



Robotics 1

Position and orientation of rigid bodies

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Position and orientation

right-handed orthogonal Reference Frames

rigid body

position: ${}^A p_{AB}$ (vector $\in \mathbb{R}^3$)
Cartesian coordinates of vector \overrightarrow{AB} expressed in RF_A

orientation:
orthonormal 3×3 matrix
($R^T = R^{-1} \Rightarrow R^T R = I$), with $\det = +1$

$${}^A R_B = \begin{bmatrix} {}^A x_B & {}^A y_B & {}^A z_B \end{bmatrix}$$

- $x_A \ y_A \ z_A$ ($x_B \ y_B \ z_B$) are axis vectors (of unitary norm) of frame RF_A (RF_B)
- components in ${}^A R_B$ are the **direction cosines** of the axes of RF_B with respect to (w.r.t.) RF_A



Position of a rigid body

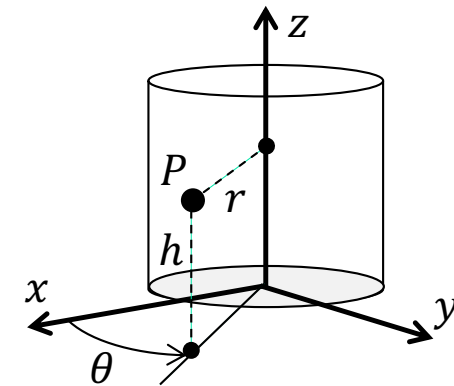
- for position representation, use of other coordinates than the Cartesian ones is possible, e.g., cylindrical or spherical
- direct transformation from cylindrical to Cartesian

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = h$$

is always well defined
(with $r \geq 0$ or $r \geq 0$)



- inverse transformation from Cartesian to cylindrical

$$x^2 + y^2 = r^2$$

$$\frac{y}{x} = \tan \theta$$

assuming +
($r \geq 0$ only)

$$r = \sqrt{x^2 + y^2}$$

$$\Rightarrow \theta = \text{atan2}\{y, x\}$$

$$h = z$$

with a singularity
for $x = y = 0$

four-quadrant arc tangent



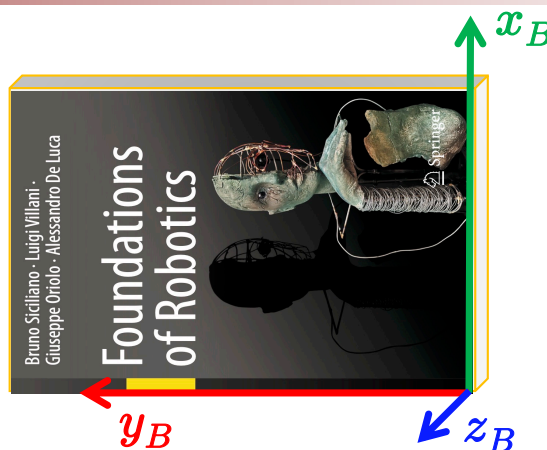
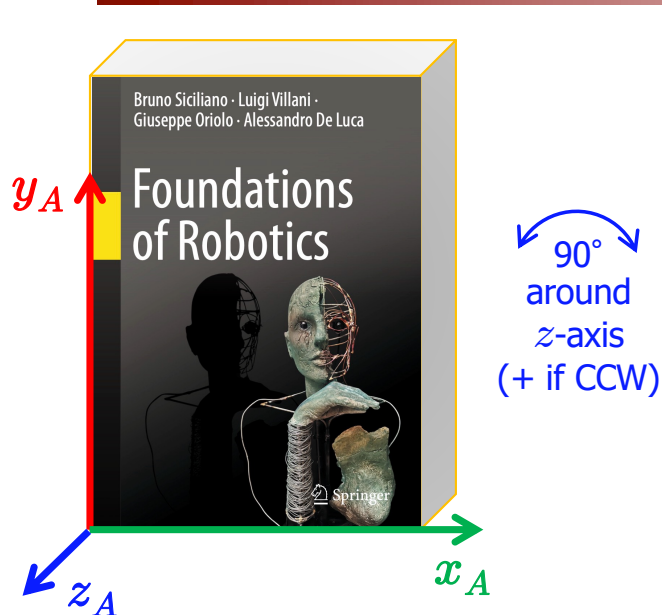
atan2 function

- arctangent with output values “in the four quadrants”
 - two input arguments
 - takes values in $[-\pi, +\pi] \Rightarrow (-\pi, +\pi]$
 - undefined only for (0, 0)
- uses the sign of both arguments to define the output quadrant
- based on **arctan** function with output values in $[-\pi/2, +\pi/2]$
- available in main languages (C++, MATLAB, ...)

$$\text{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & y \geq 0, x < 0 \\ -\pi + \arctan(\frac{y}{x}) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

← MATLAB sets
“arbitrarily”
 $\text{atan2}(0,0)=0$

Orientation of a rigid body

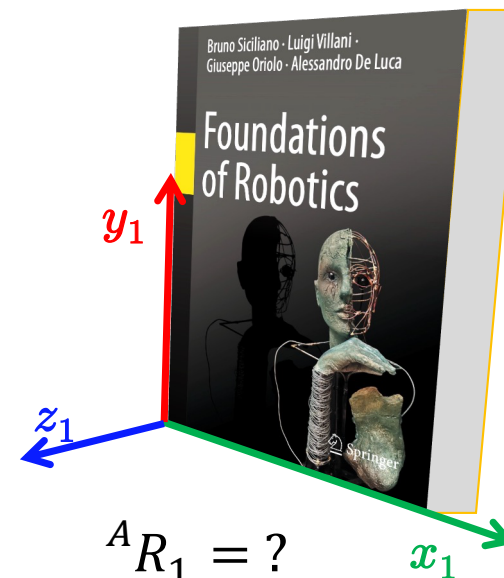


$${}^A R_B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

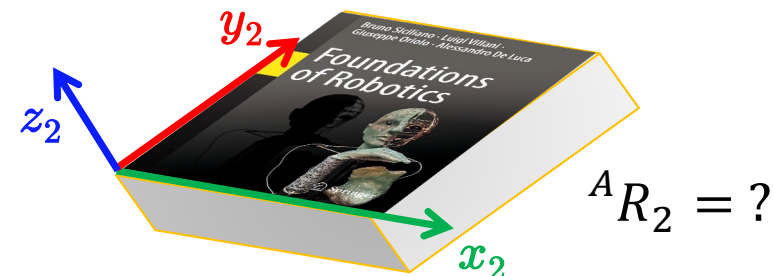
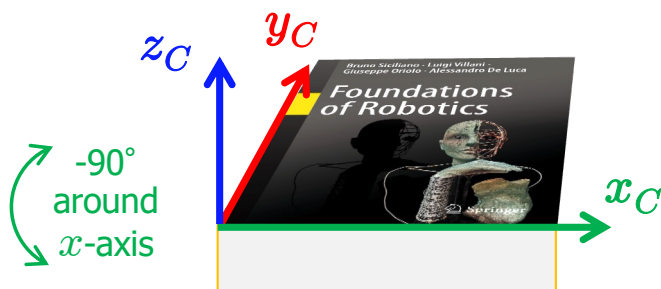
$${}^A R_A = {}^A R_B {}^B R_A = I$$

$${}^B R_C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = {}^B R_A {}^A R_C = {}^A R_B^T {}^A R_C$$

$${}^A R_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$



$${}^B R_A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = {}^A R_B^T$$





Rotation matrix

orthonormal,
with $\det = +1$

$${}^A R_B = \begin{bmatrix} x_A^T x_B & x_A^T y_B & x_A^T z_B \\ y_A^T x_B & y_A^T y_B & y_A^T z_B \\ z_A^T x_B & z_A^T y_B & z_A^T z_B \end{bmatrix}$$

direction cosine of
 z_B w.r.t. x_A

$$x_A^T z_B = \|x_A\| \|z_B\| \cos \beta = \cos \beta$$

algebraic structure
of a group $SO(3)$:
neutral element = I ,
inverse element = R^T

chain rule property

$${}^k R_i {}^i R_j = {}^k R_j$$

orientation of RF_i
w.r.t. RF_k

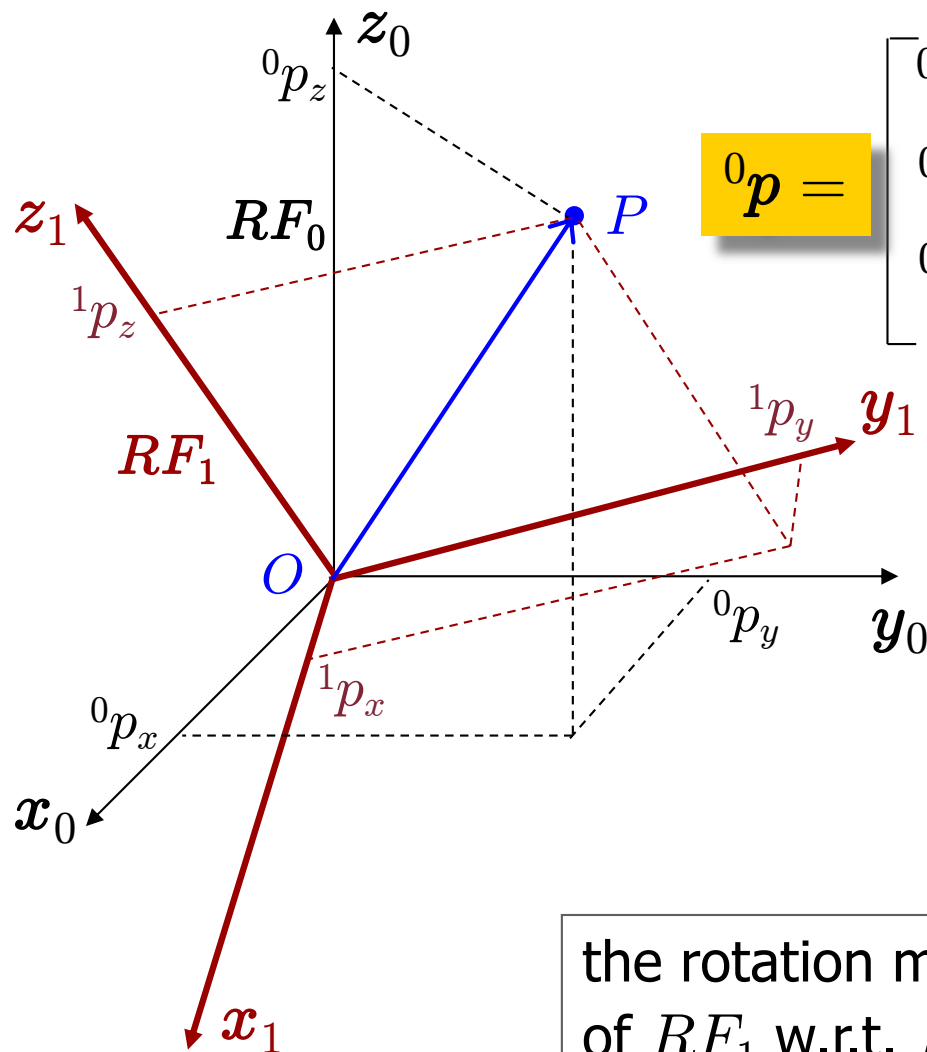
orientation of RF_j
w.r.t. RF_i

orientation of RF_j
w.r.t. RF_k

NOTE: in general, the **product** of rotation matrices does **not** commute!



Change of coordinates



$${}^0p =$$

$$\begin{bmatrix} {}^0p_x \\ {}^0p_y \\ {}^0p_z \end{bmatrix}$$

$$\begin{aligned} &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \\ &= {}^0p_x {}^0x_0 + {}^0p_y {}^0y_0 + {}^0p_z {}^0z_0 \\ &= {}^1p_x {}^0x_1 + {}^1p_y {}^0y_1 + {}^1p_z {}^0z_1 \end{aligned}$$

$$= \begin{bmatrix} {}^0x_1 & {}^0y_1 & {}^0z_1 \end{bmatrix} \begin{bmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \end{bmatrix}$$

$$= {}^0R_1 {}^1p$$

the rotation matrix 0R_1 (i.e., the orientation of RF_1 w.r.t. RF_0) represents **also** the change of coordinates of a **vector** from RF_1 to RF_0

Change of coordinates an example

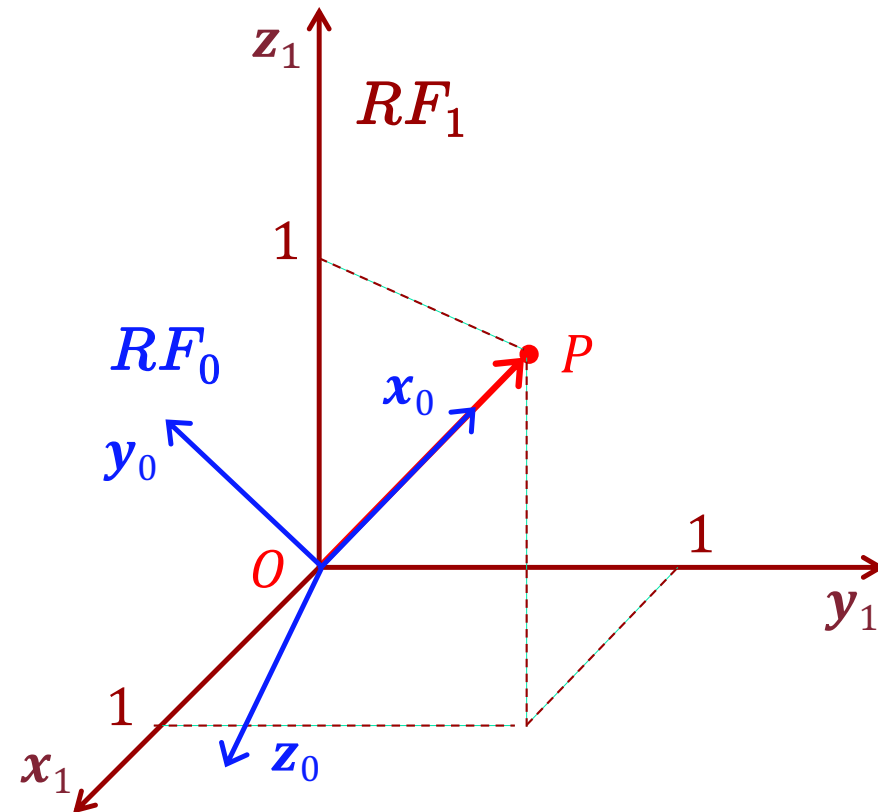
$${}^1\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$${}^0R_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$${}^0\mathbf{p} = {}^0R_1 {}^1\mathbf{p} = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}$$

$$\|\mathbf{p}\| = \|{}^0\mathbf{p}\| = \|{}^1\mathbf{p}\| = \sqrt{3}$$

... and where is RF_0 ?

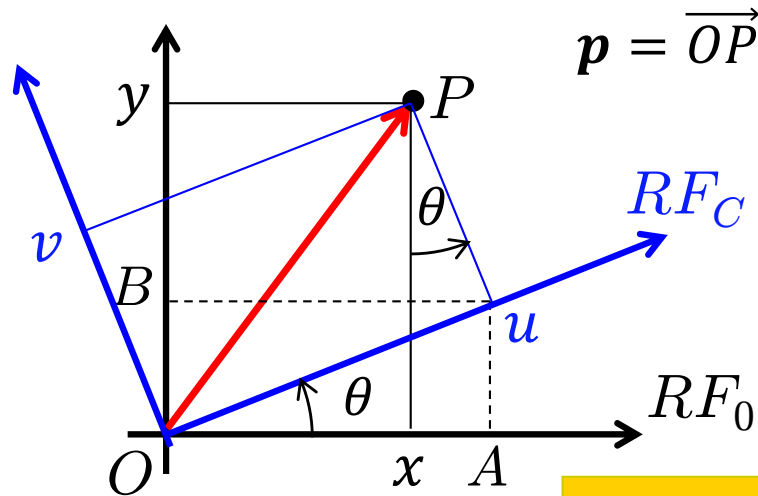


- x_0 is aligned with $\mathbf{p} = \overrightarrow{OP}$
- z_0 is orthogonal to y_1 ($z_0^T y_1 = 0$) and is positive on x_1 ($z_0^T x_1 = 1/\sqrt{2}$)
- y_0 completes a right-handed frame



Orientation of frames in a plane

(elementary rotation around z -axis)



$$\begin{aligned}x &= OA - xA = u \cos \theta - v \sin \theta \\y &= OB + By = u \sin \theta + v \cos \theta \\z &= w\end{aligned}$$

or...

$${}^0\mathbf{p} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_z(\theta) \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

${}^0\mathbf{x}_c$ ${}^0\mathbf{y}_c$ ${}^0\mathbf{z}_c$ ${}^0\mathbf{R}_c$ ${}^c\mathbf{p}$
 \downarrow \downarrow \downarrow \downarrow \downarrow

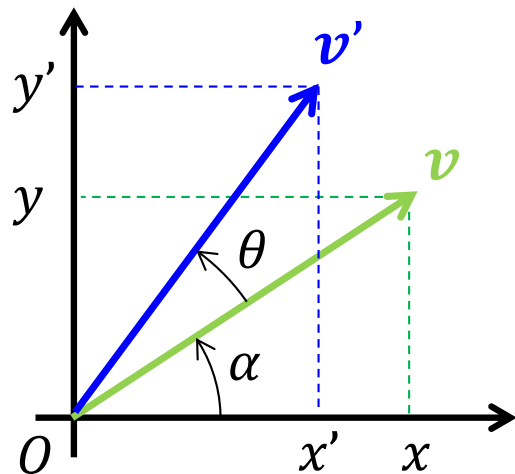
similarly:

$$R_z(-\theta) = R_z^T(\theta)$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$



Rotation of a vector around z



$$x = \|v\| \cos \alpha$$

$$y = \|v\| \sin \alpha$$

$$\begin{aligned} x' &= \|v\| \cos (\alpha + \theta) = \|v\| (\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

$$\begin{aligned} y' &= \|v\| \sin (\alpha + \theta) = \|v\| (\sin \alpha \cos \theta + \cos \alpha \sin \theta) \\ &= x \sin \theta + y \cos \theta \end{aligned}$$

$$z' = z$$

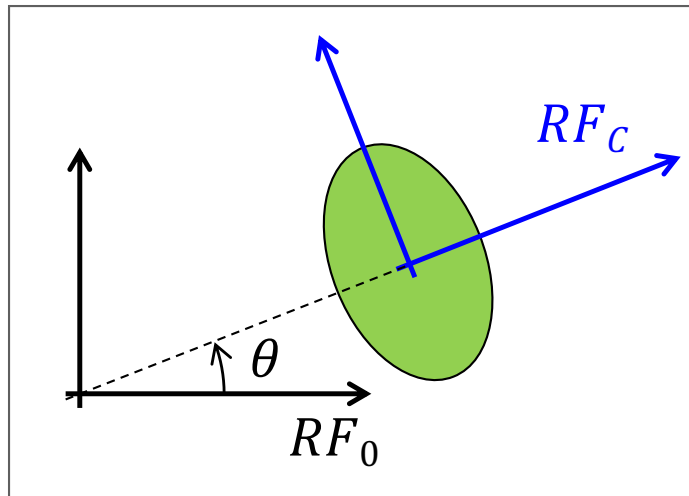
or...

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

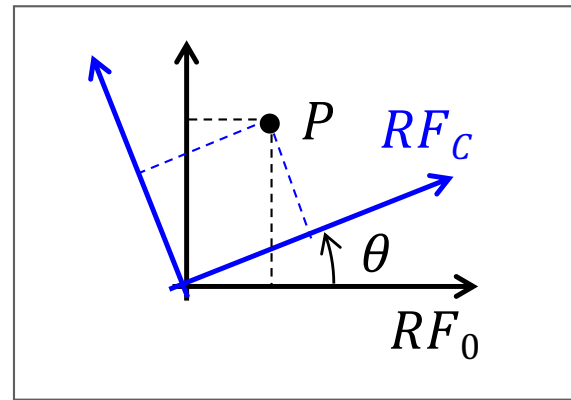
... same as before!

Equivalent interpretations of a rotation matrix

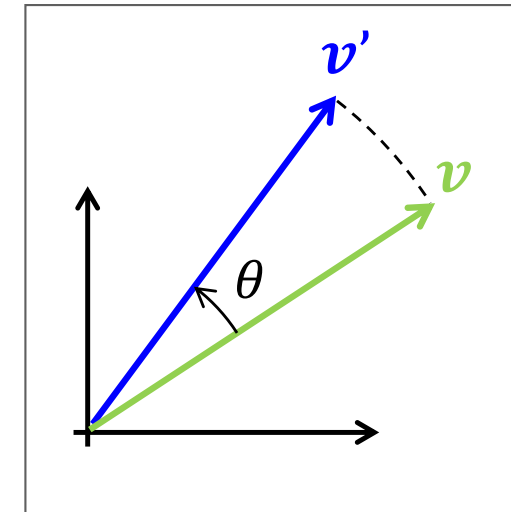
the **same** rotation matrix (e.g., $R_z(\theta)$) may represent



the orientation of a rigid
body with respect to a
reference frame RF_0
e.g., ${}^0\mathbf{x}_c \ {}^0\mathbf{y}_c \ {}^0\mathbf{z}_c = R_z(\theta)$



the change of coordinates
from RF_C to RF_0
e.g., ${}^0\mathbf{p} = R_z(\theta) {}^c\mathbf{p}$

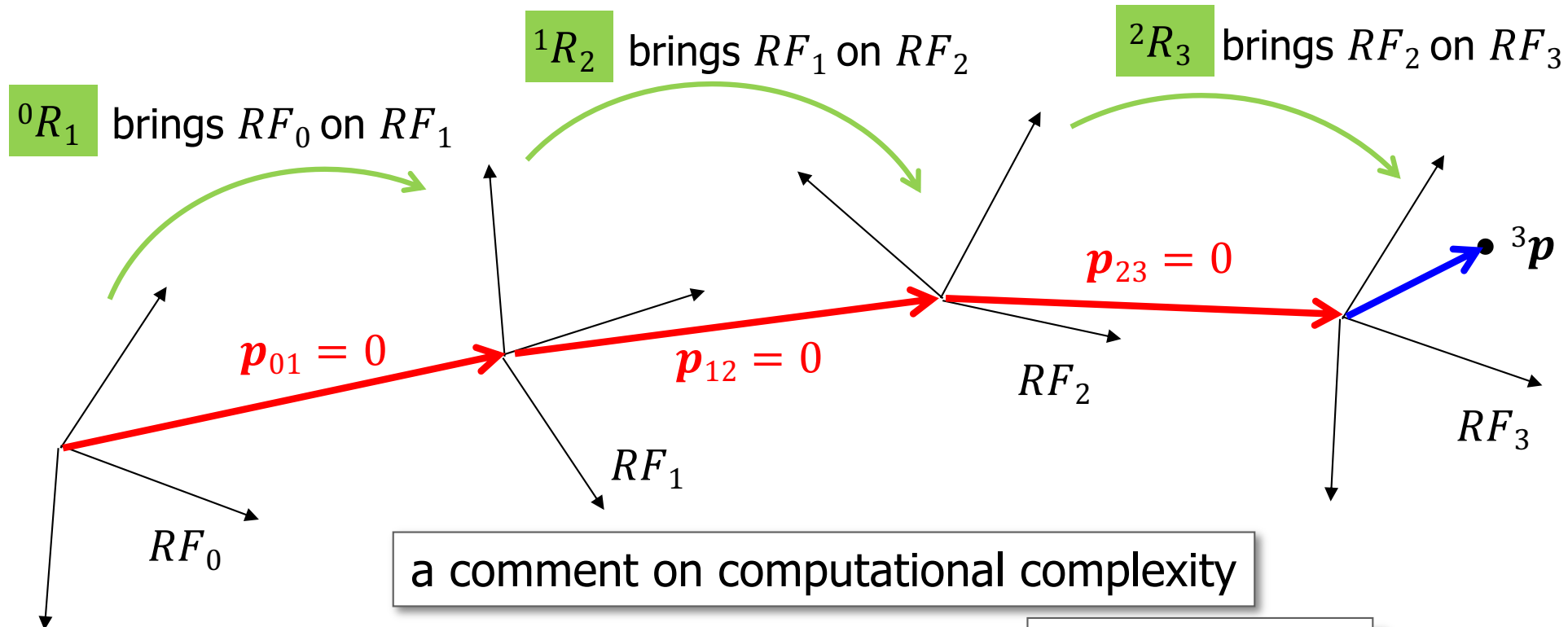


the rotation
operator on vectors
e.g., $\mathbf{v}' = R_z(\theta) \mathbf{v}$

the rotation matrix 0R_C is an operator
superposing frame RF_0 to frame RF_C



Composition of rotations



$${}^0p = ({}^0R_1 {}^1R_2 {}^2R_3) {}^3p = {}^0R_3 {}^3p$$

63 products
42 summations

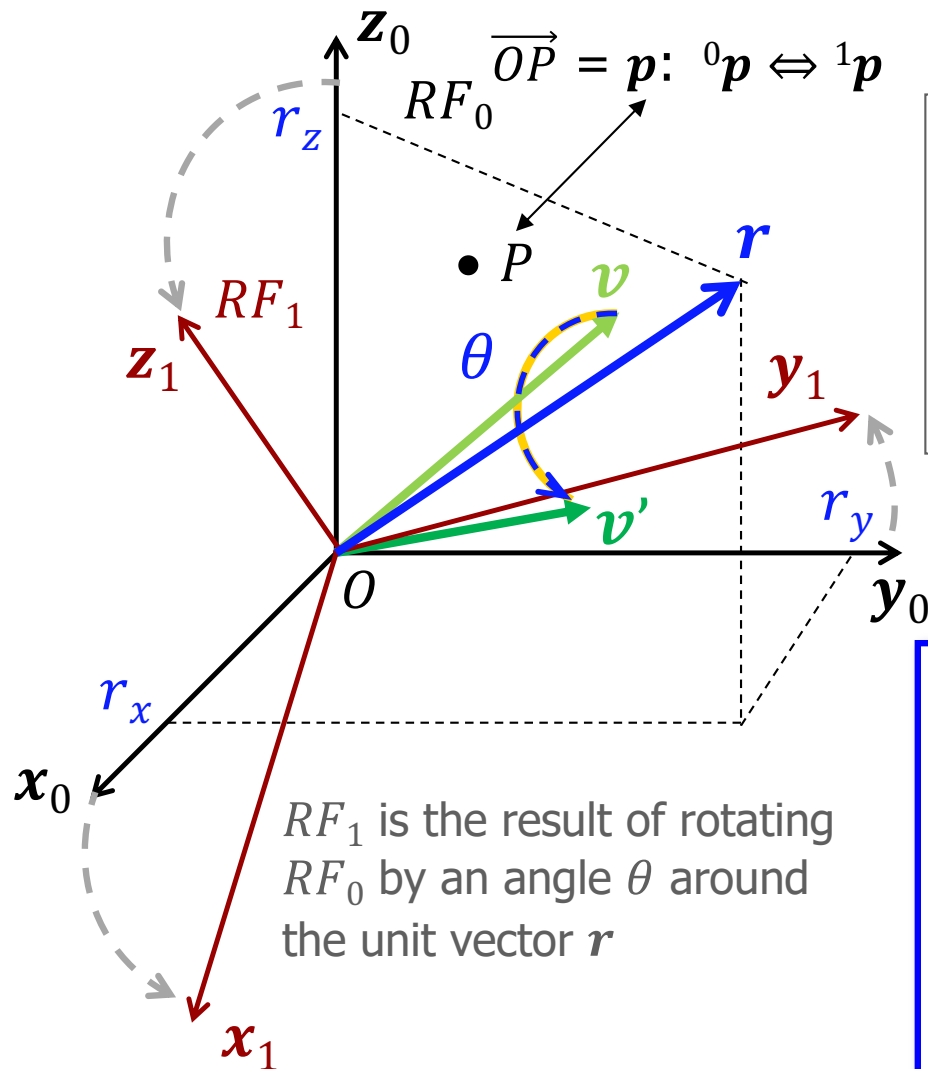
$${}^0p = {}^0R_1 ({}^1R_2 ({}^2R_3 {}^3p))$$

27 products
18 summations

$$\underbrace{{}^2p}_{{}^1p}$$



Axis/angle representation



DATA

- axis \mathbf{r} (unit vector in \mathbb{R}^3 , $\|\mathbf{r}\| = 1$)
- angle θ , positive **counterclockwise** (as seen from an "observer" oriented like \mathbf{r} with the head placed on the arrow, looking down to her/his feet)

DIRECT PROBLEM

parametrized by the given data!

find a rotation matrix $R(\theta, \mathbf{r})$ such that

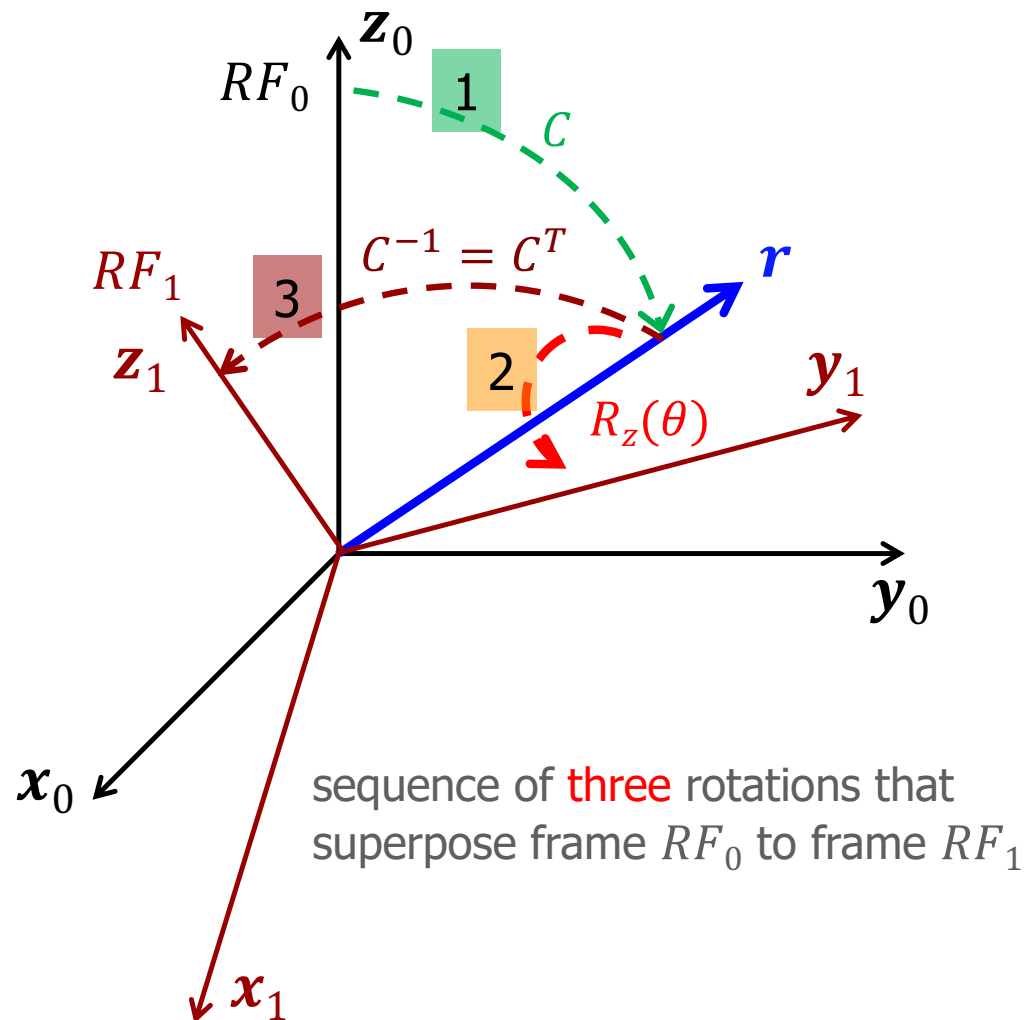
$$R(\theta, \mathbf{r}) = [{}^0\mathbf{x}_1 \ {}^0\mathbf{y}_1 \ {}^0\mathbf{z}_1]$$

$${}^0\mathbf{p} = R(\theta, \mathbf{r}) {}^1\mathbf{p}$$

$${}^0\mathbf{v}' = R(\theta, \mathbf{r}) {}^0\mathbf{v}$$



Axis/angle: Direct problem



$$R(\theta, \mathbf{r}) = \mathbf{C} R_z(\theta) \mathbf{C}^T$$

sequence of **three** rotations
(one of which is elementary)

$$\mathbf{C} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{r} \end{bmatrix}$$

after the first rotation
the z-axis coincides with \mathbf{r}

\mathbf{n} and \mathbf{s} are orthogonal
unit vectors such that
 $\mathbf{n} \times \mathbf{s} = \mathbf{r}$

Inner and outer products

whiteboard...



- (inner) **row by column** products between two 3×3 (orthonormal) matrices

$$C^T C = \begin{bmatrix} \mathbf{n}^T \\ \mathbf{s}^T \\ \mathbf{r}^T \end{bmatrix} \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

- **dyadic expansion** of a $n \times n$ generic matrix

$$\mathbf{e}_i = [0 \quad \dots \quad 1 \quad \dots \quad 0]^T, \quad i = 1, \dots, n \quad \Rightarrow \quad A = \sum_{i,j=1}^n a_{ij} \mathbf{e}_i \mathbf{e}_j^T (= I A I^T)$$

- **product** of three $n \times n$ matrices **using dyadic form**

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_{n-1} & \mathbf{b}_n \end{bmatrix} \quad \Rightarrow \quad B A B^T = \sum_{i,j=1}^n a_{ij} \mathbf{b}_i \mathbf{b}_j^T$$

- (outer) **column by row** products between two 3×3 matrices

$$\begin{aligned} C C^T = I &\Rightarrow C C^T = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{r} \end{bmatrix} \begin{bmatrix} \mathbf{n}^T \\ \mathbf{s}^T \\ \mathbf{r}^T \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{r} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{n}^T \\ \mathbf{s}^T \\ \mathbf{r}^T \end{bmatrix} \\ &= \mathbf{n} \mathbf{n}^T + \mathbf{s} \mathbf{s}^T + \mathbf{r} \mathbf{r}^T = I \end{aligned}$$



Skew-symmetric matrices

whiteboard...

- properties of a **skew-symmetric matrix**

- a square matrix S is skew-symmetric iff $S^T = -S$

$$\Leftrightarrow s_{ij} = -s_{ji} \Rightarrow s_{ii} = 0 \text{ (zeros on the diagonal)}$$

- any square matrix A can be decomposed into its symmetric and skew-symmetric parts

$$A = \frac{A+A^T}{2} + \frac{A-A^T}{2} = A_{\text{symm}} + A_{\text{skew}}$$

- in quadratic forms the skew-symmetric part vanishes (only the symmetric part matters)

$$x^T A x = \frac{1}{2}[x^T A x + (x^T A x)^T] = \frac{1}{2}[x^T A x + x^T A^T x] = x^T \frac{A + A^T}{2} x = x^T A_{\text{symm}} x$$

- canonical form of a **3x3** skew-symmetric matrix

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow S(v) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \quad S = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

also called **vee map** v
 $v = S^v$

- expression of the **vector product** between two vectors $\in \mathbb{R}^3$

$$n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, s = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \Rightarrow r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = n \times s = \begin{bmatrix} n_y s_z - s_y n_z \\ n_z s_x - s_z n_x \\ n_x s_y - s_x n_y \end{bmatrix} = S(n) s$$

Sarrus rule for determinant of $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ n_x & n_y & n_z \\ s_x & s_y & s_z \end{bmatrix}$

$$v_1 \times v_2 = S(v_1)v_2 = -v_2 \times v_1 = -S(v_2)v_1 = S^T(v_2)v_1$$



Axis/angle: Direct problem solution

$$R(\theta, \mathbf{r}) = C R_z(\theta) C^T$$

$$R(\theta, \mathbf{r}) = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{r} \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{n}^T \\ \mathbf{s}^T \\ \mathbf{r}^T \end{bmatrix}$$
$$= \mathbf{r}\mathbf{r}^T + (\mathbf{n}\mathbf{n}^T + \mathbf{s}\mathbf{s}^T) c\theta + (\mathbf{s}\mathbf{n}^T - \mathbf{n}\mathbf{s}^T) s\theta$$

taking into account

$$C C^T = \mathbf{n}\mathbf{n}^T + \mathbf{s}\mathbf{s}^T + \mathbf{r}\mathbf{r}^T = I$$

$$\mathbf{s}\mathbf{n}^T - \mathbf{n}\mathbf{s}^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\mathbf{r})$$

depends only
on \mathbf{r} and θ !

$$\rightarrow R(\theta, \mathbf{r}) = \mathbf{r}\mathbf{r}^T + (I - \mathbf{r}\mathbf{r}^T) c\theta + S(\mathbf{r}) s\theta$$



Final expression of $R(\theta, \mathbf{r})$

developing computations...

$$R(\theta, \mathbf{r}) =$$

$$\begin{bmatrix} r_x^2(1 - \cos \theta) + \cos \theta & r_x r_y(1 - \cos \theta) - r_z \sin \theta & r_x r_z(1 - \cos \theta) + r_y \sin \theta \\ r_x r_y(1 - \cos \theta) + r_z \sin \theta & r_y^2(1 - \cos \theta) + \cos \theta & r_y r_z(1 - \cos \theta) - r_x \sin \theta \\ r_x r_z(1 - \cos \theta) - r_y \sin \theta & r_y r_z(1 - \cos \theta) + r_x \sin \theta & r_z^2(1 - \cos \theta) + \cos \theta \end{bmatrix}$$

note that

sum of the diagonal
elements of a matrix



$$\text{trace } R(\theta, \mathbf{r}) = 1 + 2 \cos \theta$$

$$R(\theta, \mathbf{r}) = R(-\theta, -\mathbf{r}) = R^T(-\theta, \mathbf{r})$$



Axis/angle: a simple example

$$R(\theta, \mathbf{r}) = \mathbf{r}\mathbf{r}^T + (I - \mathbf{r}\mathbf{r}^T) c\theta + S(\mathbf{r}) s\theta$$

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{z}_0$$

$$\begin{aligned} R(\theta, \mathbf{r}) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} c\theta + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s\theta \\ &= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_z(\theta) \end{aligned}$$



Properties of $R(\theta, \mathbf{r})$

1. $R(\theta, \mathbf{r})\mathbf{r} = \mathbf{r}$ (\mathbf{r} is the **invariant** axis in this rotation)
2. when \mathbf{r} is one of the coordinate axes, R boils down to one of the known elementary rotation matrices
3. $(\theta, \mathbf{r}) \rightarrow R$ is **not** an **injective** map: $R(\theta, \mathbf{r}) = R(-\theta, -\mathbf{r})$
4. $\det R = +1 = \prod_i \lambda_i$ (eigenvalues)
5. $\text{trace } R = \text{trace } \mathbf{r}\mathbf{r}^T + \text{trace } (I - \mathbf{r}\mathbf{r}^T)c\theta = 1 + 2c\theta = \sum_i \lambda_i$

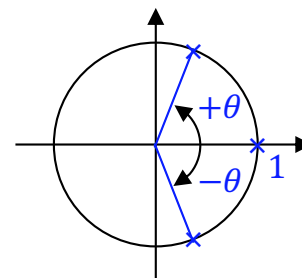
$$1. \Rightarrow \lambda_1 = 1$$

identities in green hold for any matrix!

$$4. \ \& \ 5. \Rightarrow \lambda_2 + \lambda_3 = 2c\theta \Rightarrow \lambda^2 - 2c\theta\lambda + 1 = 0$$

$$\Rightarrow \lambda_{2,3} = c\theta \pm \sqrt{c^2\theta^2 - 1} = c\theta \pm i s\theta = e^{\pm i\theta}$$

all eigenvalues λ have unitary module ($\Leftarrow R$ orthonormal)





Axis/angle: Inverse problem

GIVEN a rotation matrix $R = \{R_{ij}\}$,
FIND a unit vector \mathbf{r} and an angle θ such that

$$R = \mathbf{r}\mathbf{r}^T + (I - \mathbf{r}\mathbf{r}^T) \cos \theta + S(\mathbf{r}) \sin \theta = R(\theta, \mathbf{r})$$

note first that trace $R = R_{11} + R_{22} + R_{33} = 1 + 2 \cos \theta$; so, one could solve

$$\theta = \arccos \frac{R_{11} + R_{22} + R_{33} - 1}{2}$$

but

- this formula provides only values in $[0, \pi]$ (thus, never negative angles θ)
- loss of numerical accuracy for $\theta \rightarrow 0$ (sensitivity of $\cos \theta$ is low around 0)
- also, we better use more of the input data..



Axis/angle: Inverse problem solution

from the **data**



from $R(\theta, \mathbf{r})$

$$\mathbf{R} - \mathbf{R}^T = \begin{bmatrix} 0 & R_{12} - R_{21} & R_{13} - R_{31} \\ R_{21} - R_{12} & 0 & R_{23} - R_{32} \\ R_{31} - R_{13} & R_{32} - R_{23} & 0 \end{bmatrix} = 2 \sin \theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

it follows

$$\|\mathbf{r}\| = 1 \Rightarrow \sin \theta = \pm \frac{1}{2} \sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2} \quad (*)$$

thus

(**)

$$\theta = \text{atan2} \left\{ \pm \sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1 \right\}$$

see the slide
with its definition!

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2 \sin \theta} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

can be used **only** if

$$\sin \theta \neq 0$$

test is made on (*)
using the data $\{R_{ij}\}$



Singular cases

(use when $\sin \theta = 0$)

- if $\theta = 0$ from (**), there is **no solution** for \mathbf{r} (rotation axis undefined)
- if $\theta = \pm\pi$ from (**), then set $\sin \theta = 0$, $\cos \theta = -1$ and solve

$$\Rightarrow \mathbf{R} = 2\mathbf{r}\mathbf{r}^T - \mathbf{I}$$

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \pm \sqrt{(R_{11} + 1)/2} \\ \pm \sqrt{(R_{22} + 1)/2} \\ \pm \sqrt{(R_{33} + 1)/2} \end{bmatrix} \quad \text{with} \quad \begin{cases} r_x r_y = R_{12}/2 \\ r_x r_z = R_{13}/2 \\ r_y r_z = R_{23}/2 \end{cases} \Leftrightarrow \begin{array}{l} \text{used to resolve} \\ \text{sign ambiguities} \\ \Rightarrow \text{two solutions} \\ \text{of opposite sign} \end{array}$$

homework: write a code that determines the two solutions (θ, \mathbf{r})

$$\text{for } \mathbf{R} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$



Unit quaternion

- to eliminate non-uniqueness and singular cases of the axis/angle (θ, \mathbf{r}) representation, the **unit quaternion** can be used

$$Q = \{\eta, \epsilon\} = \{\cos(\theta/2), \sin(\theta/2) \mathbf{r}\}$$

a scalar \nearrow \nwarrow 3-dim vector

$$\theta \in (-\pi, +\pi]$$

$$\eta \geq 0$$

- $\eta^2 + \|\epsilon\|^2 = 1$ (thus, "unit ...")
- (θ, \mathbf{r}) and $(-\theta, -\mathbf{r})$ are associated to the **same** quaternion Q
- the **rotation** matrix R associated to a given quaternion Q is (**prove it!**)

$$R(\eta, \epsilon) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x \epsilon_y - \eta \epsilon_z) & 2(\epsilon_x \epsilon_z + \eta \epsilon_y) \\ 2(\epsilon_x \epsilon_y + \eta \epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y \epsilon_z - \eta \epsilon_x) \\ 2(\epsilon_x \epsilon_z - \eta \epsilon_y) & 2(\epsilon_y \epsilon_z + \eta \epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

- no** rotation is $Q = \{1, \mathbf{0}\}$, while the **inverse** rotation is $Q^{-1} = \{\eta, -\epsilon\}$
- unit quaternions are **composed** with the special rule (**prove it!**)

$$Q_1 * Q_2 = \{\eta_1 \eta_2 - \epsilon_1^T \epsilon_2, \eta_1 \epsilon_2 + \eta_2 \epsilon_1 + \epsilon_1 \times \epsilon_2\}$$



Unit quaternion: Inverse problem

GIVEN a rotation matrix $R = \{R_{ij}\}$,
FIND a quaternion $Q = \{\eta, \epsilon\}$ such that $R\{\eta, \epsilon\} = R$

summing the diagonal elements of R and $R\{\eta, \epsilon\}$ gives

$$R_{11} + R_{22} + R_{33} = 6\eta^2 + 2(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) - 3 = 4\eta^2 + 2(\eta^2 + \|\epsilon\|^2) - 3 = 4\eta^2 - 1$$

thus

$$\eta = \frac{1}{2} \sqrt{R_{11} + R_{22} + R_{33} + 1} \quad (\geq 0)$$

moreover, since

$$R_{11} - R_{22} - R_{33} + 1 = -2\eta^2 + 2(\epsilon_x^2 - (\epsilon_y^2 + \epsilon_z^2)) + 2 = 4\epsilon_x^2 - 2(\eta^2 + \|\epsilon\|^2) + 2 = 4\epsilon_x^2$$

$$R_{32} - R_{23} = 4\eta\epsilon_x \quad \Rightarrow \quad \text{sign}(R_{32} - R_{23}) = \text{sign}(\epsilon_x) \quad \text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$$\epsilon_x = \frac{1}{2} \text{sign}(R_{32} - R_{23}) \sqrt{R_{11} - R_{22} - R_{33} + 1}$$

and similarly $\epsilon_y = \frac{1}{2} \text{sign}(R_{13} - R_{32}) \sqrt{R_{22} - R_{11} - R_{33} + 1}$

$$\epsilon_z = \frac{1}{2} \text{sign}(R_{21} - R_{12}) \sqrt{R_{33} - R_{11} - R_{22} + 1}$$