

$$q_2 = \text{ATAN2}\left\{s_2, c_2\right\}, \quad \text{with } c_2 = \frac{p_x^2 + p_y^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}, \quad s_2 = \pm\sqrt{1 - c_2^2} \tag{2}$$

and

$$q_1 = \text{ATAN2}\left\{s_1, c_1\right\}, \quad \text{with } c_1 = p_x\left(\ell_1 + \ell_2 c_2\right) + p_y \ell_2 s_2, \quad s_1 = p_y\left(\ell_1 + \ell_2 c_2\right) - p_x \ell_2 s_2.$$

We denote by $\boldsymbol{q}_a = \begin{pmatrix} q_{1a} & q_{2a} \end{pmatrix}^T$ the solution that considers the '+' sign for s_2 in (2). Similarly, $\boldsymbol{q}_b$ will be the solution corresponding to the choice of the '−' sign for s_2.

Using the problem data, we obtain

$$\boldsymbol{q}_a = \begin{pmatrix} -0.5643 \\ 1.7722 \end{pmatrix} \text{[rad]} \qquad \boldsymbol{q}_b = \begin{pmatrix} 1.2078 \\ -1.7722 \end{pmatrix} \text{[rad]},$$

or in degrees

$$\boldsymbol{q}_a = \begin{pmatrix} -32.3335° \\ 101.5370° \end{pmatrix}, \qquad \boldsymbol{q}_b = \begin{pmatrix} 69.2034° \\ -101.5370° \end{pmatrix}.$$

Step 2. The 2×2 robot Jacobian matrix can be written as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell_1 \sin q_1 - \ell_2 \sin(q_1+q_2) & -\ell_2 \sin(q_1+q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1+q_2) & \ell_2 \cos(q_1+q_2) \end{pmatrix} = \begin{pmatrix} -p_y & -\ell_2 \sin(q_1+q_2) \\ p_x & \ell_2 \cos(q_1+q_2) \end{pmatrix},$$

where the direct kinematic relationship (1) has been used. Its evaluation gives

$$\boldsymbol{J}_a = \boldsymbol{J}(\boldsymbol{q}_a) = \begin{pmatrix} -0.2 & -0.4674 \\ 0.6 & 0.1775 \end{pmatrix}, \quad \boldsymbol{J}_b = \boldsymbol{J}(\boldsymbol{q}_b) = \begin{pmatrix} -0.2 & 0.2674 \\ 0.6 & 0.4225 \end{pmatrix}.$$

We obtain then the two joint velocities

$$\dot{\boldsymbol{q}}_a = \boldsymbol{J}_a^{-1}\dot{\boldsymbol{p}} = \begin{pmatrix} 0.5918 \\ 0.8165 \end{pmatrix} \text{[rad/s]} \qquad \dot{\boldsymbol{q}}_b = \boldsymbol{J}_b^{-1}\dot{\boldsymbol{p}} = \begin{pmatrix} 1.4082 \\ -0.8165 \end{pmatrix} \text{[rad/s]}.$$

Step 3. The time derivative of the robot Jacobian (still a 2×2 matrix) takes the form

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{p}_y & -\ell_2 \cos(q_1+q_2)\cdot(\dot{q}_1+\dot{q}_2) \\ \dot{p}_x & -\ell_2 \sin(q_1+q_2)\cdot(\dot{q}_1+\dot{q}_2) \end{pmatrix}.$$

Evaluating terms leads to

$$\dot{\boldsymbol{J}}_a = \dot{\boldsymbol{J}}(\boldsymbol{q}_a) = \begin{pmatrix} -0.5 & -0.25 \\ -0.5 & -0.6582 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_b = \dot{\boldsymbol{J}}(\boldsymbol{q}_b) = \begin{pmatrix} -0.5 & -0.25 \\ -0.5 & 0.1582 \end{pmatrix},$$

and thus

$$\boldsymbol{n}_a = \dot{\boldsymbol{J}}_a \dot{\boldsymbol{q}}_a = \begin{pmatrix} -0.5 \\ -0.8333 \end{pmatrix}, \qquad \boldsymbol{n}_b = \dot{\boldsymbol{J}}_b \dot{\boldsymbol{q}}_b = \begin{pmatrix} -0.5 \\ -0.8333 \end{pmatrix}.$$

We obtain finally the two joint acceleration commands as

$$\ddot{\boldsymbol{q}}_{d,a} = \boldsymbol{J}_a^{-1}\left(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}_a\right) = \begin{pmatrix} 3.3535 \\ -1.0070 \end{pmatrix} \text{[rad/s}^2\text{]} \qquad \ddot{\boldsymbol{q}}_{d,b} = \boldsymbol{J}_b^{-1}\left(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}_b\right) = \begin{pmatrix} 2.3465 \\ 1.0070 \end{pmatrix} \text{[rad/s}^2\text{]}.$$

Exercise 2

Since the velocity of the contact point between two successive wheels W_i and W_{i+1} is the same, we have for each transmission ratio of the sub-gears

$$\dot{\theta}_i r_i = \dot{\theta}_{i+1} r_{i+1} \quad \Rightarrow \quad N_i = \frac{\dot{\theta}_i}{\dot{\theta}_{i+1}} = \frac{r_{i+1}}{r_i}, \qquad i = 1,2,3. \tag{3}$$

The total transmission ratio is thus the product of the single transmission ratios N_i:

$$N = \frac{\dot{\theta}_1}{\dot{\theta}_4} = \frac{\dot{\theta}_1}{\dot{\theta}_2}\cdot\frac{\dot{\theta}_2}{\dot{\theta}_3}\cdot\frac{\dot{\theta}_3}{\dot{\theta}_4} = N_1 N_2 N_3. \tag{4}$$

When considering the direction of rotation, each pair of wheels inverts the motion (from CW to CCW, and vice versa). With three pairs of wheels, the link axis will rotate in the opposite direction of the motor axis. Indeed, using the second expression of N_i in (3), we can also write that

$$N = N_1 N_2 N_3 = \frac{r_2}{r_1}\cdot\frac{r_3}{r_2}\cdot\frac{r_4}{r_3} = \frac{r_4}{r_1}. \tag{5}$$

Thus, the total transmission ratio will be independent from the radius of any of the intermediate wheels (each rotating around its own axis, also called *idler* axis). In order to have $N > 1$, i.e., a reduction of the speed by $1/N$ from the motor to the link axis with an associated increase by N of the produced torque, it is necessary and sufficient that $r_4 > r_1$. Using the given data in (5), we have

$$N = \frac{0.035}{0.025} = 1.4\,. \tag{6}$$

Doubling the radius of the two idler wheels will produce no change in N. In general, we should limit the number of intermediate gears because of the additional friction at multiple contacts (dissipating power) and the increased inertia (resulting in a reduced acceleration available at the link axis for a given torque produced on the motor axis).

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