Robotics I – Homework 2

Hand out: December 13, 2010; Return by: January 12, 2011

Consider the 3R anthropomorphic robot manipulator with a trunk/shoulder offset shown in Fig. 1. The DH parameters are given in Tab. 1, where a_1, a_2, a_3, and d_1 are all strictly positive.

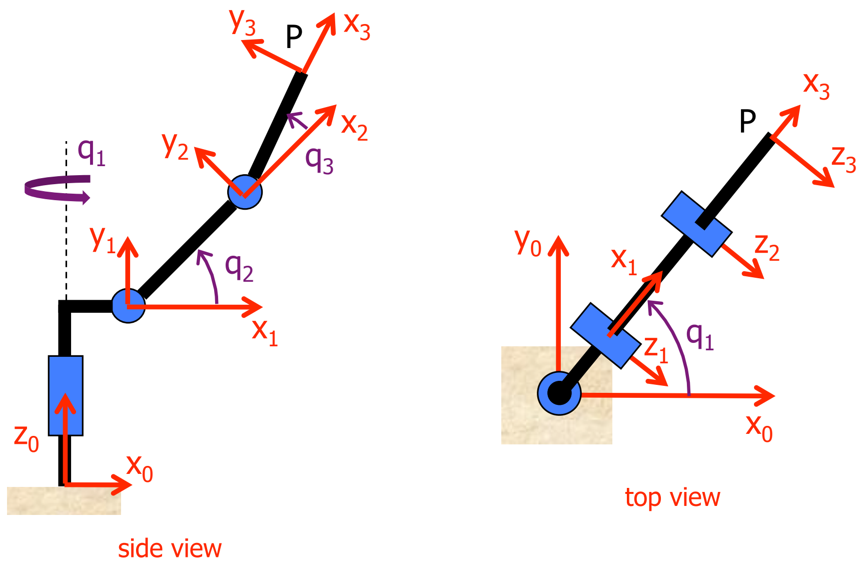

Figure 1: Side and top views of the 3R robot, with DH frames assignment (side view is for $q_1 = 0$)

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	d_1	a_1	q_1
2	0	0	a_2	q_2
3	0	0	a_3	q_3

Table 1: Table of DH parameters

We are interested only in the position $\boldsymbol{p}$ of point P, i.e., the origin of frame 3 attached to the robot end-effector. The direct kinematics is computed as

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos q_1 \,(a_1 + a_2 \cos q_2 + a_3 \cos(q_2 + q_3)) \\ \sin q_1 \,(a_1 + a_2 \cos q_2 + a_3 \cos(q_2 + q_3)) \\ d_1 + a_2 \sin q_2 + a_3 \sin(q_2 + q_3) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

i) Find the solution of the inverse kinematics problem in closed form, as a function of $\boldsymbol{p}$.

ii) Draw a section of the (primary) workspace in the vertical plane $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ in parametric form w.r.t. a_1, a_2, a_3, and d_1, assuming $a_2 > a_3$ and $a_1 > a_2 - a_3$ for the sake of simplicity. Indicate the regions in this workspace where the inverse kinematics problem has 4 solutions, 2 solutions, 1 solution, or an infinite number of solutions.

iii) Test your solution formulas using the data (in cm)

$$d_1 = 200, \qquad a_1 = 30, \qquad a_2 = 100, \qquad a_3 = 80,$$

for the following desired positions (in cm) of P:

a) $\boldsymbol{p} = \begin{pmatrix} -20 & 20 & 310 \end{pmatrix}^T$

b) $\boldsymbol{p} = \begin{pmatrix} 195 & 0 & 135 \end{pmatrix}^T$

c) $\boldsymbol{p} = \begin{pmatrix} 30 + 90\sqrt{3} & 0 & 110 \end{pmatrix}^T$ (here, $p_x \simeq 185.8845$)

d) $\boldsymbol{p} = \begin{pmatrix} 0 & 200 & 35 \end{pmatrix}^T$

e) $\boldsymbol{p} = \begin{pmatrix} 0 & 0 & 285 \end{pmatrix}^T$

Verify your results, when appropriate, by plugging the obtained configurations in the direct kinematics (1). In general, allow for some numerical tolerance in the resulting end-effector position due to truncation of decimals in the joint space solution.

iv) Consider again case *a)* of *iii)*, and include the presence of the following joint limits (in deg):

$$q_1 \in [-150°, +150°], \qquad q_2 \in [-140°, +140°], \qquad q_3 \in [-100°, +100°].$$

How many feasible inverse kinematic solutions are left?

v) Write a (Matlab) program that takes $\boldsymbol{p}$ as input and provides as output a complete answer (all solutions in any case), including a warning when there is no solution ($\boldsymbol{p}$ is out of the workspace).

[estimated time for items *i)-ii)* : 210 minutes (open books)]

Solution of Homework 2

December 13, 2010

Squaring and summing the first two equations in (1), one obtains a relation which is independent from q_1:

$$p_x^2 + p_y^2 = (a_1 + a_2 \cos q_2 + a_3 \cos(q_2 + q_3))^2 .$$

Therefore,

$$a_1 + a_2 \cos q_2 + a_3 \cos(q_2 + q_3) = \pm\sqrt{p_x^2 + p_y^2}. \tag{2}$$

Provided that $p_x^2 + p_y^2 \neq 0$, replacing (2) back into the first two equations of (1) yields

$$c_1 = \cos q_1 = \frac{\pm\, p_x}{\sqrt{p_x^2 + p_y^2}}, \qquad s_1 = \sin q_1 = \frac{\pm\, p_y}{\sqrt{p_x^2 + p_y^2}},$$

and the two solutions for q_1 are (choosing, respectively, the positive or the negative sign in '$\pm$')

$$q_1^I = \text{ATAN2}\left\{p_y, p_x\right\}, \qquad q_1^{II} = \text{ATAN2}\left\{-p_y, -p_x\right\}. \tag{3}$$

Note that the angles q_1^I and q_1^{II} differ by π (both are automatically in the interval $(-\pi, \pi]$). In the first solution, the vertical trunk of the robot (its first link) is 'facing' the desired position of point P, while in the second it is 'backing' point P (the trunk is oriented toward the opposite quadrant). Moreover, it is

$$p_x^2 + p_y^2 = 0 \quad \Longleftrightarrow \quad q_1 \text{ undefined.}$$

Thus, when the end-effector is placed on the axis of joint 1 the robot is in a singularity.

Consider again eq. (2), and rewrite it as

$$\sqrt{p_x^2 + p_y^2} \mp a_1 = \mp\left(a_2 \cos q_2 + a_3 \cos(q_2 + q_3)\right). \tag{4}$$

Also, rewrite the last equation in (1) as

$$p_z - d_1 = a_2 \sin q_2 + a_3 \sin(q_2 + q_3). \tag{5}$$

The right-hand sides of the above two equations are functions of q_2 and q_3 only. Squaring and summing eqs. (4) and (5) leads to

$$\begin{aligned}\left(\sqrt{p_x^2 + p_y^2} \mp a_1\right)^2 + (p_z - d_1)^2 &= a_2^2 + a_3^2 + 2a_2a_3\left(\cos q_2 \cos(q_2 + q_3) + \sin q_2 \sin(q_2 + q_3)\right)\\ &= a_2^2 + a_3^2 + 2a_2a_3 \cos q_3,\end{aligned}$$

which is a function of q_3 only. From this,

$$c_3 = \cos q_3 = \frac{\left(\sqrt{p_x^2 + p_y^2} \mp a_1\right)^2 + (p_z - d_1)^2 - \left(a_2^2 + a_3^2\right)}{2a_2a_3}, \quad s_3 = \sin q_3 = \pm\sqrt{1 - c_3^2}. \tag{6}$$

There are four independent combinations of signs involved in the expressions (6). The first '$\mp$' (in black) should be selected according to the (opposite) choice in the solution for q_1, namely the upper '$-$' is associated to the '$+$' in q_1^I and the lower '$+$' is associated to the '$-$' in q_1^{II}. For each of these two choices, one has two alternatives for the second '±' (in red). Thus, there will be four solutions for q_3, obtained from

$$q_3 = \mathrm{ATAN2}\{s_3, c_3\}, \tag{7}$$

and labeled as follows:

$$q_3^{I,+}, \qquad q_3^{I,-}, \qquad q_3^{II,+}, \qquad q_3^{II,-}. \tag{8}$$

When all four solutions exist (see below), the following identities hold:

$$q_3^{I,+} = -q_3^{I,-}, \qquad q_3^{II,+} = -q_3^{II,-}.$$

Indeed, in (6) it must be

$$-1 \le c_3 \le 1.$$

This condition characterizes the fact that the desired position $\boldsymbol{p}$ belongs to the robot (primary) workspace. The inequalities are made explicit as

$$(a_2 - a_3)^2 = a_2^2 + a_3^2 - 2a_2a_3 \le \left(\sqrt{p_x^2 + p_y^2} \mp a_1\right)^2 + (p_z - d_1)^2 \le a_2^2 + a_3^2 + 2a_2a_3 = (a_2 + a_3)^2. \tag{9}$$

Thus, before applying eq. (7), one should check whether the given data $\boldsymbol{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T$ satisfy or not (9) —this test can be done right at the beginning of the solution procedure, so as to discard requests that are out of the workspace (no solution). If the inequalities in (9) are strictly satisfied for both choices of signs '∓', then there will be four inverse kinematic solutions. If they are strictly satisfied only for one of the two signs, then there will be only two solutions. Furthermore, if one of the two inequalities is active (i.e., is satisfied with the equality sign), then $|c_3| = 1$ and a pair of solutions in (8) will collapse into one. This is again a singular situation. In particular, at the outer boundary of the robot workspace (for $c_3 = 1$, i.e., $q_3 = 0$), the other pair of solutions in (8) will never be present.

For each numerical solution (q_1, q_3) found so far, the associated value of q_2 is determined as follows. Adding the first equation in (1) multiplied by c_1 to the second one multiplied by s_1 yields

$$c_1p_x + s_1p_y - a_1 = a_2 \cos q_2 + a_3 \cos(q_2 + q_3) = (a_2 + a_3c_3) \cos q_2 - (a_3s_3) \sin q_2. \tag{10}$$

Equation (5) can be manipulated similarly as

$$p_z - d_1 = a_2 \sin q_2 + a_3 \sin(q_2 + q_3) = (a_2 + a_3c_3) \sin q_2 + (a_2s_3) \cos q_2. \tag{11}$$

Equations (10–11) constitute a linear system in the two unknowns $\cos q_2$ and $\sin q_2$,

$$\begin{pmatrix} a_2 + a_3c_3 & -a_3s_3 \\ a_3s_3 & a_2 + a_3c_3 \end{pmatrix} \begin{pmatrix} \cos q_2 \\ \sin q_2 \end{pmatrix} = \begin{pmatrix} c_1p_x + s_1p_y - a_1 \\ p_z - d_1 \end{pmatrix},$$

which can be solved in a unique way, provided that the coefficient matrix $\boldsymbol{A}$ on the left-hand side is non-singular. Since $\det \boldsymbol{A} = a_2^2 + a_3^3 + 2a_2a_3c_3$, this determinant is always positive unless $a_2 = a_3$ and $c_3 = -1$. In this case, q_2 will be undefined[1]. When $\det \boldsymbol{A} > 0$, the unique solution is

$$\begin{pmatrix} c_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} \cos q_2 \\ \sin q_2 \end{pmatrix} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} a_2 + a_3c_3 & a_3s_3 \\ -a_3s_3 & a_2 + a_3c_3 \end{pmatrix} \begin{pmatrix} c_1p_x + s_1p_y - a_1 \\ p_z - d_1 \end{pmatrix}$$

$$= \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} (a_2 + a_3c_3)(c_1p_x + s_1p_y - a_1) + a_3s_3(p_z - d_1) \\ (a_2 + a_3c_3)(p_z - d_1) - a_3s_3(c_1p_x + s_1p_y - a_1) \end{pmatrix}.$$

[1] In this special situation, point P is on joint axis 2, at a distance a_1 from joint axis 1. Then, the input data will necessarily be such that $\sqrt{p_x^2 + p_y^2} = a_1$ and $p_z = d_1$. An infinite number of inverse solutions for q_2 results.

From this, we obtain as usual[2]

$$q_2 = \text{ATAN2}\{s_2, c_2\}, \tag{12}$$

and label each of the (potential) solutions as

$$q_2^{I,+}, \qquad q_2^{I,-}, \qquad q_2^{II,+}, \qquad q_2^{II,-}, \tag{13}$$

where the same notation has been used as in (8). Indeed, some of these values may not need to be computed, depending on the number of inverse kinematic solutions for q_3. This completes item *i)*.

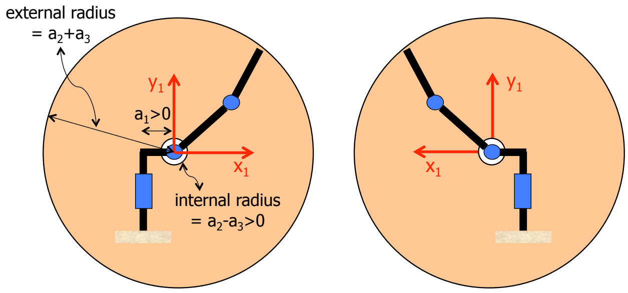

Figure 2: Two sections of the workspace obtained for a given q_1 (left) and for $q_1 + \pi$ (right)

For item *ii)*, the previous analysis shows that the workspace of the robot can be divided in regions where the number of inverse kinematics solutions is the same. With reference to Fig. 2, consider the case of a given q_1. The workspace section is a circular annulus with external radius $a_2 + a_3$ and internal radius $|a_2 - a_3|$ (in the given assumptions, simply $a_2 - a_3 > 0$), as depicted on the left of Fig. 2. Proceeding from the origin of the first frame ($x_1 = y_1 = z_1 = 0$) and moving outwards, the number of inverse kinematic solutions in this situation will be, respectively, 0, 1 (on the inner boundary), 2, 1 (on the outer boundary), and 0. However, in the same vertical plane there is another part of the workspace (obtained by symmetric reflection with respect to z_0) corresponding to a value of $q_1 \pm \pi$ (right of Fig. 2). These two parts may intersect or not, depending on the relative size of a_1 w.r.t. a_2 and a_3. Accordingly, the number of solutions will change.

Figure 3 shows the total workspace section for $a_1 < a_2 + a_3$. The shown picture refers to the case when the additional condition $a_1 < 1.5a_3 - 0.5a_2$ holds, in which case the inner empty circle of the configurations with q_1 is fully reached by two configurations obtained for $q_1 + \pi$ (and viceversa, exchanging the role of q_1 and $q_1 + \pi$). The number of resulting inverse kinematic solutions in the different regions of the total workspace section is also indicated. In particular, on the vertical axis z_0 there will be an infinite number of inverse kinematic solutions (for any given q_1, there are in general two inverse solutions). The robot workspace is the 3D volume obtained by rotating the planar figure around $\boldsymbol{z}_0$.

It should be noted that the absence of a shoulder offset ($a_1 = 0$) would largely simplify the analysis. For instance, the two sections obtained for q_1 and $q_1 + \pi$ would coincide, and the number of inverse kinematic solutions would be either 4 (in the generic case, when P is strictly internal to the workspace), 1 (on the internal and and external boundaries of the workspace, except for the two points lying on the $\boldsymbol{z}_0$ axis), or infinite (when P is on the $\boldsymbol{z}_0$ axis).

[2]Being the common denominator $\det \boldsymbol{A}$ to s_2 and c_2 positive, it can be discarded in the evaluation of the four-quadrant arctangent function.

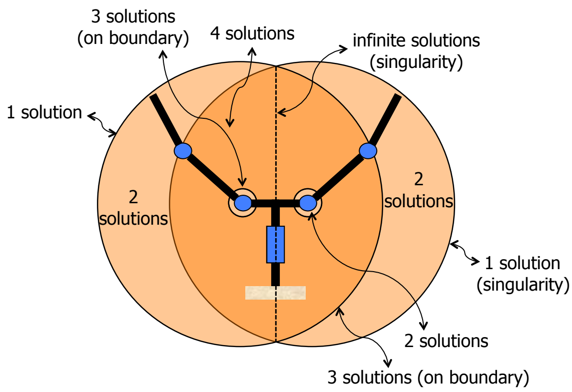

Figure 3: The total workspace section, as union of the two sections in Fig. 2, and the number of inverse kinematic solutions in the different regions

With the numerical data given in *iii)*, the inverse kinematic solutions are computed as follows. They are illustrated in Figs. 4–7.

a) For $\boldsymbol{p} = \begin{pmatrix} -20 & 20 & 310 \end{pmatrix}^T$, there are 4 distinct inverse kinematic solutions:

$$
\begin{aligned}
\boldsymbol{q}^{I,+} &= \begin{pmatrix} 2.3562 & 0.8103 & 1.8427 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} 135.00 & 46.43 & 105.58 \end{pmatrix}^T \text{ [deg]} \\
\boldsymbol{q}^{I,-} &= \begin{pmatrix} 2.3562 & 2.3624 & -1.8427 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} 135.00 & 135.36 & -105.58 \end{pmatrix}^T \text{ [deg]} \\
\boldsymbol{q}^{II,+} &= \begin{pmatrix} -0.7854 & 1.3614 & 1.6273 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} -45.00 & 78.00 & 93.23 \end{pmatrix}^T \text{ [deg]} \\
\boldsymbol{q}^{II,-} &= \begin{pmatrix} -0.7854 & 2.7546 & -1.6273 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} -45.00 & 157.83 & -93.23 \end{pmatrix}^T \text{ [deg]}
\end{aligned}
$$

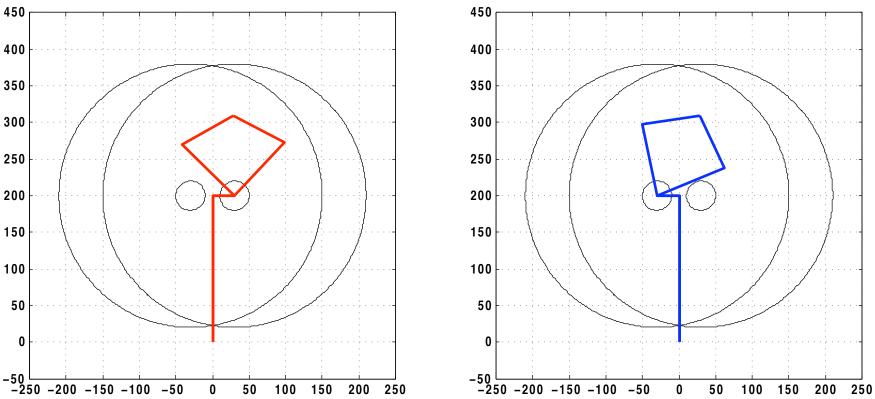

Figure 4: The four inverse kinematic solutions for case *a)*, shown in the vertical plane specified by $\theta_1 = 135°$. On the left: $\boldsymbol{q}^{I,+}$ and $\boldsymbol{q}^{I,-}$. On the right: $\boldsymbol{q}^{II,+}$ and $\boldsymbol{q}^{II,-}$

b) For $\boldsymbol{p} = \begin{pmatrix} 195 & 0 & 135 \end{pmatrix}^T$, there are only 2 distinct inverse kinematic solutions since point P is in one of the two 'far' parts of the workspace, which can be reached only for one value of $q_1 (= q_1^I)$. The two solutions are:

$$\boldsymbol{q}^{I,+} = \begin{pmatrix} 0 & -0.5290 & 0.3463 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} 0 & -30.31 & 19.84 \end{pmatrix}^T \text{ [deg]}$$
$$\boldsymbol{q}^{I,-} = \begin{pmatrix} 0 & -0.2215 & -0.3463 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} 0 & -12.69 & -19.84 \end{pmatrix}^T \text{ [deg]}$$

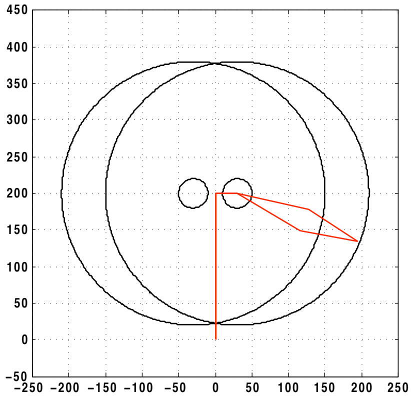

Figure 5: The two inverse kinematic solutions $\boldsymbol{q}^{I,+}$ and $\boldsymbol{q}^{I,-}$ for case *b)*, shown in the vertical plane specified by $\theta_1 = 0$

c) For $\boldsymbol{p} = \begin{pmatrix} 30 + 90\sqrt{3} & 0 & 110 \end{pmatrix}^T$, there is a unique solution (the second and third link are stretched, with point P on the outer boundary of the workspace). The solution is:

$$\boldsymbol{q}^{I} = \begin{pmatrix} 0 & -0.5236 & 0 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} 0 & -30 & 0 \end{pmatrix}^T \text{ [deg]}$$

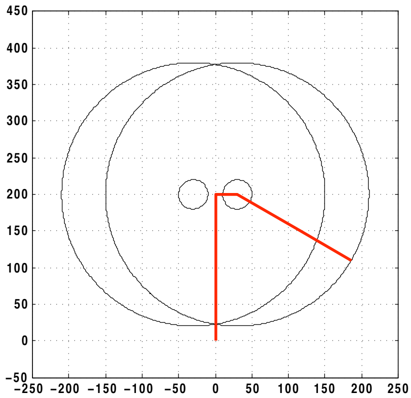

Figure 6: The single inverse kinematic solution $\boldsymbol{q}^{I}$ for case *c)*, shown in the vertical plane specified by $\theta_1 = 0$

d) For $\boldsymbol{p} = \begin{pmatrix} 0 & 200 & 35 \end{pmatrix}^T$, there are no solutions ($\boldsymbol{p}$ is out of the workspace).

e) For $\boldsymbol{p} = \begin{pmatrix} 0 & 0 & 285 \end{pmatrix}^T$, there is an infinite number of solutions since q_1 is undefined (arbitrary). However, all solutions have one of the two following forms:

$$\boldsymbol{q}^{any,+} = \begin{pmatrix} any & 1.0474 & 2.1144 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} any & 60.01 & 121.14 \end{pmatrix}^T \text{ [deg]}$$
$$\boldsymbol{q}^{any,-} = \begin{pmatrix} any & 2.7728 & -2.1144 \end{pmatrix}^T \text{ [rad]} = \begin{pmatrix} any & 158.87 & -121.14 \end{pmatrix}^T \text{ [deg]}$$

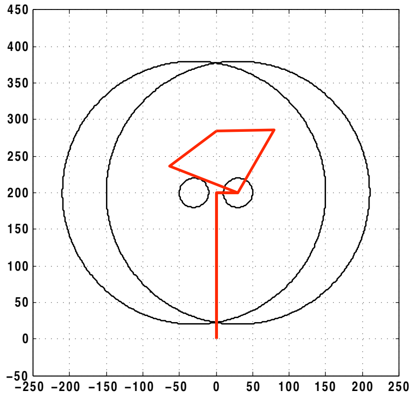

Figure 7: Two (out of the infinite number of) inverse kinematic solutions for case *e)*, shown in the vertical plane specified by a generic value of θ_1. All other solutions are obtained from this pair, by varying $\theta_1 \in (-\pi, \pi]$

Finally, in the presence of the joint limits given at item *iv)* the only solution that remains feasible in case *a)* is $\boldsymbol{q}^{II,+} = \begin{pmatrix} -0.7854 & 1.3614 & 1.6273 \end{pmatrix}^T$ [rad] $= \begin{pmatrix} -45.00 & 78.00 & 93.23 \end{pmatrix}^T$ [deg].

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