Robotics I

June 17, 2011

Consider a 2R planar robot having link lengths $\ell_1 = 3$ and $\ell_2 = 2$ [m]. The joint velocities are limited by

$$|\dot{q}_1| \leq 1\,\text{rad/s}, \qquad |\dot{q}_2| \leq 1.5\,\text{rad/s}.$$

Determine the feasible Cartesian velocity $\boldsymbol{v} = (v_x, v_y)$ of the end-effector which is the largest in norm at the configuration $\boldsymbol{q}_a = (\pi/6, \pi/3)$ [rad], providing also the joint velocities $\dot{\boldsymbol{q}}$ realizing it and the resulting norm $\|\boldsymbol{v}\|$. Repeat this analysis for a second configuration $\boldsymbol{q}_b = (\pi/6, 7\pi/8)$ [rad]. Draw a figure illustrating all feasible Cartesian velocities of the end-effector at least for the first case. Also, illustrate what happens to this figure when the robot is in a singular configuration $\boldsymbol{q}_s$.

[90 minutes; open books]

Solution

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The solution is a straightforward application of linear algebra. Figure 1 shows a rectangle R_j in the $(\dot{q}_1, \dot{q}_2)$ space representing the region of feasible joint velocities, having the four vertices $\dot{\boldsymbol{q}}_A$ to $\dot{\boldsymbol{q}}_D$ with both joints at their maximum (positive) or minimum (negative, symmetric) bounds.

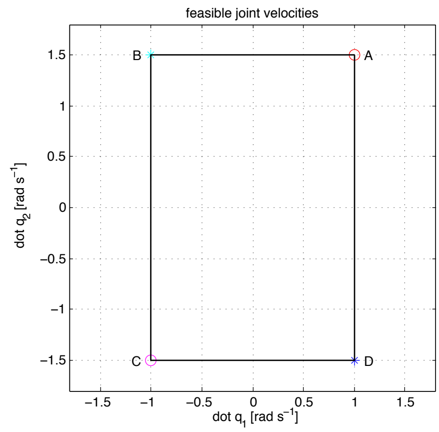

Figure 1: Joint velocity limits — the vertices of the rectangle R_j are labeled as A to D with reference to the following Cartesian plots

The Jacobian of the 2R planar robot is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 \sin q_1 + \ell_2 \sin(q_1+q_2)) & -\ell_2 \sin(q_1+q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1+q_2) & \ell_2 \cos(q_1+q_2) \end{pmatrix}.$$

When evaluated at the configuration $\boldsymbol{q}_a = (\pi/6, \pi/3)$, and with the given link lengths, we have the constant matrix

$$\boldsymbol{J}_a = \boldsymbol{J}(\boldsymbol{q}_a) = \begin{pmatrix} -3.5 & -2 \\ 2.5981 & 0 \end{pmatrix}$$

that will generate all feasible Cartesian velocities $\boldsymbol{v} \in P_c$ as

$$\boldsymbol{v} = \boldsymbol{J}_a \dot{\boldsymbol{q}}, \qquad \forall \boldsymbol{q} \in R_j.$$

In particular, the vertices A to D of R_j will map respectively into the homonymous vertices of the region P_c (see Fig. 2):

$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{J}_a \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -6.5 \\ 2.5981 \end{pmatrix} & \boldsymbol{v}_B &= \boldsymbol{J}_a \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -2.5981 \end{pmatrix} \\ \boldsymbol{v}_C &= \boldsymbol{J}_a \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 6.5 \\ -2.5981 \end{pmatrix} & \boldsymbol{v}_D &= \boldsymbol{J}_a \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 2.5981 \end{pmatrix}. \end{aligned} \tag{1}$$

The four boundaries of P_c are characterized by points that are linear combinations of the above four vertices, taken two by two in alphabetic sequence. Therefore, P_c will be a convex polytope.

The largest Cartesian velocity in norm occurs at one (or more) vertex of P_c which is the farthest away from the origin (in the following figures, the origin of the (v_x, v_y) space is located at the robot end-effector for a more intuitive visualization). Hence,

$$V_{\text{max,a}} = \max\{\|\boldsymbol{v}\|, \text{for } \boldsymbol{v} \in P_c\} = \max\{\|\boldsymbol{v}_A\|, \|\boldsymbol{v}_B\|, \|\boldsymbol{v}_C\|, \|\boldsymbol{v}_D\|\} = \|\boldsymbol{v}_A\| = \|\boldsymbol{v}_C\| = 7 \text{ [m/s]}.$$

There are indeed two opposite and saturated joint velocities, $\dot{\boldsymbol{q}}_A$ and $\dot{\boldsymbol{q}}_C$, that provide the maximum norm of the Cartesian velocity.

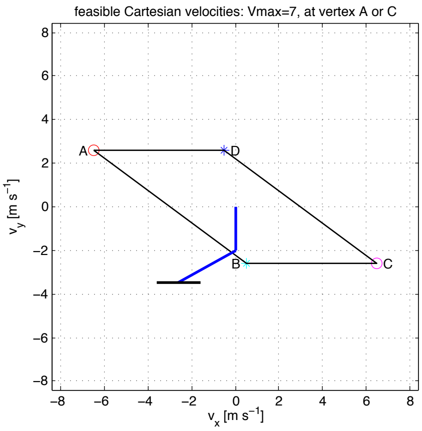

Figure 2: Polytope of feasible Cartesian velocities at $\boldsymbol{q}_a = (\pi/6, \pi/3)$, centered at the end-effector of the robot (shown in blue on its fixed base) — the vertices A to D are the images of those in Fig. 1

The analysis is identical at the configuration $\boldsymbol{q}_b = (\pi/6, 7\pi/8)$, see Fig. 3 (note the different scale). There, the Jacobian takes the numerical values

$$\boldsymbol{J}_b = \boldsymbol{J}(\boldsymbol{q}_b) = \begin{pmatrix} -1.2389 & 0.2611 \\ 0.6152 & -1.9829 \end{pmatrix}$$

and we obtain, in place of (1),

$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{J}_b \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -0.8474 \\ -2.3591 \end{pmatrix} & \boldsymbol{v}_B &= \boldsymbol{J}_b \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1.6305 \\ -3.5895 \end{pmatrix} \\ \boldsymbol{v}_C &= \boldsymbol{J}_b \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 0.8474 \\ 2.3591 \end{pmatrix} & \boldsymbol{v}_D &= \boldsymbol{J}_b \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} -1.6305 \\ 3.5895 \end{pmatrix}, \end{aligned} \tag{2}$$

from which

$$V_{\text{max,b}} = \|\boldsymbol{v}_B\| = \|\boldsymbol{v}_D\| = 3.9425 \text{ [m/s]}.$$

The two opposite and saturated joint velocities that provide the maximum norm of the Cartesian velocity are now $\dot{\boldsymbol{q}}_B$ and $\dot{\boldsymbol{q}}_D$. This change is due to the different configuration, in much the same way as in the analysis of manipulability (the associated velocity ellipsoid is not related to the presence of hard bounds on the joint velocities, since in that case we map all possible $\dot{\boldsymbol{q}}$, normalized with $\|\dot{\boldsymbol{q}}\| = 1$). However, the maximum norm of the Cartesian velocity is now almost halved w.r.t.

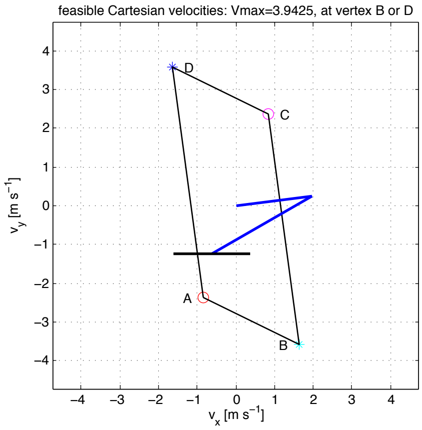

Figure 3: Polytope of feasible Cartesian velocities at $\boldsymbol{q}_b = (\pi/6, 7\pi/8)$, centered at the robot end-effector

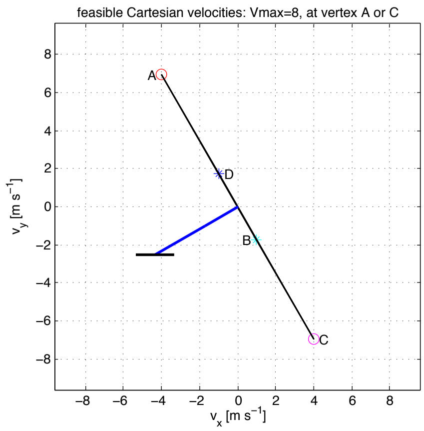

Figure 4: Polytope of feasible Cartesian velocities at the singular configuration $\boldsymbol{q}_s = (\pi/6, 0)$ — the area of the polytope vanishes and only one possible direction for the end-effector velocity is left, with limited amplitude

the previous case and also the area of the polytope is reduced. This is because the robot is close to a singular configuration, the folded one.

To complete the analysis, we consider the singular configuration $\boldsymbol{q}_s = (\pi/6, 0)$, i.e., with the arm stretched (see Fig. 4). In this case, the Jacobian

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} -2.5 & -1 \\ 4.3301 & 1.7321 \end{pmatrix}$$

is singular and the polytope collapses. It is

$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{J}_s \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -4 \\ 6.9282 \end{pmatrix} = -\boldsymbol{v}_C \\ \boldsymbol{v}_B &= \boldsymbol{J}_s \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.7321 \end{pmatrix} = -\boldsymbol{v}_D, \end{aligned} \tag{3}$$

and

$$V_{\text{max,s}} = \|\boldsymbol{v}_A\| = \|\boldsymbol{v}_C\| = 8 \text{ [m/s]}.$$

This value is the largest of all cases. Indeed, the saturated joint velocity $\dot{\boldsymbol{q}}_A$ (or $\dot{\boldsymbol{q}}_C$) is maximally contributing to the Cartesian velocity in just one single direction when the arm is fully stretched, similarly to when a human is throwing of a ball.

The Matlab source code generating the solution and the plots is available upon request.

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