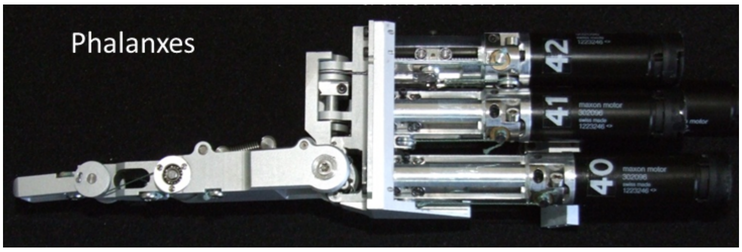
# Robotics I

**February 6, 2013**

## Exercise 1



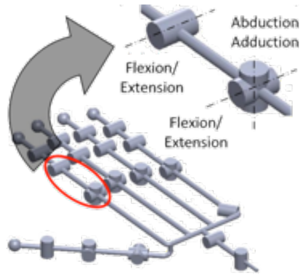




Figure 1: A 4-DOF finger of a robotic hand (left) and the kinematics of the full hand (right). The first three DOFs of the index finger are circled in red

Figure 1 shows a picture of the index finger of a prototype robotic hand, as well as the kinematics scheme of the full hand. The index finger has 4 revolute degrees of freedom (DOFs), two at its base, allowing abduction/adduction and flexion/extension, and two other for flexion/extension of the phalanxes. For simplicity, assume that each joint is independently actuated. A reference frame is placed at the base of the finger, attached to the palm of the hand (see Fig. 2).

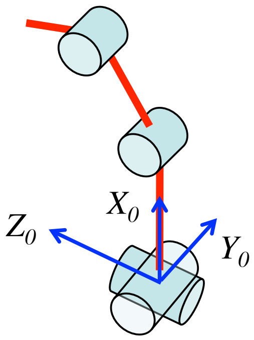



Figure 2: Kinematics scheme of the 4-DOF index finger

**a)** Assign a set of Denavit-Hartenberg frames to the robotic finger and define the associated table of parameters, introducing symbolic quantities as needed. The origin of the last frame should be located at the tip of the finger.

**b)** By mimicking the mobility of the index finger of your hand, define reasonable numerical values for the joint ranges of the robotic finger.

**c)** Derive the $6 \times 4$ geometric Jacobian of the robotic finger in symbolic form.

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**Exercise 2**

Consider the class of trigonometric functions of time

$$q(t) = \sum_{h=1}^{m} \left[ a_h \sin\left( (2h-1)\,\frac{\pi}{2}\frac{t}{T} \right) + b_h \cos\left( (2h-1)\,\frac{\pi}{2}\frac{t}{T} \right) \right], \qquad t \in [0, T] \tag{1}$$

parametrized by the $2m$ coefficients $a_h$, $b_h$, for $h = 1, \ldots, m$, and let a trajectory planning problem be specified by the following boundary conditions to be interpolated

$$q(0) = q_0, \quad q(T) = q_1, \quad \dot{q}(0) = v_0, \quad \dot{q}(T) = v_1, \tag{2}$$

where $T > 0$ is the motion time.

**a)** Address problem (2) using (1), so that a solution trajectory $q(t)$ always exists and is uniquely specified by the given data. Write a program that computes such solution and plots the resulting position, velocity, and acceleration profiles.

**b)** Test your program on the two data sets:

*i)* $q_0 = -40°$, $q_1 = 40°$, $v_0 = v_1 = 0$, $T = 2$ s;

*ii)* $q_0 = 20°$, $q_1 = 60°$, $v_0 = v_1 = 0$, $T = 2$ s.

Verify that in the second case the resulting trajectory *wanders*[1], while no under- or over-shoot occurs to the position profile in the first case. Which is the source of this different behavior?

**c)** Expand the class of trajectories $q(t)$ in (1) by adding a constant term, namely

$$q'(t) = \bar{q} + q(t). \tag{3}$$

The previous method can be modified in a simple way so that, at least in the rest-to-rest case ($v_0 = v_1 = 0$), the solution $q'(t)$ will always be a non-wandering trajectory. Show how to obtain this, and test the modified solution again on case *ii)* above.

**[180 minutes; open books]**

[1] In general, *wandering* in case of a rest-to-rest trajectory between two points $q_0$ and $q_1$ means that the position exceeds the interval $[\min\{q_0, q_1\}, \max\{q_0, q_1\}]$ at some instant during motion.

# Solutions

February 6, 2013

**Exercise 1**

Figure 3 shows a possible assignment of Denavit-Hartenberg frames for the robotic finger. Table 1 contains the associated parameters, where $L_2$, $L_3$, and $L_4$ are the lengths of the three phalanxes of the finger.

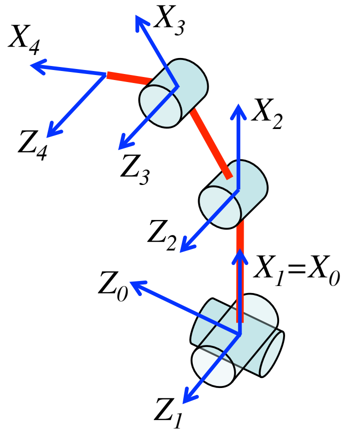


Figure 3: Denavit-Hartenberg frames for the 4-DOF index finger

| $i$ | $\alpha_i$ | $d_i$ | $a_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $\pi/2$ | 0 | 0 | $q_1$ |
| 2 | 0 | 0 | $L_2$ | $q_2$ |
| 3 | 0 | 0 | $L_3$ | $q_3$ |
| 4 | 0 | 0 | $L_4$ | $q_4$ |

Table 1: Denavit-Hartenberg parameters associated to Fig. 3

With the above choices, the zero configuration $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T = \mathbf{0}$ corresponds to the finger pointing straight upward. Accordingly, possible estimates of the joint ranges that mimic the mobility of a human index finger are

$$q_1 \in [-20°, +20°]\,, \quad q_2 \in [-10°, +80°]\,, \quad q_2 \in [0°, +95°]\,, \quad q_4 \in [0°, +45°]\,.$$

The desired $6 \times 4$ Jacobian matrix is most efficiently computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{p}_{04}}{\partial q_1} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_2} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_3} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_4} \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix},$$

namely:

- for the first three rows (linear components), by analytic derivation of the finger tip position vector $\boldsymbol{p}_{04}$;
- for the last three rows (angular components), by the standard geometric expressions for revolute joints.

From Table 1, we obtain the homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} \boldsymbol{R}_1(q_1) & \boldsymbol{p}_{01} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{A}_i(q_i) = \begin{pmatrix} \boldsymbol{R}_i(q_i) & \boldsymbol{p}_{i-1,i}(q_i) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_i & -\sin q_i & 0 & L_i \cos q_i \\ \sin q_i & \cos q_i & 0 & L_i \sin q_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for } i = 2, 3, 4.$$

The position of the tip finger is then

$$\begin{aligned} \boldsymbol{p}_{04,\text{hom}}(\boldsymbol{q}) &= \boldsymbol{A}_1(q_1) \left( \boldsymbol{A}_2(q_2) \left( \boldsymbol{A}_3(q_3) \left( \boldsymbol{A}_4(q_4) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right)\right)\right) \\ &= \begin{pmatrix} \cos q_1 \left( L_2 \cos q_2 + L_3 \cos(q_2 + q_3) + L_4 \cos(q_2 + q_3 + q_4) \right) \\ \sin q_1 \left( L_2 \cos q_2 + L_3 \cos(q_2 + q_3) + L_4 \cos(q_2 + q_3 + q_4) \right) \\ L_2 \sin q_2 + L_3 \sin(q_2 + q_3) + L_4 \sin(q_2 + q_3 + q_4) \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_{04}(\boldsymbol{q}) \\ 1 \end{pmatrix}. \end{aligned}$$

In the first row above, brackets have been used to indicate the most convenient order of products, especially for symbolic computations.

Using the usual compact notations, we obtain $\boldsymbol{J}_L(\boldsymbol{q})$ as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_{04}(\boldsymbol{q})}{\partial \boldsymbol{q}} =$$

$$\begin{pmatrix} -(L_2 c_2 + L_3 c_{23} + L_4 c_{234}) s_1 & -(L_2 s_2 + L_3 s_{23} + L_4 s_{234}) c_1 & -(L_3 s_{23} + L_4 s_{234}) c_1 & -L_4 s_{234} c_1 \\ (L_2 c_2 + L_3 c_{23} + L_4 c_{234}) c_1 & -(L_2 s_2 + L_3 s_{23} + L_4 s_{234}) s_1 & -(L_3 s_{23} + L_4 s_{234}) s_1 & -L_4 s_{234} s_1 \\ 0 & L_2 c_2 + L_3 c_{23} + L_4 c_{234} & L_3 c_{23} + L_4 c_{234} & L_4 c_{234} \end{pmatrix},$$

while for $\boldsymbol{J}_A(\boldsymbol{q})$ we have

$$\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_1 = \boldsymbol{R}_1(q_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{z}_2 = \boldsymbol{R}_1(q_1)\boldsymbol{R}_2(q_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{z}_3 = \boldsymbol{R}_1(q_1)\boldsymbol{R}_2(q_2)\boldsymbol{R}_3(q_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}.$$

Indeed, $\boldsymbol{z}_1 = \boldsymbol{z}_2 = \boldsymbol{z}_3$ since the three joints axes 2, 3, and 4 are always parallel.

**Exercise 2**

By considering $m = 2$ in eq. (1), the function $q(t)$ will contain the four coefficients $a_1$, $b_1$, $a_2$, and $b_2$, which are necessary and also sufficient, as we will see, to impose four arbitrary boundary conditions in (2). We have thus

$$q(t) = a_1 \sin\left(\frac{\pi}{2}\frac{t}{T}\right) + a_2 \sin\left(\frac{3\pi}{2}\frac{t}{T}\right) + b_1 \cos\left(\frac{\pi}{2}\frac{t}{T}\right) + b_2 \cos\left(\frac{3\pi}{2}\frac{t}{T}\right),$$

and

$$\dot{q}(t) = \left(\frac{\pi}{2T}\right) \cdot \left(a_1 \cos\left(\frac{\pi}{2}\frac{t}{T}\right) + 3a_2 \cos\left(\frac{3\pi}{2}\frac{t}{T}\right) - b_1 \sin\left(\frac{\pi}{2}\frac{t}{T}\right) - 3b_2 \sin\left(\frac{3\pi}{2}\frac{t}{T}\right)\right).$$

Note that the arguments of the trigonometric terms are properly scaled with respect to the motion time $T$, so that all these terms take only the values 0, +1 or −1 at the boundaries $t = 0$ and $t = T$. Therefore,

$$q(0) = q_0 \quad \Rightarrow \quad b_1 + b_2 = q_0, \qquad q(T) = q_1 \quad \Rightarrow \quad a_1 - a_2 = q_1,$$

and

$$\dot{q}(0) = v_0 \quad \Rightarrow \quad a_1 + 3a_2 = v_0 \frac{2T}{\pi}, \qquad \dot{q}(T) = v_1 \quad \Rightarrow \quad -b_1 + 3b_2 = v_1 \frac{2T}{\pi}.$$

The resulting linear system of equations has a simple block diagonal structure

$$\begin{pmatrix} 1 & -1 & & \\ 1 & 3 & & \\ & & 1 & 1 \\ & & -1 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ (2T/\pi)\, v_0 \\ q_0 \\ (2T/\pi)\, v_1 \end{pmatrix}$$

and is always solvable (the determinant of both $2 \times 2$ blocks is equal to 4), leading to

$$\begin{aligned} a_1 &= \frac{1}{4}\left(3q_1 + (2T/\pi)\, v_0\right) & a_2 &= \frac{1}{4}\left((2T/\pi)\, v_0 - q_1\right) \\ b_1 &= \frac{1}{4}\left(3q_0 - (2T/\pi)\, v_1\right) & b_2 &= \frac{1}{4}\left(q_0 + (2T/\pi)\, v_1\right). \end{aligned} \tag{4}$$

We note that, by increasing the order $m$, additional boundary conditions on higher order derivatives could be handled similarly (e.g., with $m = 3$, we can match also desired initial and final accelerations). Just for reference, the expression of the time derivative of (1) for a generic $m$ is

$$\dot{q}(t) = \frac{\pi}{2T} \sum_{h=1}^{m} (2h-1) \left[a_h \cos\left((2h-1)\frac{\pi}{2}\frac{t}{T}\right) - b_h \cos\left((2h-1)\frac{\pi}{2}\frac{t}{T}\right)\right], \qquad t \in [0, T]$$

and the four boundary conditions (2) at $t = 0$ and $t = T$ are written in general as

$$\sum_{h=1}^{m} b_h = q_0, \quad \sum_{h=1}^{m} (-1)^{h-1} a_h = q_1, \quad \sum_{h=1}^{m} (2h-1) a_h = v_0 \frac{2T}{\pi}, \quad \sum_{h=1}^{m} (2h-1)(-1)^h b_h = v_1 \frac{2T}{\pi}.$$

While this flexibility is a nice feature of the considered class of trajectories, there is still an issue concerning the full predictability of the interpolating motion.

For case *i)*, which is a rest-to-rest motion in $T = 2$ s from $q_0 = -40°$ to $q_1 = 40°$, Figure 4 shows the position and velocity profiles of the solution trajectory, obtained for $a_1 = -b_1 = 30$

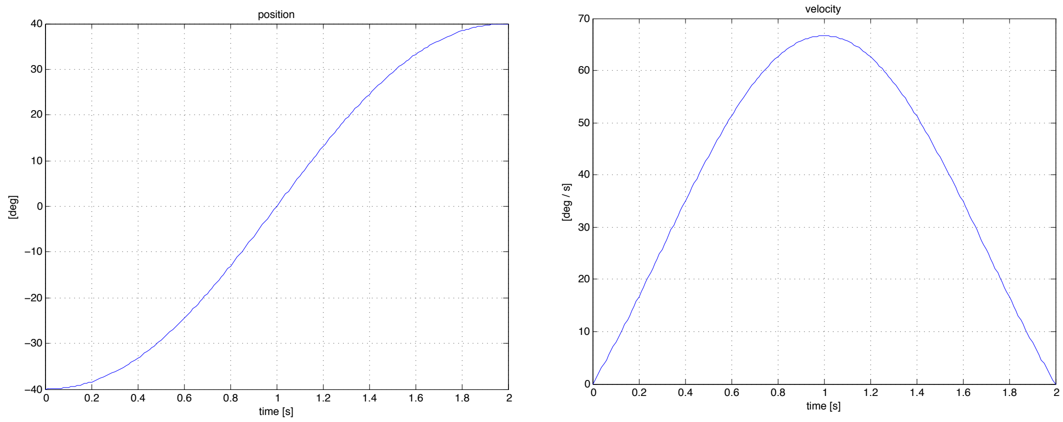


Figure 4: Rest-to-rest motion case *i)*: Balanced data give no wandering motion

and $a_2 = b_2 = -10$. The complete motion is satisfactory, with no wandering (neither under- nor over-shoot in position). We call this a situation with *balanced* data: the initial and final positions have opposite values, and motion occurs symmetrically around the zero average position. As a matter of fact, it can be shown analytically that $q_0 \cdot q_1 \leq 0$ is a sufficient condition for the velocity *not* to change sign in the interval $[0, T]$.

Figure 5 shows the position and velocity profiles obtained for the rest-to-rest case *ii)*, where $q_0 = 20°$ and $q_1 = 60°$. From (4), we have $a_1 = 45$, $b_1 = 15$, $a_2 = -15$, and $b_2 = 5$. Despite the total displacement is now halved with respect to the first case, a wandering behavior is present: the position undershoots the initial value $q_0 = 20°$ during the first 0.6 s (velocity is negative for 0.4 s from the motion start). The only apparent difference is that the initial and final position data are now *unbalanced* with respect to zero.

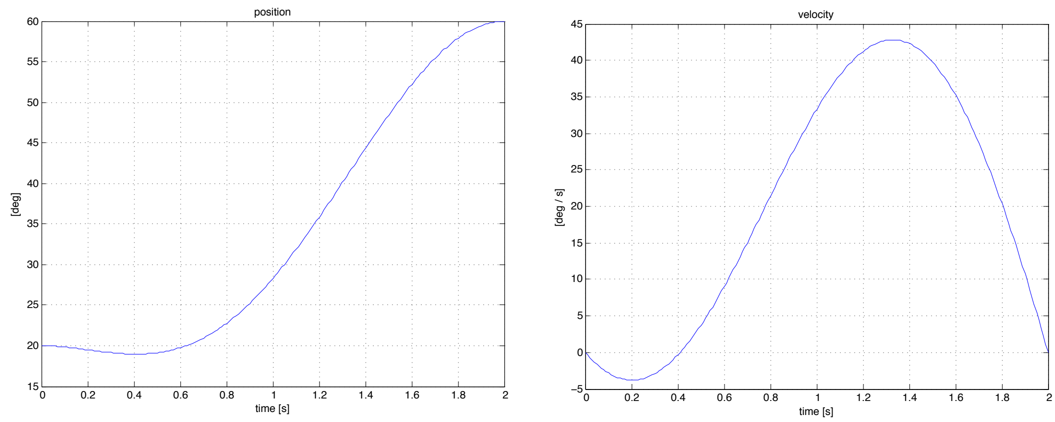


Figure 5: Rest-to-rest motion case *ii)*: Unbalanced data give rise to undershoot in position

With reference to the rest-to-rest case ($v_0 = v_1 = 0$), the above considerations allows to enforce a nice behavior in general (i.e., also in he unbalanced case). The addition of a suitable constant term, as in the modified trajectory (3), overcomes in fact the wandering problem.

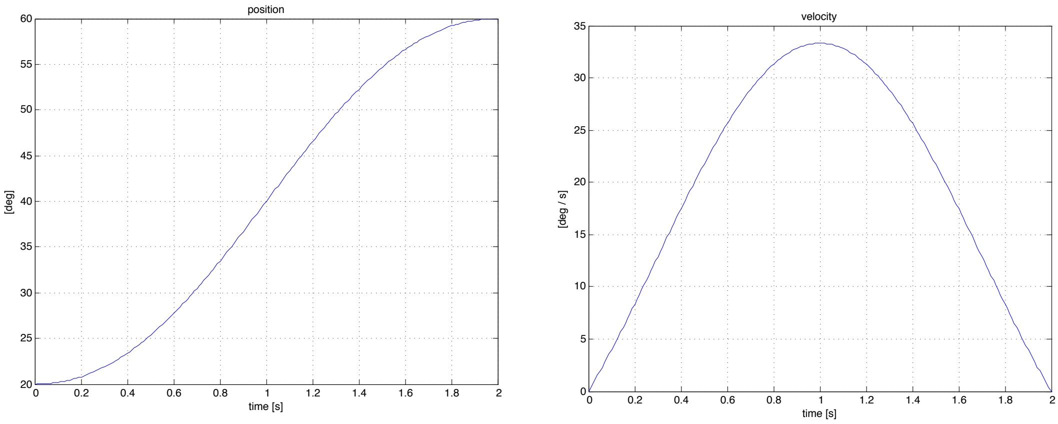


Figure 6: The modified solution for the rest-to-rest motion case *ii)* eliminates wandering, reduces accordingly the peak velocity, and provides a symmetric velocity profile

Denote now the given initial and final position as $p_0$ and $p_1$ (this redefinition allows using the previous formulas, with minimal intervention), and compute

$$\bar{q} = \frac{p_0 + p_1}{2}, \qquad q_0 = -\frac{p_1 - p_0}{2}, \qquad q_1 = \frac{p_1 - p_0}{2}. \tag{5}$$

The original data are thus transformed so that motion occurs around the average position $\bar{q}$, with opposite $q_1 = -q_0$ as desired. Taking into account the boundary conditions (2) for $q(t)$, the interpolation problem for $q'(t)$ is correctly formulated since

$$\text{at } t = 0 \quad \Rightarrow \quad q'(0) = \bar{q} + q_0 = p_0 \qquad\qquad \text{at } t = T \quad \Rightarrow \quad q'(T) = \bar{q} + q_1 = p_1.$$

As a result, the modified solution $q'(t)$ will be defined by $\bar{q}$ in (5) and by the same expressions (4) for $a_1$, $a_2$, $b_1$, and $b_2$, where $q_0$ and $q_1$ are now taken from (5).

The obtained trajectory will have a symmetric velocity profile with respect to the mid-motion instant $t = T/2$. We also note that the modified trajectory will provide a natural solution even in the degenerate case of no displacement ($p_0 = p_1 = p$): since we would have then $q_0 = q_1 = 0$, the solution will be a constant trajectory $q'(t) = \bar{q} = p$ (the previous method would generate instead a trajectory that moves out from the value $p$, returning to it at the final instant). Figure 6 shows the trajectory obtained with the modified method for case *ii)*. Wandering is no longer present. For completeness, Figure 7 reports the acceleration profiles of the various computed trajectories.

A Matlab code follows.

```
clear all; close all; clc
% boundary data
p0=20; % deg
p1=60;
v0=0; % deg/s
v1=0;
T=2; % total motion time (s)
% balanced solution does not wander (symmetric around average position)
pm=(p0+p1)/2;
```

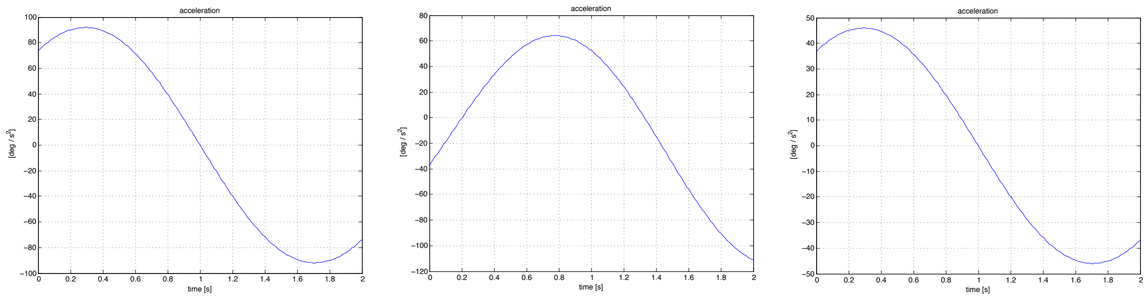


Figure 7: Acceleration profiles for case *i)* (left), case *ii)* with original solution (center), and case *ii)* with modified solution (right)

```
pd=p1-p0;
q0=-pd/2;
q1=pd/2;
% unbalanced solution leads to wandering ---uncomment to see this result
% q0=p0;
% q1=p1;
% pm=0;
% solution
a1=0.25*(3*q1+v0*2*T/pi);
a2=0.25*(v0*2*T/pi-q1);
b1=0.25*(3*q0-v1*2*T/pi);
b2=0.25*(q0+v1*2*T/pi);
t=[0:0.01:T];
tau=(pi/2)*t/T;
p=pm+a1*sin(tau)+a2*sin(3*tau)+b1*cos(tau)+b2*cos(3*tau);
pdot=pi/(2*T)*(a1*cos(tau)+3*a2*cos(3*tau)-b1*sin(tau)-3*b2*sin(3*tau));
pddot=-(pi/(2*T))^2*(a1*sin(tau)+9*a2*sin(3*tau)+b1*cos(tau)+9*b2*cos(3*tau));
plot(t,p);grid;title('position');xlabel('time [s]');ylabel('[deg]'); pause;
plot(t,pdot);grid;title('velocity');xlabel('time [s]');ylabel('[deg / s]'); pause;
plot(t,pddot);grid;title('acceleration');xlabel('time [s]');ylabel('[deg / s^2]')
% end
```

We conclude with a final remark. An alternative way to avoid trajectory wandering in the rest-to-rest case ($v_0 = v_1 = 0$) would be to replace eqs. (5) with

$$\bar{q} = p_0, \qquad q_0 = 0, \qquad q_1 = p_1 - p_0, \tag{6}$$

namely resetting the initial position to zero and working with the total displacement $p_1 - p_0$. When inserted in (4), these choices yield

$$a_1 = \frac{3}{4}\left(p_1 - p_0\right), \qquad a_2 = -\frac{1}{4}\left(p_1 - p_0\right), \qquad b_1 = b_2 = 0,$$

discarding the presence of cosinusoidal functions in (1). Figure 8 shows the resulting motion for case *ii)*. Wandering has been eliminated as expected (being $q_0 \cdot q_1 = 0$, the sufficient condition is in fact satisfied), but the velocity profile is no longer symmetric and has a higher peak (compare with Fig. 6, right). Thus, the modified solution in (5) is to be preferred to (6).

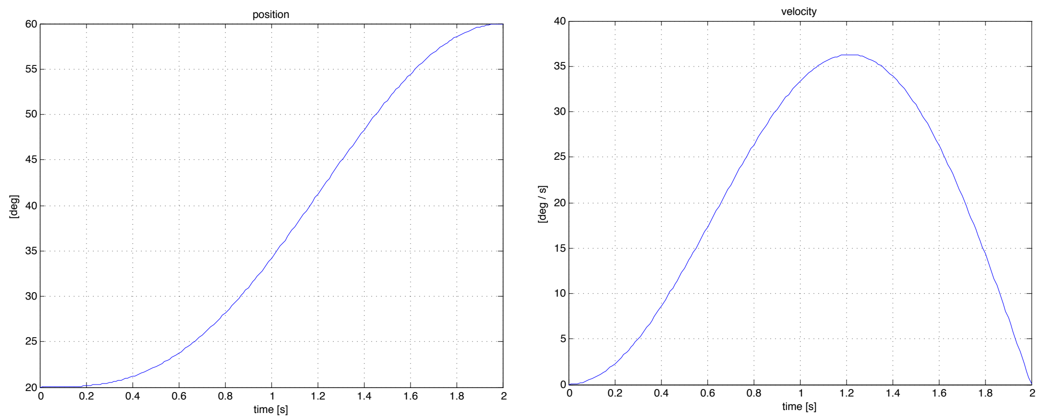


Figure 8: The alternative solution (6) for the rest-to-rest motion case *ii)* eliminates wandering, but yields a non-symmetric velocity profile as opposed to the use of (5)

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