Robotics I
June 11, 2012

Exercise 1

The time derivative of a rotation matrix can be given the following two alternative expressions:

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\boldsymbol{S}(\boldsymbol{\Omega}), \qquad \dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}.$$

Prove the correctness of both expressions and give the physical interpretation of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$.

Exercise 2

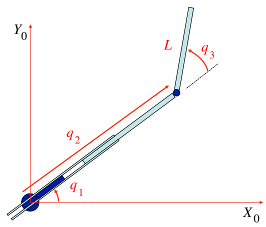

Figure 1: Planar RPR robot

For the planar RPR robot shown in Fig. 1, derive the 2×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the Cartesian velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^2$ of the end effector, and find all its singularities. Keeping q_1 as arbitrary, choose a singular configuration of this robot and denote the Jacobian in this configuration as $\bar{\boldsymbol{J}} = \bar{\boldsymbol{J}}(q_1)$. For each of the following linear subspaces,

$$\mathcal{R}\left(\bar{\boldsymbol{J}}\right) \qquad \mathcal{N}\left(\bar{\boldsymbol{J}}\right) \qquad \mathcal{R}\left(\bar{\boldsymbol{J}}^T\right) \qquad \mathcal{N}\left(\bar{\boldsymbol{J}}^T\right),$$

provide the symbolic expression of a unitary basis (i.e., a set of linearly independent unit vectors spanning the whole subspace).

Exercise 3

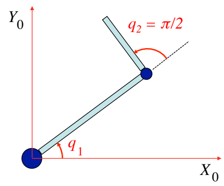

Figure 2: A planar 2R robot with the second link at $q_2 = \pi/2$

Consider a planar 2R robot, with links of length $\ell_1 = 1$ and $\ell_2 = 0.5$ [m], in the configuration shown in Fig. 2. The two motors at the joints are equipped with incremental encoders, respectively providing r_1 and r_2 pulses/turn. The gear ratios of the transmission/reduction systems of the two motors are $N_1 = 100$ and $N_2 = 80$. Determine the minimum resolutions of the two encoders so that they can be used to sense a displacement at the robot end-effector level as small as $\Delta p = 10^{-4}$ [m], alternatively in one of two arbitrary orthogonal directions.

[150 minutes; open books]

Solution

June 11, 2012

Exercise 1

As presented in the lecture slides, we consider first the identity $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$. Taking the time derivative:

$$\frac{d}{dt}\left(\boldsymbol{R}\boldsymbol{R}^T\right) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = \left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right) + \left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right)^T = \boldsymbol{O}.$$

Therefore, the matrix $\dot{\boldsymbol{R}}\boldsymbol{R}^T$ is skew symmetric. We can write

$$\dot{\boldsymbol{R}}\boldsymbol{R}^T = \boldsymbol{S}(\boldsymbol{\omega}) \qquad \Rightarrow \qquad \dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R},$$

where the angular velocity $\boldsymbol{\omega}$ is expressed in the *base* (unrotated) frame.

Similarly, consider the identity $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$. Taking the time derivative:

$$\frac{d}{dt}\left(\boldsymbol{R}^T\boldsymbol{R}\right) = \dot{\boldsymbol{R}}^T\boldsymbol{R} + \boldsymbol{R}^T\dot{\boldsymbol{R}} = \left(\boldsymbol{R}^T\dot{\boldsymbol{R}}\right) + \left(\boldsymbol{R}^T\dot{\boldsymbol{R}}\right)^T = \boldsymbol{O}.$$

Therefore, the matrix $\boldsymbol{R}^T\dot{\boldsymbol{R}}$ is skew symmetric. We can write

$$\boldsymbol{R}^T\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\Omega}) \qquad \Rightarrow \qquad \dot{\boldsymbol{R}} = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega}),$$

where the angular velocity $\boldsymbol{\Omega}$ is now expressed in the *body* (rotated) frame.

From this interpretation, it also follows that

$$\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Omega},$$

and so

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R} = \boldsymbol{S}(\boldsymbol{R}\boldsymbol{\Omega})\boldsymbol{R} = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega}).$$

This implies that (see Exercise 3.1 in the textbook)

$$\boldsymbol{S}(\boldsymbol{R}\boldsymbol{\Omega}) = \boldsymbol{R}\,\boldsymbol{S}(\boldsymbol{\Omega})\boldsymbol{R}^T.$$

Conversely,

$$\boldsymbol{S}(\boldsymbol{R}^T\boldsymbol{\omega}) = \boldsymbol{R}^T\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}.$$

Exercise 2

The direct kinematics of the considered RPR planar robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2c_1 + Lc_{13} \\ q_2s_1 + Ls_{13} \end{pmatrix}.$$

Therefore, the Jacobian of interest is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(q_2s_1 + Ls_{13}\right) & c_1 & -Ls_{13} \\ q_2c_1 + Lc_{13} & s_1 & Lc_{13} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix}.$$

To check the singularities (i.e., where rank $\boldsymbol{J} < 2$), we consider the three 2×2 minors:

$$\det\begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 \end{pmatrix} = (q_2 + Lc_3), \quad \det\begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_3 \end{pmatrix} = Lq_2 s_3, \quad \det\begin{pmatrix} \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix} = Lc_3.$$

They are simultaneously zero iff $\{q_2 = 0 \text{ .AND. } c_3 = 0\}$. Let then q_1 be arbitrary, $q_2 = 0$, and choose for instance $q_3 = +\pi/2$. In this configuration,

$$\bar{\boldsymbol{J}} = \bar{\boldsymbol{J}}(q_1) = \begin{pmatrix} -Lc_1 & c_1 & -Lc_1 \\ -Ls_1 & s_1 & -Ls_1 \end{pmatrix}$$

Unitary bases for the range and null spaces of interest are provided as follows:

$$\mathcal{R}\left(\bar{\boldsymbol{J}}\right) = \text{span}\left\{\begin{pmatrix} c_1 \\ s_1 \end{pmatrix}\right\} \qquad \mathcal{N}\left(\bar{\boldsymbol{J}}^T\right) = \text{span}\left\{\begin{pmatrix} s_1 \\ -c_1 \end{pmatrix}\right\},$$

$$\mathcal{R}\left(\bar{\boldsymbol{J}}^T\right) = \text{span}\left\{\frac{1}{\sqrt{1+2L^2}}\begin{pmatrix} L \\ -1 \\ L \end{pmatrix}\right\} \quad \mathcal{N}\left(\bar{\boldsymbol{J}}\right) = \text{span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{1+L^2}}\begin{pmatrix} 1 \\ L \\ 0 \end{pmatrix}\right\}.$$

Exercise 3

Denote by $\boldsymbol{\Delta\theta}$ the vector of motor position variations (the increments measured by the encoders at the motor sides), by $\boldsymbol{\Delta q}$ the associated vector of link position variations, by $\boldsymbol{\Delta p}$ the resulting vector of end-effector position variations, and by $\boldsymbol{N}$ the diagonal matrix of reduction ratios

$$\boldsymbol{N} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} = \begin{pmatrix} 100 & 0 \\ 0 & 80 \end{pmatrix}.$$

We have

$$\Delta\boldsymbol{p} = \boldsymbol{J}(\boldsymbol{q})\Delta\boldsymbol{q} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{N}^{-1}\Delta\boldsymbol{\theta}, \tag{1}$$

with the Jacobian of the 2R planar robot given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 s_1 + \ell_2 s_{12}) & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{pmatrix}.$$

To eliminate the appearance of q_1, it is convenient to work in the rotated frame 1 attached to the first link. Since we are working in the plane $(\boldsymbol{x}, \boldsymbol{y})$, it is

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell_2 s_2 & -\ell_2 s_2 \\ \ell_1 + \ell_2 c_2 & \ell_2 c_2 \end{pmatrix}.$$

Therefore, we replace eq. (1) by

$${}^1\Delta\boldsymbol{p} = {}^1\boldsymbol{J}(\boldsymbol{q})\boldsymbol{N}^{-1}\Delta\boldsymbol{\theta}. \tag{2}$$

At the given configuration $q_2 = \pi/2$,

$${}^1\boldsymbol{J}\boldsymbol{N}^{-1} = \begin{pmatrix} -\ell_2/N_1 & -\ell_2/N_2 \\ \ell_1/N_1 & 0 \end{pmatrix}.$$

The end-effector displacement Δp that is requested to be sensed in either of two orthogonal directions can be defined using again the coordinate axes of frame 1. Let

$${}^1\Delta\boldsymbol{p}_I = \begin{pmatrix} \Delta p \\ 0 \end{pmatrix}, \qquad {}^1\Delta\boldsymbol{p}_{II} = \begin{pmatrix} 0 \\ \Delta p \end{pmatrix}.$$

In case I, we solve from eq. (2)

$$\Delta\boldsymbol{\theta}_I = \boldsymbol{N} \cdot {}^1\boldsymbol{J}^{-1} \cdot {}^1\Delta\boldsymbol{p}_I = \begin{pmatrix} 0 \\ -N_2\Delta p/\ell_2 \end{pmatrix}.$$

Plugging the data $\ell_2 = 0.5$, $N_2 = 80$, and $\Delta p = 10^{-4}$, we obtain the minimum increment that should be sensed by the encoder at motor 2 in case I:

$$|\Delta\theta_{2,I}| = 16 \cdot 10^{-3}\,[\text{rad}].$$

Since the resolution of this encoder is $|\Delta\theta_2| = 2\pi/r_2$, the minimum number of pulses/turn needed is

$$r_2 = \frac{\pi}{8} \cdot 10^3 \simeq 392.7.$$

Being the number r of pulses/turn typically a power of 2, an incremental encoder with $512 = 2^9$ pulses/turn would be sufficient for joint 2 in this case.

Similarly, in case II

$$\Delta\boldsymbol{\theta}_{II} = \boldsymbol{N} \cdot {}^1\boldsymbol{J}^{-1} \cdot {}^1\Delta\boldsymbol{p}_{II} = \begin{pmatrix} N_1\Delta p/\ell_1 \\ -N_2\Delta p/\ell_1 \end{pmatrix}.$$

Plugging the data $\ell_1 = 1$, $N_1 = 100$, $N_2 = 80$, and $\Delta p = 10^{-4}$, we obtain the minimum increments that should be sensed by the two encoders in case II:

$$|\Delta\theta_{1,II}| = 10^{-2}\,[\text{rad}], \qquad |\Delta\theta_{2,II}| = 8 \cdot 10^{-3}\,[\text{rad}].$$

Note that the obtained condition on the second encoder will be more stringent in case II than in case I. From $|\Delta\theta_1| = 2\pi/r_1$ and $|\Delta\theta_2| = 2\pi/r_2$, the minimum resolutions for the two encoders will be

$$r_1 = 2\pi \cdot 10^2 \simeq 614, \qquad r_2 = \frac{\pi}{4} \cdot 10^3 \simeq 785.4.$$

Being the number of pulses/turn typically a power of 2, two equal incremental encoders with $1024 = 2^{10}$ pulses/turn mounted at the motor sides of the two robot joints would be sufficient to satisfy the requested end-effector sensing accuracy.

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