Robotics 1

Differential kinematics

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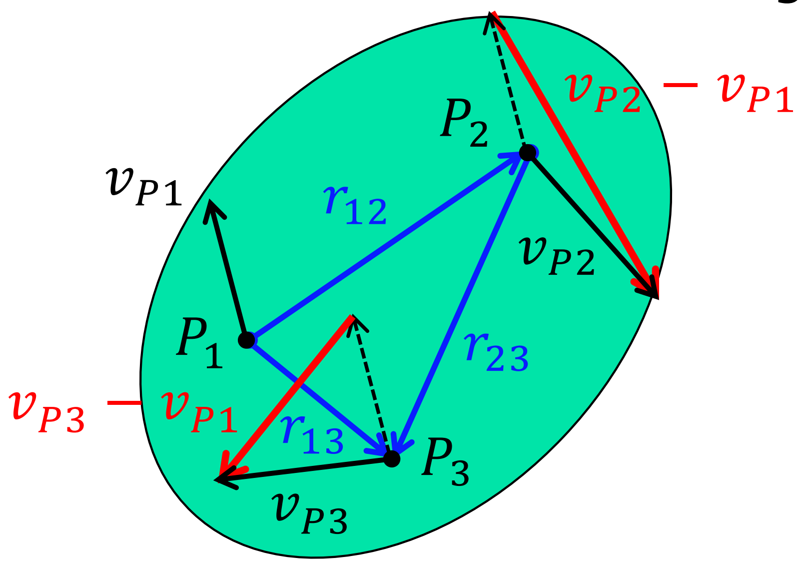
Differential kinematics

- relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)
- instantaneous velocity mappings can be obtained through time differentiation of the direct kinematics or in a geometric way, directly at the differential level
  - different treatments arise for rotational quantities
  - establish the relation between angular velocity and
    - time derivative of a rotation matrix
    - time derivative of the angles in a minimal representation of orientation

Angular velocity of a rigid body



"rigidity" constraint on distances among points:

$\|r_{ij}\|$ = constant

$\Rightarrow$ $v_{Pi} - v_{Pj}$ orthogonal to r_{ij}

1 $\quad v_{P2} - v_{P1} = \omega_1 \times r_{12}$

2 $\quad v_{P3} - v_{P1} = \omega_1 \times r_{13}$

3 $\quad v_{P3} - v_{P2} = \omega_2 \times r_{23}$

$\forall P_1, P_2, P_3$

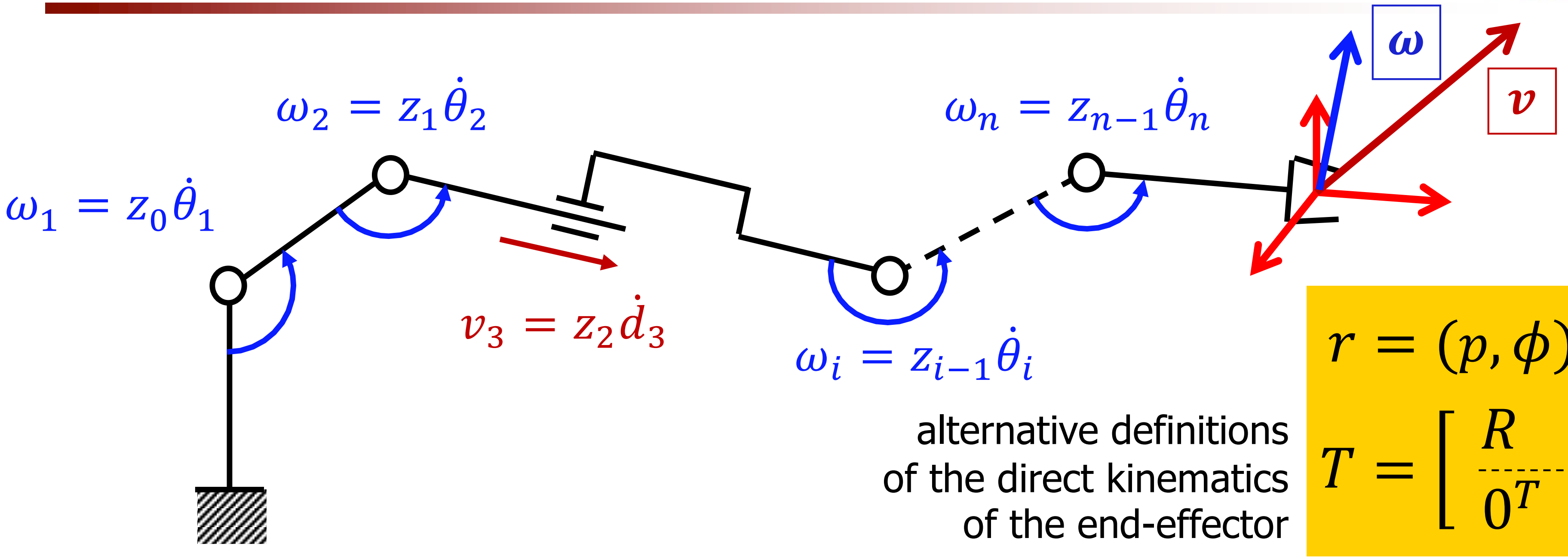
2 - 1 = 3 $\Rightarrow$ $\omega_1 = \omega_2 = \omega$

aka, "(fundamental) kinematic equation" of rigid bodies

$$\boxed{v_{Pj} = v_{Pi} + \omega \times r_{ij} = v_{Pi} + S(\omega)\, r_{ij}} \Leftrightarrow \boxed{\dot{r}_{ij} = \omega \times r_{ij}}$$

- the angular velocity ω is associated to the whole body (**not** to a point)
- if $\exists P_1, P_2$: $v_{P1} = v_{P2} = 0 \Rightarrow$ pure rotation (circular motion of all $P_j \notin$ line $P_1 P_2$)
- $\omega = 0 \Rightarrow$ pure translation (**all** points have the same velocity v_P)

Linear and angular velocity of the robot end-effector



alternative definitions
of the direct kinematics
of the end-effector

$$r = (p, \phi)$$

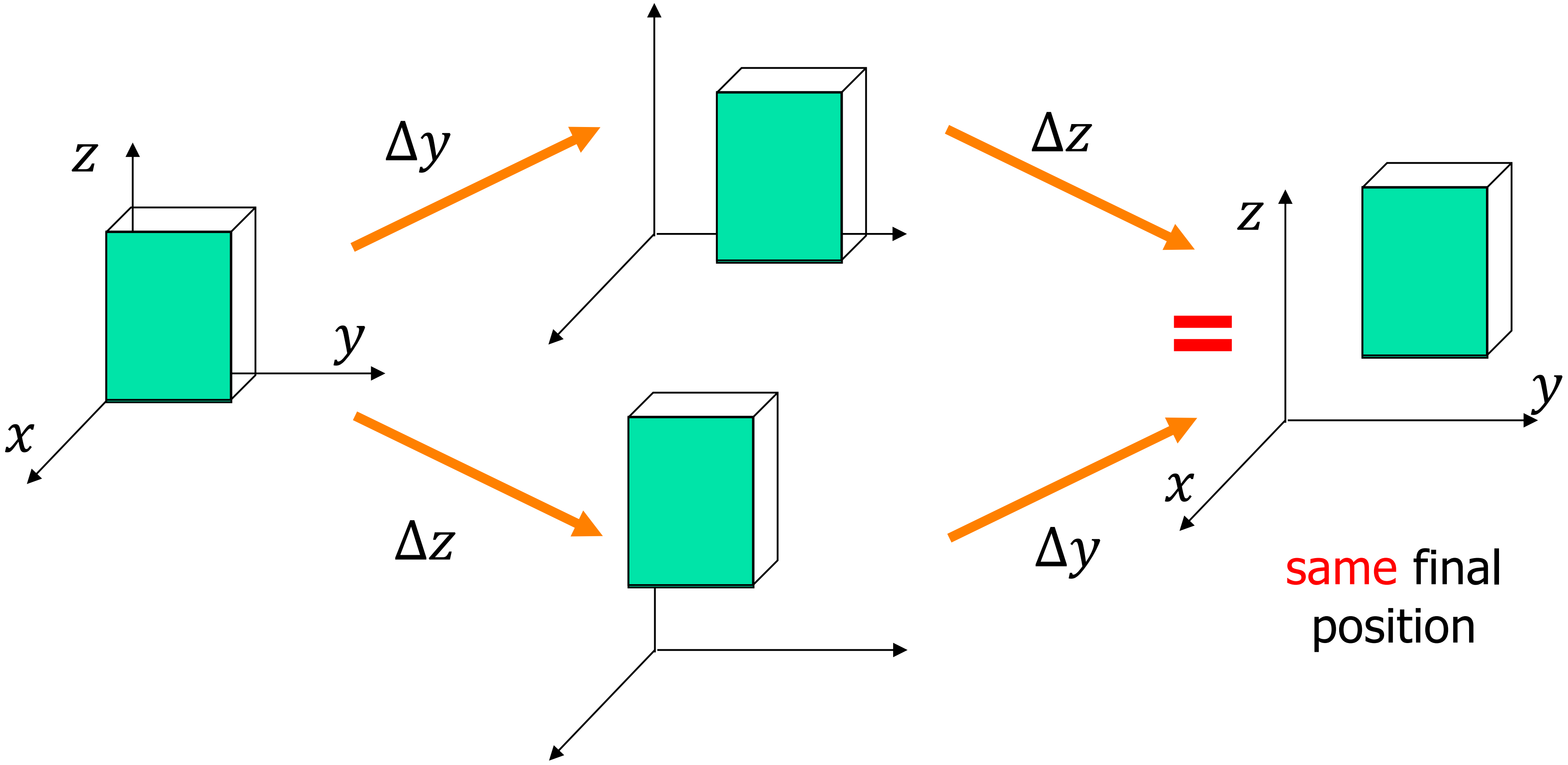
$$T = \left[\begin{array}{c:c} R & p \\ \hdashline 0^T & 1 \end{array}\right]$$

- v and ω are "vectors", namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - such contributions will be given by the single (linear or angular) joint velocities
- on the other hand, ϕ (and $\dot{\phi}$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

in general, $\omega \neq \dot{\phi}$

Finite and infinitesimal translations

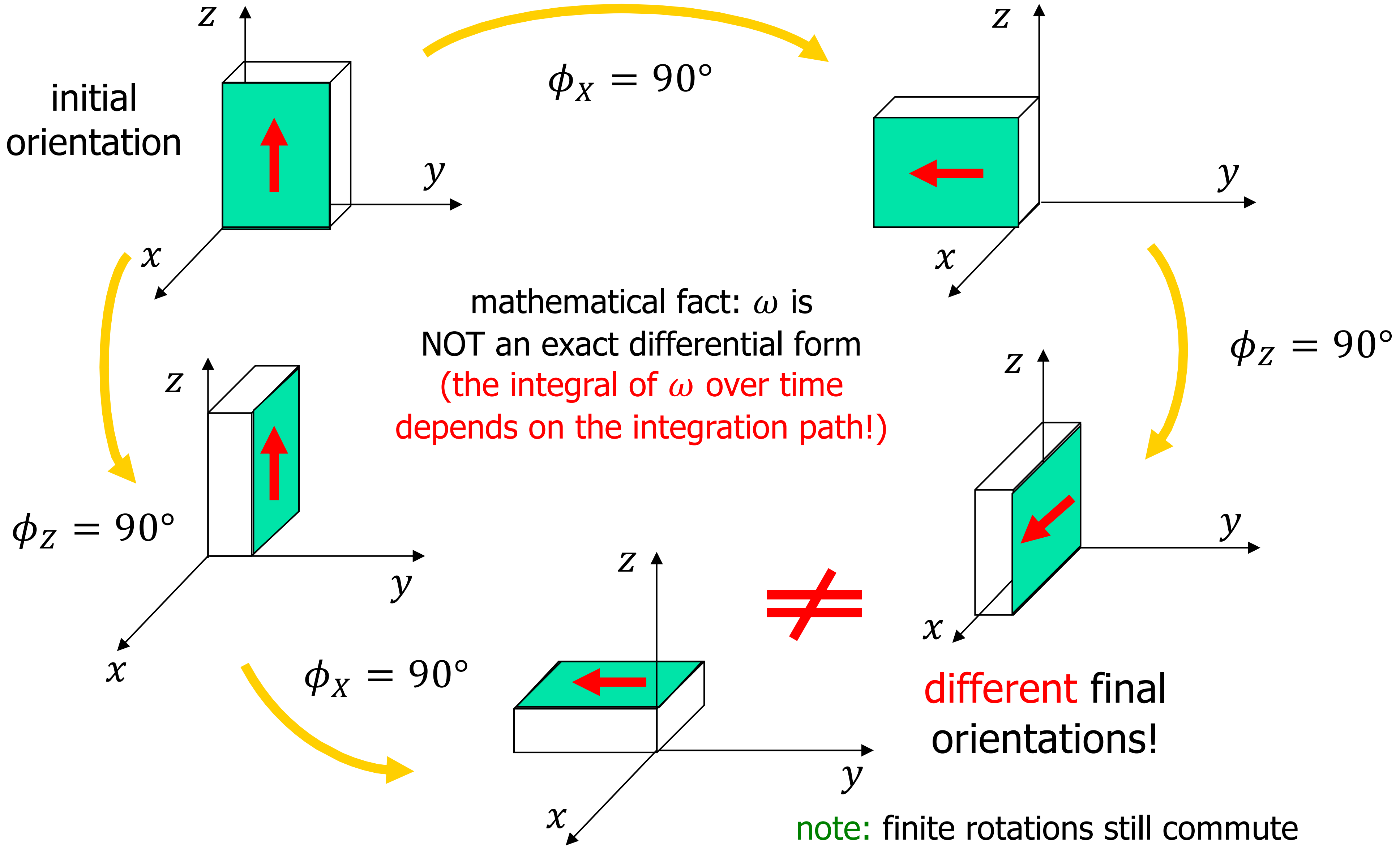
- finite $\Delta x, \Delta y, \Delta z$ or infinitesimal dx, dy, dz translations (linear displacements) always commute



same final position

Finite rotations do not commute

example



initial orientation

mathematical fact: ω is NOT an exact differential form (the integral of ω over time depends on the integration path!)

different final orientations!

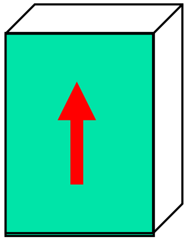
note: finite rotations still commute when made around the same fixed axis

ω is not an exact differential

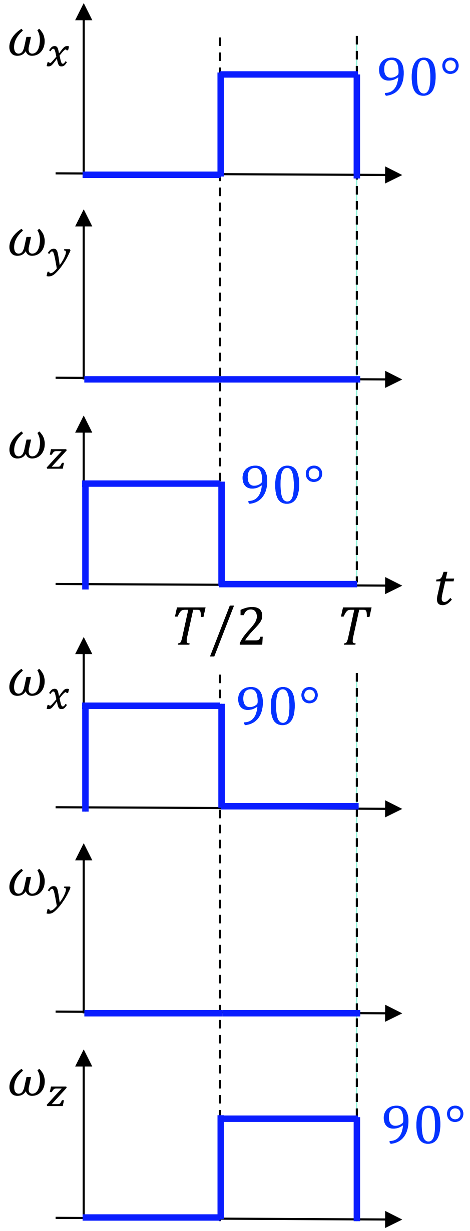
whiteboard …

$T = 2\ s$

initial
orientation



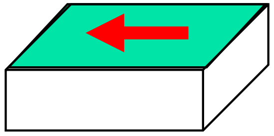
$R_i = I$



$$\int_0^T \omega(t)dt = \int_0^T \begin{pmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{pmatrix} dt$$

$$= \begin{pmatrix} 90° \\ 0 \\ 90° \end{pmatrix}$$

first final
orientation



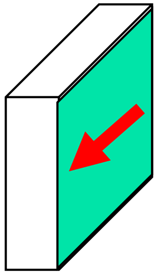
$R_{f,ZX}$

$$\int_0^T \dot{\phi}(t)dt = \int_0^T \frac{d\phi}{dt}dt = \int_{\phi(0)}^{\phi(T)} d\phi = \phi_f - \phi_i$$

an exact differential form

$$\int_0^T \omega(t)dt = \cdots = \begin{pmatrix} 90° \\ 0 \\ 90° \end{pmatrix}$$

$R_{f,XZ}$



…the same value
but a different…

…final
orientation

Infinitesimal rotations commute!

- infinitesimal rotations $d\phi_X, d\phi_Y, d\phi_Z$ around x, y, z axes

$$R_X(\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_X & -\sin\phi_X \\ 0 & \sin\phi_X & \cos\phi_X \end{bmatrix} \Rightarrow R_X(d\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_X \\ 0 & d\phi_X & 1 \end{bmatrix}$$

$$R_Y(\phi_Y) = \begin{bmatrix} \cos\phi_Y & 0 & \sin\phi_Y \\ 0 & 1 & 0 \\ -\sin\phi_Y & 0 & \cos\phi_Y \end{bmatrix} \Rightarrow R_Y(d\phi_Y) = \begin{bmatrix} 1 & 0 & d\phi_Y \\ 0 & 1 & 0 \\ -d\phi_Y & 0 & 1 \end{bmatrix}$$

$$R_Z(\phi_Z) = \begin{bmatrix} \cos\phi_Z & -\sin\phi_Z & 0 \\ \sin\phi_Z & \cos\phi_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_Z(d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & 0 \\ d\phi_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$R(d\phi) = \underset{\substack{\uparrow \\ \text{in any order}}}{R(d\phi_X, d\phi_Y, d\phi_Z)} = \begin{bmatrix} 1 & -d\phi_Z & d\phi_Y \\ d\phi_Z & 1 & -d\phi_X \\ -d\phi_Y & d\phi_X & 1 \end{bmatrix} \leftarrow$$

neglecting second- and third-order (infinitesimal) terms

$$= I + S(d\phi)$$

Time derivative of a rotation matrix

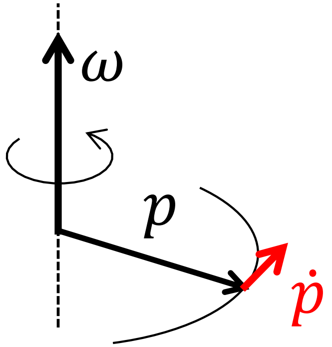
- let $R = R(t)$ be a rotation matrix, given as a function of time
- since $I = R(t)R^T(t)$, taking the time derivative of both sides yields

$$0 = d\big(R(t)R^T(t)\big)/dt = (dR(t)/dt)R^T(t) + R(t)(dR^T(t)/dt)$$
$$= (dR(t)/dt)R^T(t) + \big((dR(t)/dt)\, R^T(t)\big)^T$$

thus $(dR(t)/dt)\, R^T(t) = S(t)$ is a skew-symmetric matrix

- let $p(t) = R(t)p'$ a vector (with constant norm) rotated over time
- comparing

$$\dot{p}(t) = (dR(t)/dt)p' = S(t)R(t)p' = S(t)p(t)$$
$$\dot{p}(t) = \omega(t) \times p(t) = S\big(\omega(t)\big)p(t)$$



we get $S = S(\omega)$

$$\boxed{\dot{R} = S(\omega)R} \Longleftrightarrow \boxed{S(\omega) = \dot{R}\, R^T}$$

Example

Time derivative of an elementary rotation matrix

$$R_X\big(\phi(t)\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t) & \cos\phi(t) \end{bmatrix}$$

$$\dot{R}_X(\phi)R_X^T(\phi) = \dot{\phi}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{bmatrix} = S(\omega) \quad \Longrightarrow \quad \omega = \omega_X = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

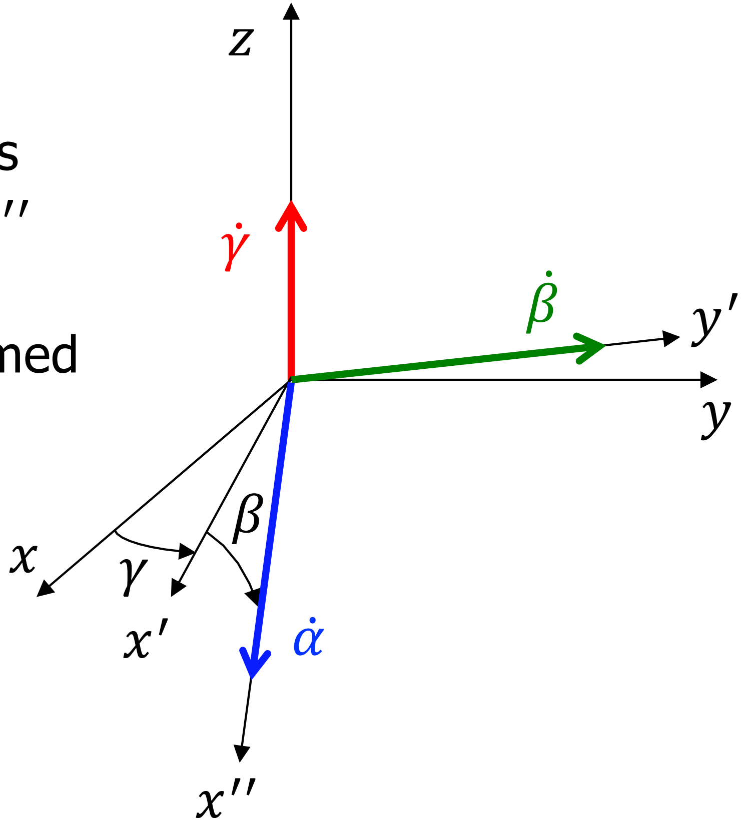
more in general, for the axis/angle rotation matrix

$$R\big(r, \theta(t)\big) \implies \dot{R}(r,\theta)R^T(r,\theta) = S(\omega) \quad \Longrightarrow \quad \omega = \omega_r = \dot{\theta}\, r = \dot{\theta}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

Time derivative of RPY angles and ω

$$R_{RPY}(\alpha_X, \beta_Y, \gamma_Z) = R_{ZY'X''}(\gamma_Z, \beta_Y, \alpha_X) = R_Z(\gamma) R_{Y'}(\beta) R_{X''}(\alpha)$$

the three contributions $\dot{\gamma}Z, \dot{\beta}Y', \dot{\alpha}X''$ to ω are simply summed as vectors



$$\omega = \overbrace{\begin{bmatrix} c\beta c\gamma & -s\gamma & 0 \\ c\beta s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \end{bmatrix}}^{T_{RPY}(\beta,\gamma)} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

$$\begin{array}{ccc} X'' & Y' & Z \\ \uparrow & \uparrow & \\ \text{1st col in } R_Z(\gamma)R_{Y'}(\beta) & \text{2nd col in } R_Z(\gamma) & \end{array}$$

det $T_{RPY}(\beta,\gamma) = \cos\beta = 0$

for $\beta = \pm\pi/2$

(singularity of the RPY representation)

similar treatment for the other 11 minimal representations...

Robot Jacobian matrices

- analytic Jacobian (obtained by time differentiation)

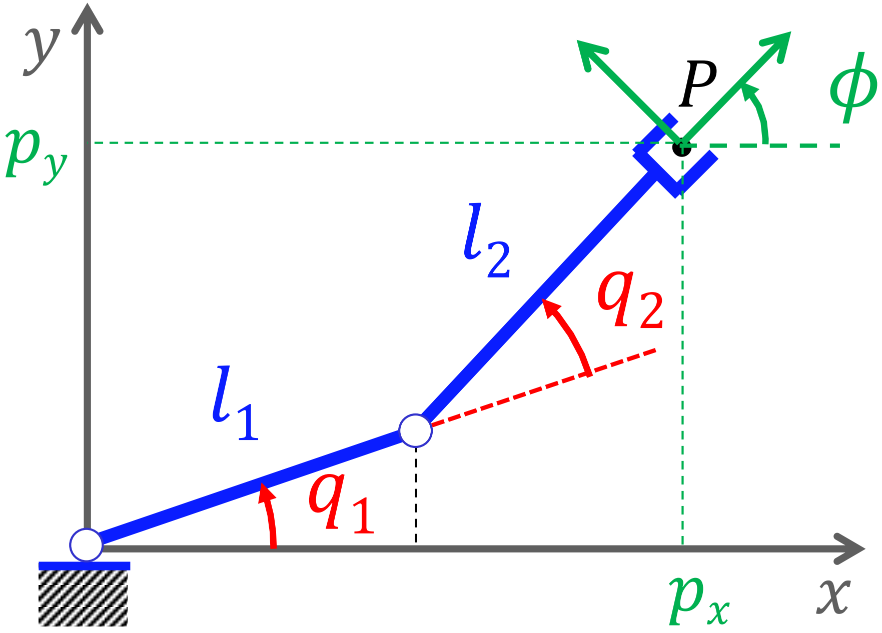
$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} = f_r(q) \quad \Rightarrow \quad \dot{r} = \begin{pmatrix} \dot{p} \\ \dot{\phi} \end{pmatrix} = \frac{\partial f_r(q)}{\partial q}\,\dot{q} = J_r(q)\,\dot{q}$$

- geometric or basic Jacobian (no derivatives)

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = J(q)\,\dot{q}$$

- in both cases, the Jacobian matrix depends on the (current) configuration of the robot

Analytic Jacobian of planar 2R arm



direct kinematics

$$r \left\{ \begin{array}{l} p_x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ p_y = l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \\ \hline \phi = q_1 + q_2 \end{array} \right.$$

$$\dot{p}_x = -l_1\, s_1 \dot{q}_1 - l_2 s_{12}(\dot{q}_1 + \dot{q}_2)$$

$$\dot{p}_y = l_1\, c_1 \dot{q}_1 + l_2 c_{12}(\dot{q}_1 + \dot{q}_2)$$

$$\dot{\phi} = \omega_z = \dot{q}_1 + \dot{q}_2$$

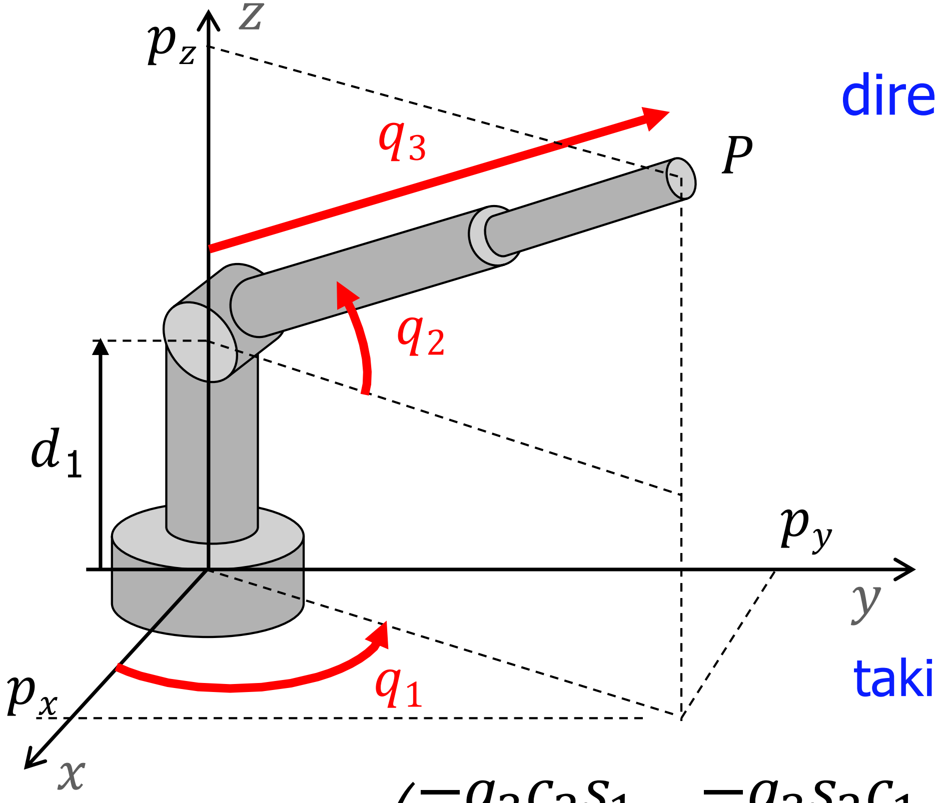
here, all rotations occur around the same fixed axis z (normal to the plane of motion)

$$\Rightarrow \quad J_r(q) = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ \hline 1 & 1 \end{pmatrix}$$

given r, this is a 3 × 2 matrix

$$\boxed{\dot{r} = J_r(q)\dot{q}}$$

Analytic Jacobian of polar (RRP) robot



direct kinematics (here, $r = p$)

$$\left.\begin{aligned} p_x &= q_3 c_2 c_1 \\ p_y &= q_3 c_2 s_1 \\ p_z &= d_1 + q_3 s_2 \end{aligned}\right\} f_r(q)$$

taking the derivative w.r.t. time t …

$$v = \dot{p} = \underbrace{\begin{pmatrix} -q_3 c_2 s_1 & -q_3 s_2 c_1 & c_2 c_1 \\ q_3 c_2 c_1 & -q_3 s_2 s_1 & c_2 s_1 \\ 0 & q_3 c_2 & s_2 \end{pmatrix}}_{\frac{\partial f_r(q)}{\partial q}} \dot{q} = J_r(q)\dot{q}$$

… requires doing only partial derivatives w.r.t. joint variables q_1 … q_n

Geometric Jacobian

always a $6 \times n$ matrix

end-effector
instantaneous
velocity

$$\begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \cdots & J_{Ln}(q) \\ J_{A1}(q) & \cdots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

superposition of effects

$$v_E = \underbrace{J_{L1}(q)\dot{q}_1}_{} + \cdots + J_{Ln}(q)\dot{q}_n \qquad \omega_E = \underbrace{J_{A1}(q)\dot{q}_1}_{} + \cdots + J_{An}(q)\dot{q}_n$$

contribution to the linear e-e velocity due to $\dot{q}_1$

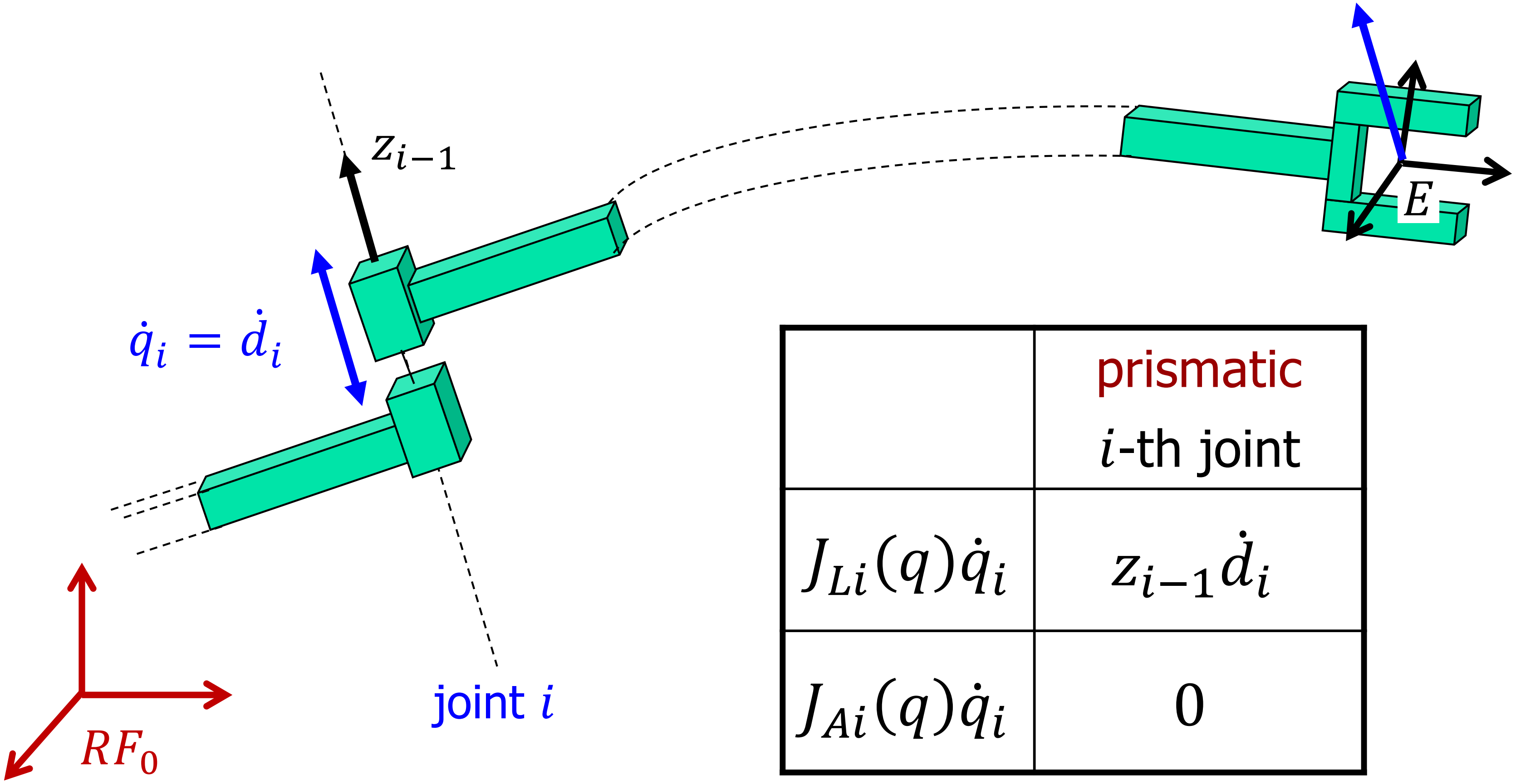
contribution to the angular e-e velocity due to $\dot{q}_1$

linear and angular velocity belong to (linear) vector spaces in $\mathbb{R}^3$

Contribution of a prismatic joint

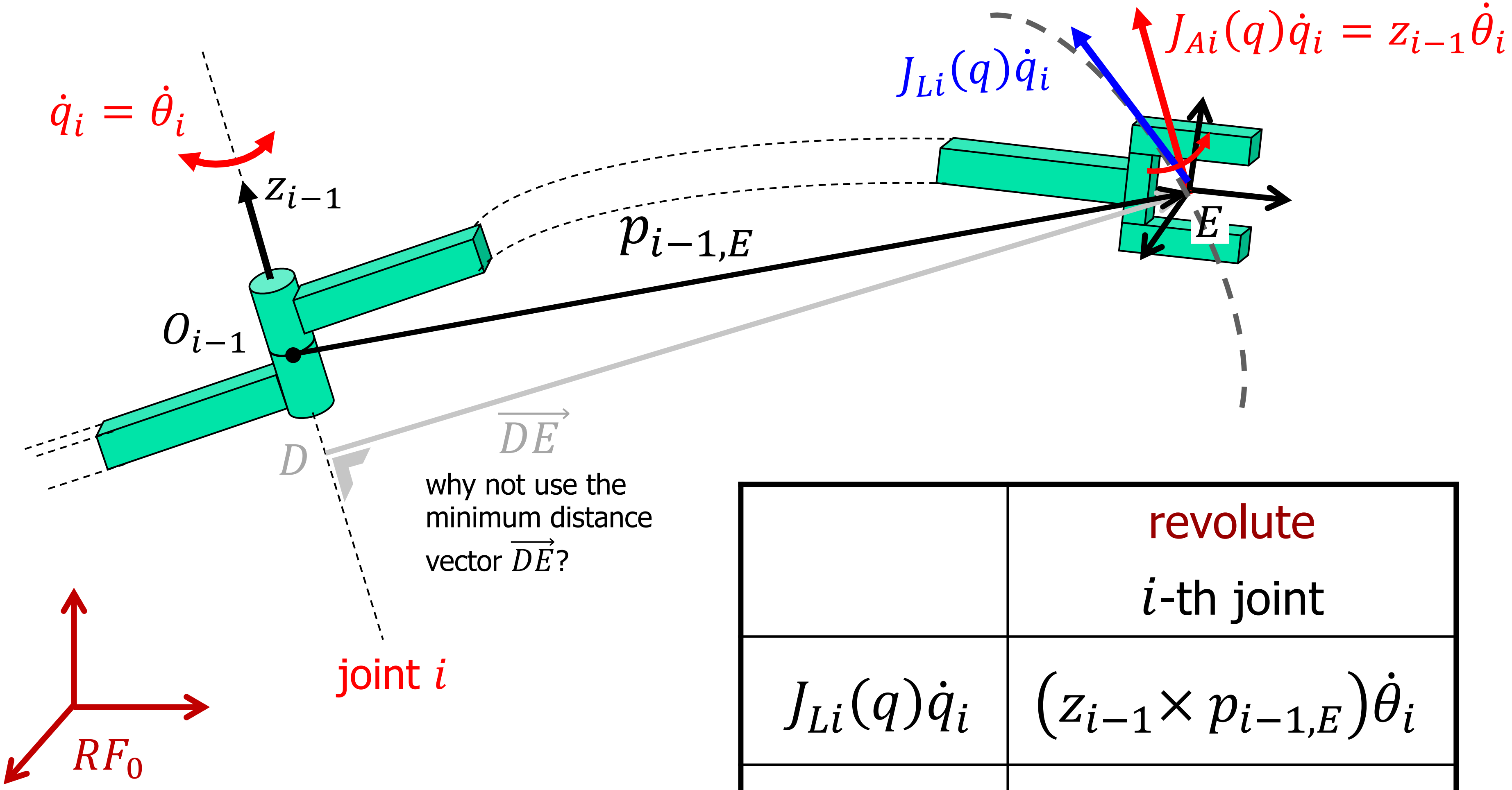
note: joints beyond the i-th one are considered to be "frozen", so that the distal part of the robot is a single rigid body

$$J_{Li}(q)\dot{q}_i = z_{i-1}\dot{d}_i$$



	prismatic i-th joint
$J_{Li}(q)\dot{q}_i$	$z_{i-1}\dot{d}_i$
$J_{Ai}(q)\dot{q}_i$	0

Contribution of a revolute joint



why not use the minimum distance vector $\overrightarrow{DE}$?

	revolute i-th joint
$J_{Li}(q)\dot{q}_i$	$(z_{i-1} \times p_{i-1,E})\dot{\theta}_i$
$J_{Ai}(q)\dot{q}_i$	$z_{i-1}\dot{\theta}_i$

Expression of geometric Jacobian

$$\left(\begin{pmatrix}\dot{p}_{0,E}\\ \omega_E\end{pmatrix} =\right)\begin{pmatrix}v_E\\ \omega_E\end{pmatrix} = \begin{pmatrix}J_L(q)\\ J_A(q)\end{pmatrix}\dot{q} = \begin{pmatrix}J_{L1}(q) & \cdots & J_{Ln}(q)\\ J_{A1}(q) & \cdots & J_{An}(q)\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{pmatrix}$$

	prismatic i-th joint	revolute i-th joint
$J_{Li}(q)$	z_{i-1}	$z_{i-1} \times p_{i-1,E}$
$J_{Ai}(q)$	0	z_{i-1}

this can be also computed as

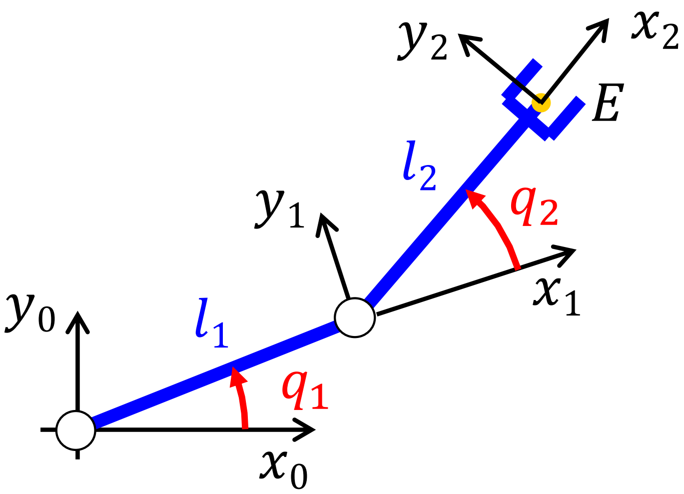
$$= \frac{\partial p_{0,E}(q)}{\partial q_i}$$

$$z_{i-1} = {}^{0}R_1(q_1) \cdots {}^{i-2}R_{i-1}(q_{i-1})\, \overset{\begin{pmatrix}0\\0\\1\end{pmatrix}}{{}^{i-1}z_{i-1}}$$

$$p_{i-1,E} = \underbrace{p_{0,E}(q_1, \cdots, q_n)}_{\text{complete kinematics for e-e position}} - \underbrace{p_{0,i-1}(q_1, \cdots, q_{i-1})}_{\text{partial kinematics for } O_{i-1} \text{ position}}$$

all vectors should be expressed in the same reference frame (here, the base frame RF_0)

Geometric Jacobian of planar 2R arm



Denavit-Hartenberg table

joint	α_i	d_i	a_i	θ_i
1	0	0	l_1	q_1
2	0	0	l_2	q_2

$$J(q) = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

$$z_0 = z_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

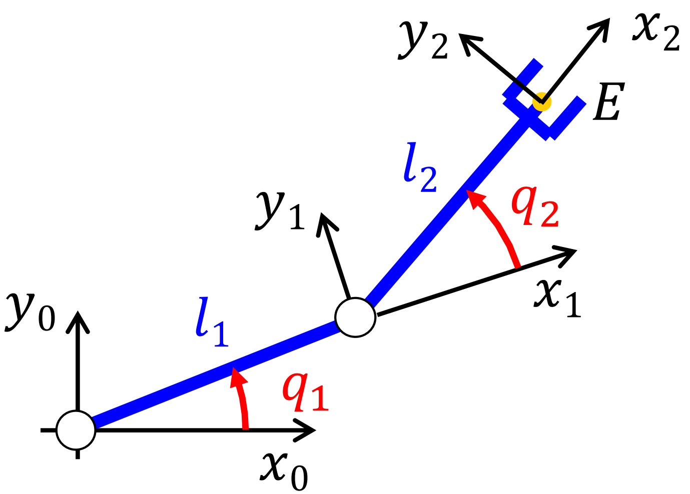
$$^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,1}$$

$$^0A_2 = \begin{pmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,E}$$

all computations can be made numerically, evaluating first the direct kinematics terms!

$$p_{1,E} = p_{0,E} - p_{0,1}$$

Geometric Jacobian of planar 2R arm



$$J(q) = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

$$\left.\begin{matrix} v_z \equiv 0 \\ \omega_x \equiv 0 \\ \omega_y \equiv 0 \end{matrix}\right\} \qquad = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

compare rows 1, 2, and 6
with the analytic
Jacobian in slide #13!

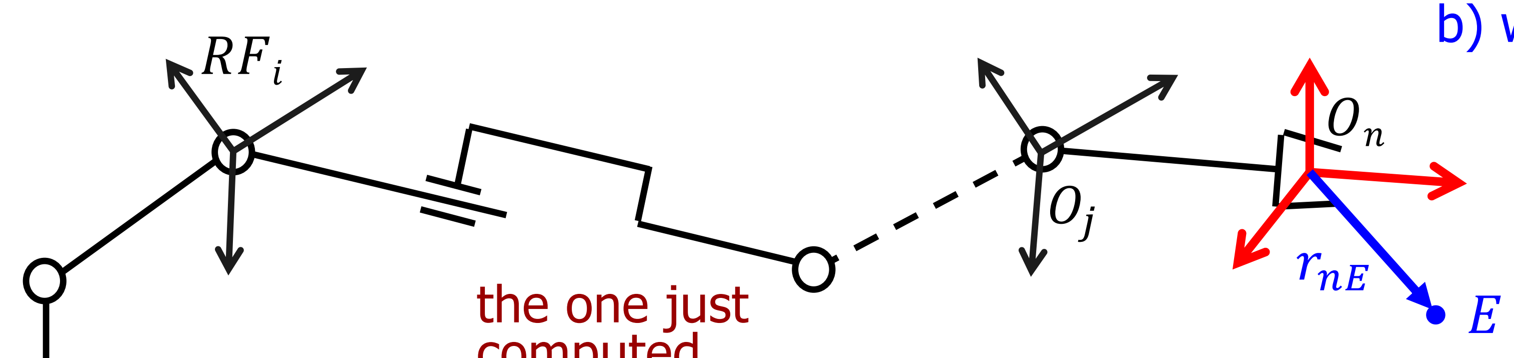
note: the Jacobian is here a 6 × 2 matrix,
thus its maximum rank is 2

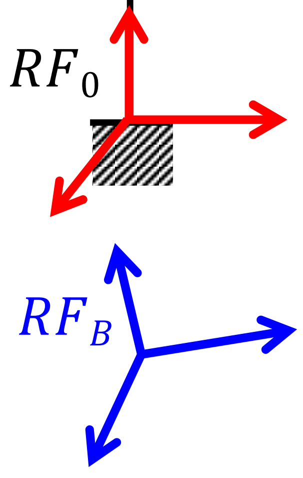
⬇

at most 2 components of the linear/angular end-effector velocity can be independently assigned

Transformations of Jacobian matrix

b) we may choose
$E \Rightarrow O_j(q)$





the one just computed …

$$\begin{pmatrix} {}^0v_n \\ {}^0\omega \end{pmatrix} = {}^0J_n(q)\,\dot{q}$$

$$\begin{aligned} v_E &= v_n + \omega \times r_{nE} \\ &= v_n + S(r_{En})\,\omega \end{aligned}$$

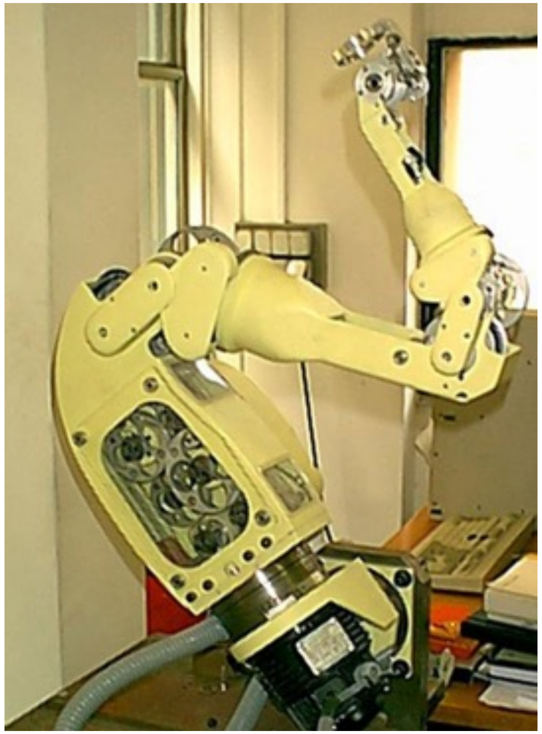
$$\begin{pmatrix} {}^Bv_E \\ {}^B\omega \end{pmatrix} = \begin{pmatrix} {}^BR_0 & 0 \\ 0 & {}^BR_0 \end{pmatrix} \begin{pmatrix} I & S({}^0r_{En}) \\ 0 & I \end{pmatrix} \begin{pmatrix} {}^0v_n \\ {}^0\omega \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} {}^BR_0 & 0 \\ 0 & {}^BR_0 \end{pmatrix} \begin{pmatrix} I & S({}^0r_{En}) \\ 0 & I \end{pmatrix}}_{\text{this part is never singular!}} {}^0J_n(q)\,\dot{q} = {}^BJ_E(q)\,\dot{q}$$

a) we may choose
$RF_B \Rightarrow RF_i(q)$

Example: Dexter robot

- 8R robot manipulator with transmissions by pulleys and steel cables (joints 3 to 8)
  - lightweight: only 15 kg in motion
  - 6 motors located inside the second link
  - incremental encoders (homing)
  - redundancy degree for e-e pose task: $n - m = 2$
  - compliant in the interaction with environment



i	a (mm)	d (mm)	α (rad)	range θ (deg)
0	0	0	$-\pi/2$	[-12.56, 179.89]
1	144	450	$-\pi/2$	[-83, 84]
2	0	0	$\pi/2$	[7, 173]
3	100	350	$\pi/2$	[65, 295]
4	0	0	$-\pi/2$	[-174, -3]
5	24	250	$-\pi/2$	[57, 265]
6	0	0	$-\pi/2$	[-129.99, -45]
7	100	0	π	[-55.05, 30]

Mid-frame Jacobian of Dexter robot

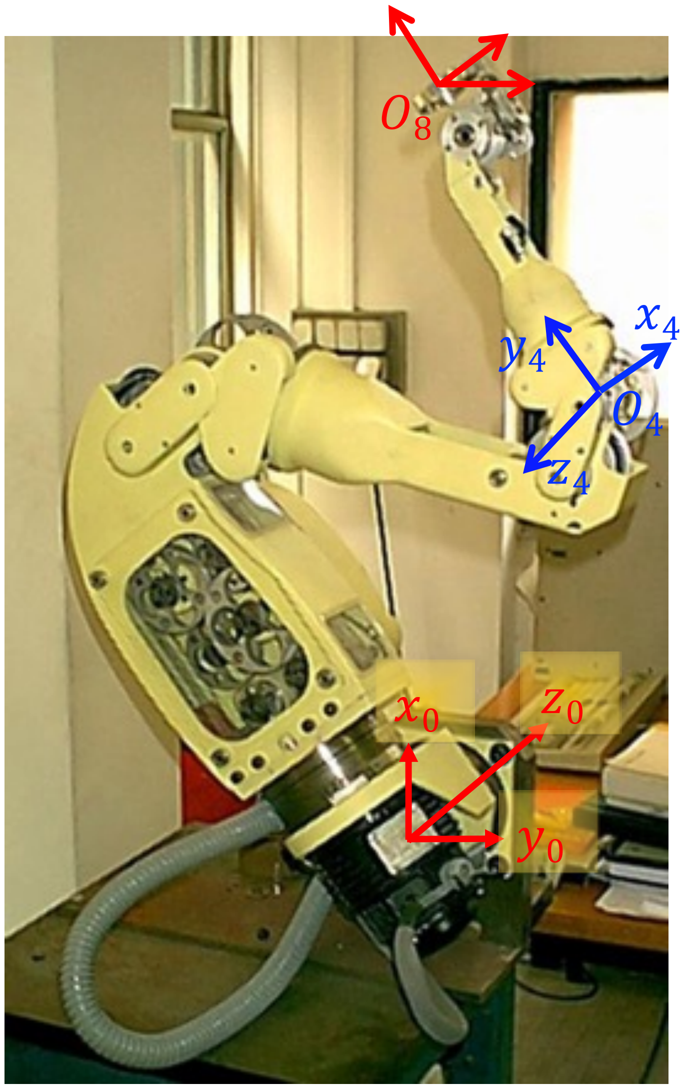
- geometric Jacobian ${}^{0}J_8(q)$ is very complex
- “mid-frame” Jacobian ${}^{4}J_4(q)$ is relatively simple!

$$
{}^4\hat{J}_4 = \left[\begin{matrix}
d_1 s_1 s_3 + d_3 s_3 c_2 s_1 - a_1 c_3 c_1 s_2 - d_1 c_3 c_1 c_2 - d_3 c_1 c_3 \\
-a_3 s_3 c_2 s_1 + a_3 c_3 c_1 + a_1 c_1 c_2 - d_1 c_1 s_2 \\
-d_3 c_3 c_2 s_1 - a_1 s_3 c_1 s_2 - d_1 s_3 c_1 c_2 - d_3 s_3 c_1 - d_1 s_1 c_3 + a_3 s_2 s_1 \\
-c_3 c_2 s_1 - s_3 c_1 \\
-s_2 s_1 \\
-s_3 c_2 s_1 + c_3 c_1
\end{matrix}\right.
$$

$$
\begin{matrix}
a_1 s_3 + d_3 s_3 s_2 & d_3 c_3 & 0 & 0 & 0 \\
-a_3 s_3 s_2 & -a_3 c_3 & 0 & 0 & 0 \\
-a_1 c_3 - d_3 c_3 s_2 - a_3 c_2 & d_3 s_3 & -a_3 & 0 & 0 \\
-c_3 s_2 & s_3 & 0 & 0 & -s_4 \\
c_2 & 0 & 1 & 0 & c_4 \\
-s_3 s_2 & -c_3 & 0 & 1 & 0
\end{matrix}
$$

6 rows,
8 columns

$$
\left.\begin{matrix}
-a_5 s_4 - d_5 c_5 c_4 & -a_5 s_5 c_4 c_6 + d_5 s_5 s_6 c_4 \\
-d_5 c_5 s_4 + a_5 c_4 & d_5 s_5 s_6 s_4 - a_5 s_5 s_4 c_6 \\
d_5 s_5 & -a_5 c_6 c_5 + d_5 c_5 s_6 \\
-c_4 s_5 & -c_4 c_5 s_6 + s_4 c_6 \\
-s_4 s_5 & -s_4 c_5 s_6 - c_4 c_6 \\
-c_5 & s_5 s_6
\end{matrix}\right]
$$



Summary of differential relations

$\dot{p} \rightleftarrows v$

$$\dot{p} = v$$

$\dot{R} \rightleftarrows \omega$

$$\dot{R} = S(\omega)R \quad \Longleftrightarrow \quad \text{for each (unit) column } r_i \text{ of } R \text{ (a frame): } \dot{r}_i = \omega \times r_i$$

$$S(\omega) = \dot{R}R^T$$

[in body frame ($\Omega = R^T\omega$): $\dot{R} = RS(\Omega)$, $S(\Omega) = R^T\dot{R} = R^T S(\omega) R$]

$\dot{\phi} \rightleftarrows \omega$

$$\omega = \omega_{\dot{\phi}_1} + \omega_{\dot{\phi}_2} + \omega_{\dot{\phi}_3} = \underset{\uparrow}{a_1}\dot{\phi}_1 + \underset{\uparrow}{a_2}(\phi_1)\dot{\phi}_2 + \underset{\uparrow}{a_3}(\phi_1, \phi_2)\dot{\phi}_3$$

$$= T(\phi)\,\dot{\phi}$$

(moving) axes of definition for the sequence of rotations ϕ_i, $i = 1,2,3$

---

special case: if the task vector r is

$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} \Rightarrow J_r(q) = \begin{pmatrix} I & 0 \\ 0 & T^{-1}(\phi) \end{pmatrix} J(q) \Longleftrightarrow J(q) = \begin{pmatrix} I & 0 \\ 0 & T(\phi) \end{pmatrix} J_r(q)$$

$J_r \rightleftarrows J$

$T(\phi)$ has always a singularity $\Leftrightarrow$ singularity of the specific minimal representation of orientation

Acceleration relations (and beyond…)

Higher-order differential kinematics

- differential relations between motion in the joint space and motion in the task space can be established at the second order, third order, ...
- the analytic Jacobian always "weights" the highest-order derivative

velocity $\dot{r} = J_r(q)\,\dot{q}$

matrix function $N_2(q,\dot{q})$

acceleration $\ddot{r} = J_r(q)\,\ddot{q} + \dot{J}_r(q)\dot{q}$

matrix function $N_3(q,\dot{q},\ddot{q})$

jerk $\dddot{r} = J_r(q)\,\dddot{q} + 2\dot{J}_r(q)\ddot{q} + \ddot{J}_r(q)\dot{q}$

snap $\ddddot{r} = J_r(q)\,\ddddot{q} + \cdots$

- the same holds true also for the geometric Jacobian $J(q)$

Primer on linear algebra

given a matrix J: $m \times n$ (m rows, n columns)

- rank $\rho(J)$ = max # of rows or columns that are linearly independent
  - $\rho(J) \leq \min(m, n) \Longleftarrow$ if equality holds, J has full rank
  - if $m = n$ and J has full rank, J is nonsingular and the inverse J^{-1} exists
  - $\rho(J)$ = dimension of the largest nonsingular square submatrix of J
- range space $\mathcal{R}(J)$ = subspace of all linear combinations of the columns of J

$$\mathcal{R}(J) = \{v \in \mathbb{R}^m : \exists \xi \in \mathbb{R}^n, v = J\xi\} \quad \longleftarrow$$ also called image of J

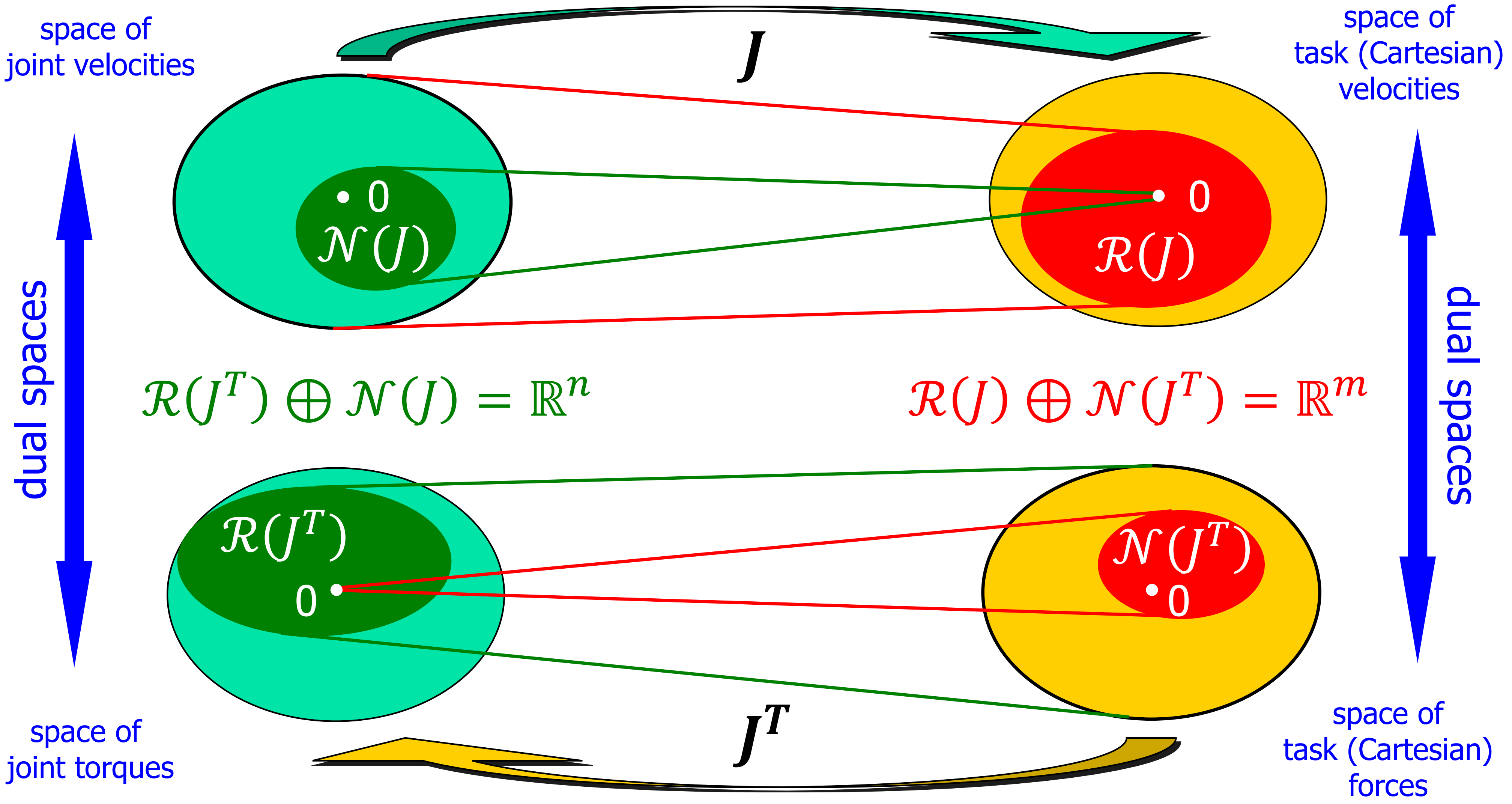
  - $\dim(\mathcal{R}(J)) = \rho(J)$
- null space $\mathcal{N}(J)$ = subspace of all vectors that are zeroed by matrix J

$$\mathcal{N}(J) = \{\xi \in \mathbb{R}^n : J\xi = 0 \in \mathbb{R}^m\} \quad \longleftarrow$$ also called kernel of J

  - $\dim(\mathcal{N}(J)) = n - \rho(J)$
- $\mathcal{R}(J) \oplus \mathcal{N}(J^T) = \mathbb{R}^m$ and $\mathcal{R}(J^T) \oplus \mathcal{N}(J) = \mathbb{R}^n$ (direct sum of subspaces)
  - any element $v \in V = V_1 + V_2$ can be written as $v = v_1 + v_2$, $v_1 \in V_1, v_2 \in V_2$
  - … in a unique way if and only if $V_1 \cap V_2 = \{0\}$ (a 'direct' sum, not just a sum!)

Robot Jacobian

decomposition in linear subspaces and duality



space of
joint velocities

J

space of
task (Cartesian)
velocities

0 $\mathcal{N}(J)$

0 $\mathcal{R}(J)$

dual spaces

dual spaces

$$\mathcal{R}(J^T) \oplus \mathcal{N}(J) = \mathbb{R}^n$$

$$\mathcal{R}(J) \oplus \mathcal{N}(J^T) = \mathbb{R}^m$$

$\mathcal{R}(J^T)$ 0

$\mathcal{N}(J^T)$ 0

J^T

space of
joint torques

space of
task (Cartesian)
forces

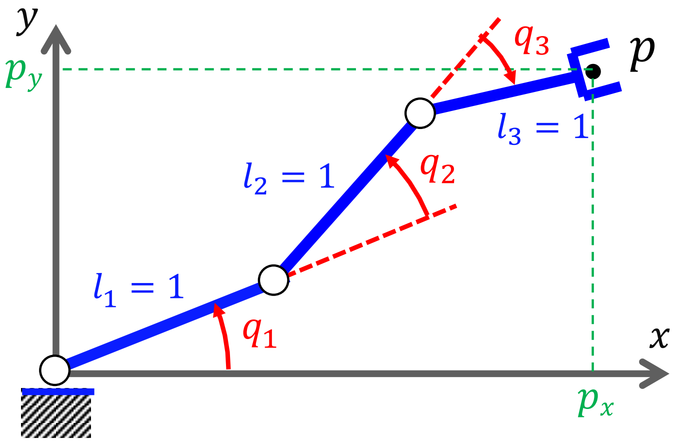
(in a given configuration q)

Mobility analysis in the task space

- $\rho(J) = \rho(J(q))$, $\mathcal{R}(J) = \mathcal{R}(J(q))$, $\mathcal{N}(J^T) = \mathcal{N}(J^T(q))$, etc. are locally defined, i.e., they depend on the current configuration q
- $\mathcal{R}(J(q))$ is the subspace of all "generalized" velocities (with linear and/or angular components) that can be instantaneously realized by the robot end-effector when varying the joint velocities $\dot{q}$ at the current q
- if $\rho\big(J(q)\big) = m$ at q ($J(q)$ has max rank, with $m \leq n$), the end-effector can be moved in any direction of the task space $\mathbb{R}^m$
- if $\rho(J(q)) < m$, there are directions in $\mathbb{R}^m$ in which the end-effector cannot move (at least, not instantaneously!)
  - these directions $\in \mathcal{N}(J^T(q))$, the complement of $\mathcal{R}(J(q))$ to task space $\mathbb{R}^m$, which is of dimension $m - \rho(J(q))$
- if $\mathcal{N}(J(q)) \neq \{0\}$, there are non-zero joint velocities $\dot{q}$ that produce zero end-effector velocity ("self motions")
  - this happens always for $m < n$, i.e., when the robot is redundant for the task

Mobility analysis for a planar 3R robot

whiteboard …



$l_1 = l_2 = l_3 = 1$ $\quad n = 3, \; m = 2$

$$WS_1 = \{p \in \mathbb{R}^2 : \|p\| \le 3\} \subset \mathbb{R}^2$$

$$WS_2 = \{p \in \mathbb{R}^2 : \|p\| \le 1\} \subset \mathbb{R}^2$$

$$p = \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix}$$

in $\mathbb{R}^2$

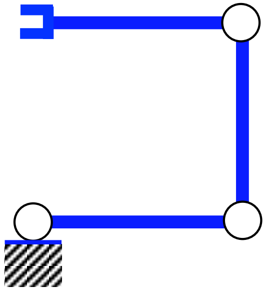
in $\mathbb{R}^3$

$$v = \dot{p} = \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} \dot{q} = J(q)\dot{q}$$

case 1)

$q = (0, \pi/2, \pi/2)$

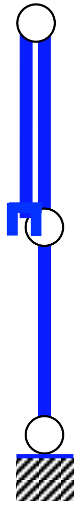
$$J = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$



case 2)

$q = (\pi/2, 0, \pi)$

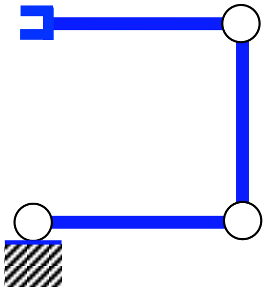
$$J = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



- run the MATLAB code subspaces_3Rplanar.m available in the course material

Mobility analysis for a planar 3R robot

whiteboard …



$q = (0, \pi/2, \pi/2)$

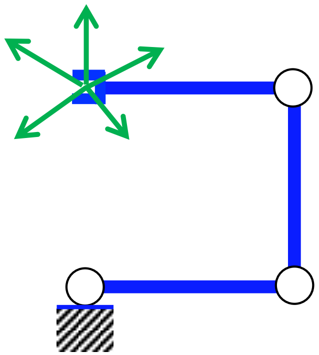
case 1)

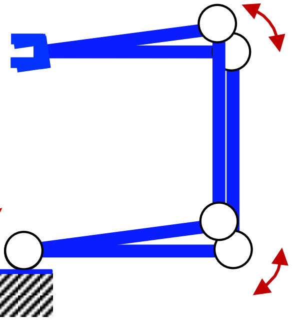
$$J = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad J^T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\rho(J) = 2 = m \qquad \rho(J^T) = \rho(J) = 2$$

$$\mathcal{R}(J) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2 \qquad \mathcal{N}(J) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\dim \mathcal{N}(J) = 1$$
$$= n - \rho(J)(= n - m)$$





$$\mathcal{R}(J) \oplus \mathcal{N}(J^T) = \mathbb{R}^2$$
$$\mathcal{R}(J^T) \oplus \mathcal{N}(J) = \mathbb{R}^3$$

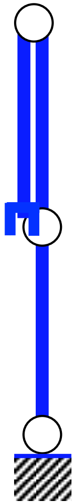
$$\mathcal{R}(J^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\dim \mathcal{R}(J^T) = 2$$
$$= \rho(J)\ (= m)$$

$$\mathcal{N}(J^T) = 0$$

Mobility analysis for a planar 3R robot

whiteboard ...



$q = (\pi/2, 0, \pi)$

case 2)

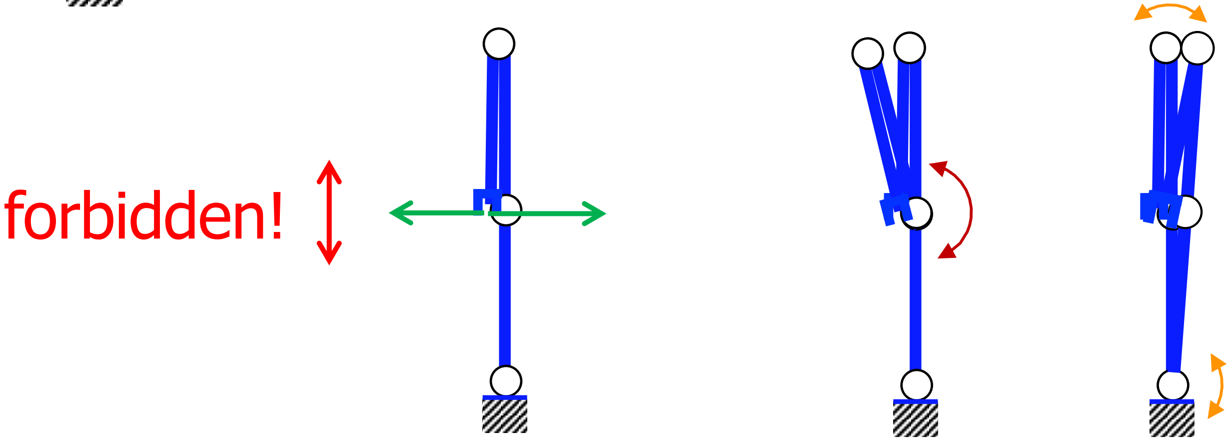
$$J = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad J^T = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\rho(J) = 1 < m \qquad \rho(J^T) = \rho(J) = 1$$

$$\mathcal{R}(J) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\dim \mathcal{R}(J) = 1 = \rho(J)$$

$$\mathcal{N}(J) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \begin{aligned} \dim \mathcal{N}(J) &= 2 \\ &= n - \rho(J) \end{aligned}$$



$$\mathcal{R}(J) \oplus \mathcal{N}(J^T) = \mathbb{R}^2$$

$$\mathcal{R}(J^T) \oplus \mathcal{N}(J) = \mathbb{R}^3$$

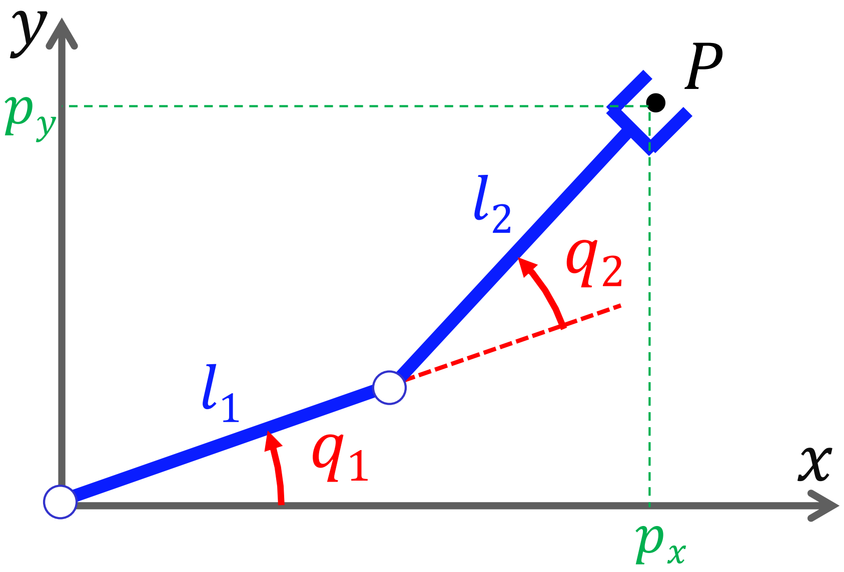
$$\mathcal{R}(J^T) = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \begin{aligned} \dim \mathcal{R}(J^T) &= 1 \\ &= \rho(J) \end{aligned}$$

$$\mathcal{N}(J^T) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \qquad \begin{aligned} \dim \mathcal{N}(J^T) &= 1 \\ &= m - \rho(J) \end{aligned}$$

Kinematic singularities

- **configurations where the Jacobian loses rank**
  $\Leftrightarrow$ **loss of instantaneous mobility** of the robot end-effector
- for $m = n$ (≤ 6), they correspond to Cartesian poses at which the number of solutions of the inverse kinematics problem differs from the generic case
- "in" a singular configuration, we cannot find any joint velocity that realizes a desired end-effector velocity in some directions of the task space
- "close" to a singularity, large joint velocities may be needed to realize even a small velocity of the end-effector in some directions of the task space
- finding and analyzing in advance the mobility of a robot helps in singularity avoidance during trajectory planning and motion control
  - when $m = n$: find the configurations q such that $\det J(q) = 0$
  - when $m < n$: find the configurations q such that all $m \times m$ minors of $J(q)$ are singular (or, equivalently, such that $\det(J(q)J^T(q)) = 0$)
- finding all singular configurations of a robot with a large number of joints, or the actual "distance" from a singularity, is a complex computational task

Singularities of planar 2R robot



$$\boxed{\det J(q) = l_1 l_2 s_2}$$

direct kinematics

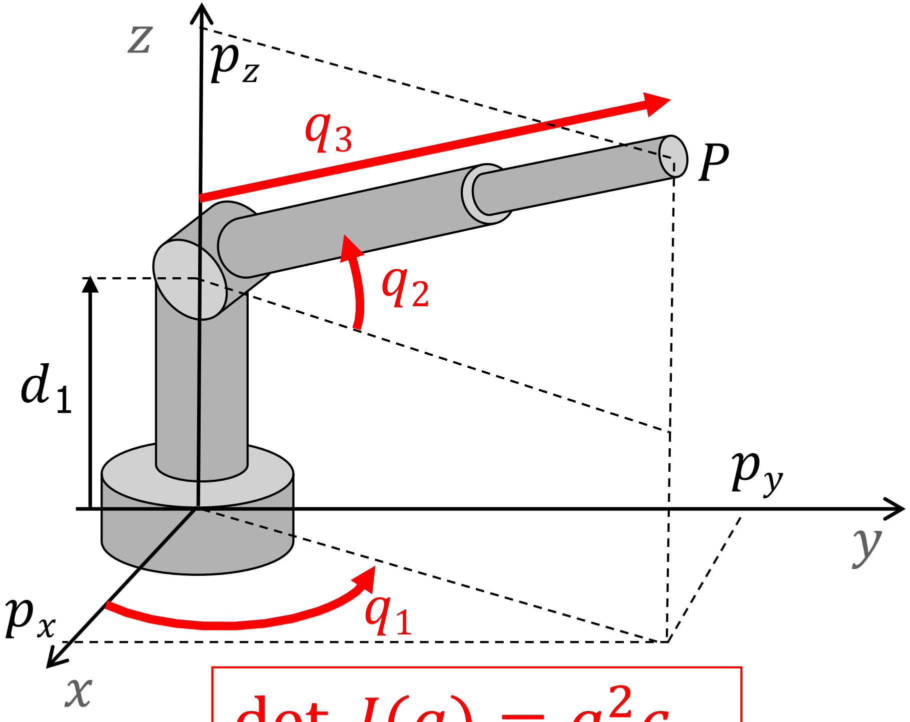
$$p_x = l_1 c_1 + l_2 c_{12}$$
$$p_y = l_1 s_1 + l_2 s_{12}$$

analytic Jacobian

$$\dot{p} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \dot{q} = J(q)\dot{q}$$

- singularities: robot arm is stretched ($q_2 = 0$) or folded ($q_2 = \pi$)
- singular configurations correspond here to Cartesian points that are on the boundary of the primary workspace (or at the center of WS_1 if $l_1 = l_2$)
- in many cases (as here), singularities separate regions of the configuration space with distinct inverse kinematic solutions (e.g., elbow “up” or “down”)

Singularities of polar (RRP) robot



$$\boxed{\det J(q) = q_3^2 c_2}$$

direct kinematics

$$p_x = q_3 c_2 c_1$$

$$p_y = q_3 c_2 s_1$$

$$p_z = d_1 + q_3 s_2$$

analytic Jacobian

$$\dot{p} = \begin{pmatrix} -q_3 s_1 c_2 & -q_3 c_1 s_2 & c_1 c_2 \\ q_3 c_1 c_2 & -q_3 s_1 s_2 & s_1 c_2 \\ 0 & q_3 c_2 & s_2 \end{pmatrix} \dot{q}$$

$$= J(q)\dot{q}$$

- singularities
  - E-E is along the z axis ($q_2 = \pm\pi/2$): simple singularity ⇒ rank $\rho(J) = 2$
  - third link is fully retracted ($q_3 = 0$): double singularity ⇒ rank $\rho(J)$ drops to 1
- all singular configurations correspond here to Cartesian points internal to the workspace (supposing no range limits for the prismatic joint)

Singularities of robots with spherical wrist

- $n = 6$, last three joints are revolute and their axes intersect at a point
- without loss of generality, we set $O_6 = W$ = center of spherical wrist (i.e., choose $d_6 = 0$ in DH table) and obtain for the geometric Jacobian

$$J(q) = \begin{pmatrix} J_{11} & 0 \\ J_{12} & J_{22} \end{pmatrix}$$

- since $\det J(q_1, \cdots, q_5) = \det J_{11} \cdot \det J_{22}$, there is a decoupling property
  - $\det J_{11}(q_1, q_2, q_3) = 0$ provides the arm singularities
  - $\det J_{22}(q_4, q_5) = 0$ provides the wrist singularities
- being in the geometric Jacobian $J_{22} = (z_3 \quad z_4 \quad z_5)$, wrist singularities correspond to when z_3, z_4 and z_5 become linearly dependent vectors

  $\Longrightarrow$ when either $q_5 = 0$ or $q_5 = \pm\pi/2$ (see Euler angles singularities!)
- inversion of $J(q)$ is simpler (block triangular structure)
- the determinant of $J(q)$ will never depend on q_1: why?