Robotics 1

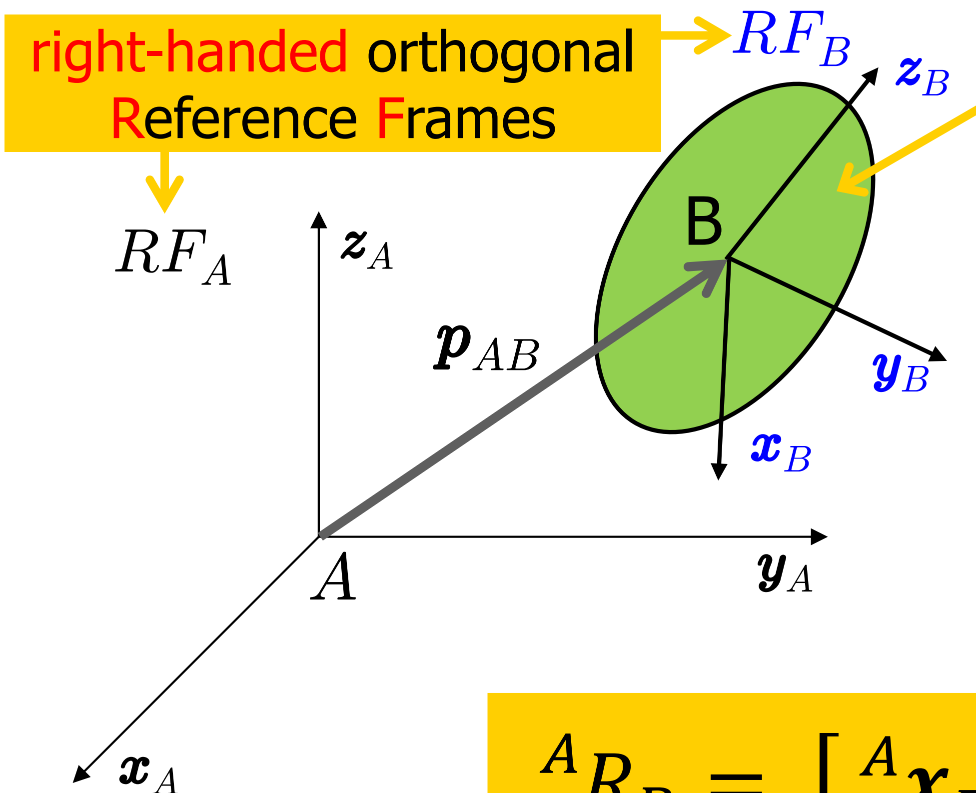
Position and orientation of rigid bodies

Prof. Alessandro De Luca

Dipartimento di Ingegneria Informatica Automatica e Gestionale Antonio Ruberti

Sapienza Università di Roma

Position and orientation



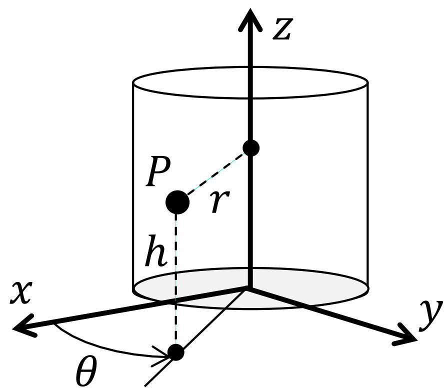
- position: ${}^A\boldsymbol{p}_{AB}$ (vector $\in \mathbb{R}^3$)
  Cartesian coordinates of vector $\overrightarrow{AB}$ expressed in RF_A
- orientation:
  orthonormal 3×3 matrix
  $(R^T = R^{-1} \Rightarrow R^T R = I)$, with det = +1

$$
{}^A R_B = \left[\, {}^A\boldsymbol{x}_B \quad {}^A\boldsymbol{y}_B \quad {}^A\boldsymbol{z}_B \, \right]
$$

- $\boldsymbol{x}_A \; \boldsymbol{y}_A \; \boldsymbol{z}_A$ ($\boldsymbol{x}_B \; \boldsymbol{y}_B \; \boldsymbol{z}_B$) are axis vectors (of unitary norm) of frame RF_A (RF_B)
- components in ${}^A R_B$ are the direction cosines of the axes of RF_B with respect to (w.r.t.) RF_A

Position of a rigid body

- for position representation, use of other coordinates than the Cartesian ones is possible, e.g., cylindrical or spherical
- direct transformation from cylindrical to Cartesian



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = h$$

is always well defined
(with $r \geq 0$ or $r \gtreqless 0$)

- inverse transformation from Cartesian to cylindrical

assuming +
($r \geq 0$ only)

$$x^2 + y^2 = r^2 \qquad \frac{y}{x} = \tan\theta \quad \Longrightarrow \quad r = \sqrt{x^2+y^2}, \quad \theta = \operatorname{atan2}\{y, x\}, \quad h = z$$

four-quadrant arc tangent

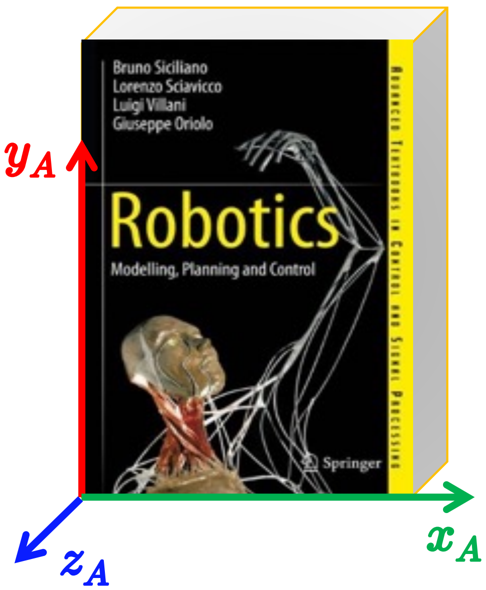
with a singularity
for $x = y = 0$

atan2 function

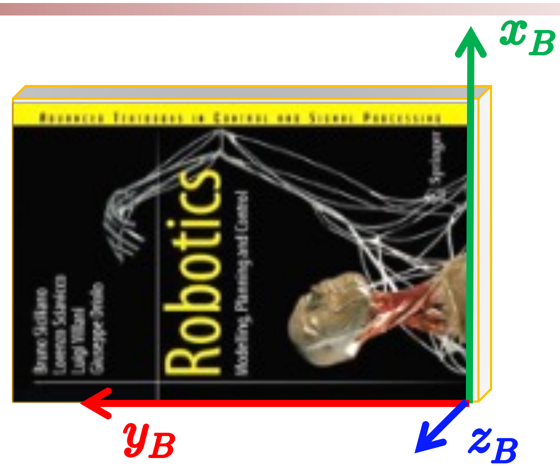
- arctangent with output values "in the four quadrants"
  - two input arguments
  - takes values in $[-\pi, +\pi]$
  - undefined only for $(0, 0)$
- uses the sign of both arguments to define the output quadrant
- based on arctan function with output values in $[-\pi/2, +\pi/2]$
- available in main languages (C++, Matlab, …)

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & y \geq 0, x < 0 \\ -\pi + \arctan(\frac{y}{x}) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

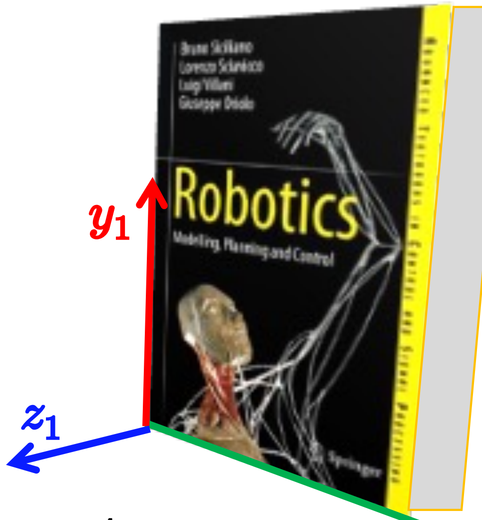
Orientation of a rigid body



90° around z-axis



$$ {}^A R_B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$



$$ {}^A R_1 = ? $$

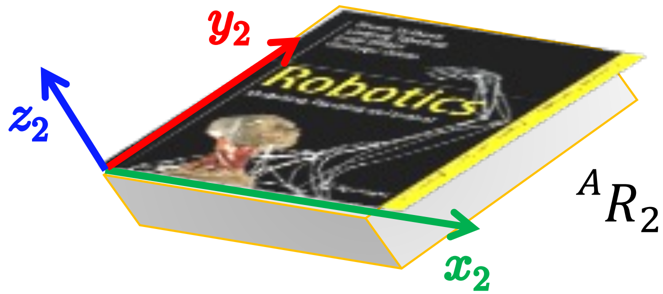
$$ {}^B R_A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = {}^A R_B^T $$

$$ {}^A R_A = {}^A R_B \, {}^B R_A = I $$

$$ {}^B R_C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = {}^B R_A \, {}^A R_C = {}^A R_B^T \, {}^A R_C $$

-90° around x-axis

$$ {}^A R_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} $$



$$ {}^A R_2 = ? $$

Rotation matrix

$$ {}^A R_B = \begin{bmatrix} \boldsymbol{x}_A{}^T \boldsymbol{x}_B & \boldsymbol{x}_A{}^T \boldsymbol{y}_B & \boldsymbol{x}_A{}^T \boldsymbol{z}_B \\ \boldsymbol{y}_A{}^T \boldsymbol{x}_B & \boldsymbol{y}_A{}^T \boldsymbol{y}_B & \boldsymbol{y}_A{}^T \boldsymbol{z}_B \\ \boldsymbol{z}_A{}^T \boldsymbol{x}_B & \boldsymbol{z}_A{}^T \boldsymbol{y}_B & \boldsymbol{z}_A{}^T \boldsymbol{z}_B \end{bmatrix} $$

direction cosine of $\boldsymbol{z}_B$ w.r.t. $\boldsymbol{x}_A$

$$ \boldsymbol{x}_A^T \boldsymbol{z}_B = \|\boldsymbol{x}_A\| \|\boldsymbol{z}_B\| \cos\beta = \cos\beta $$

orthonormal, with det = +1

algebraic structure of a group $SO(3)$: neutral element = I, inverse element = R^T

chain rule property

$$ \boxed{{}^k R_i \, {}^i R_j = {}^k R_j} $$

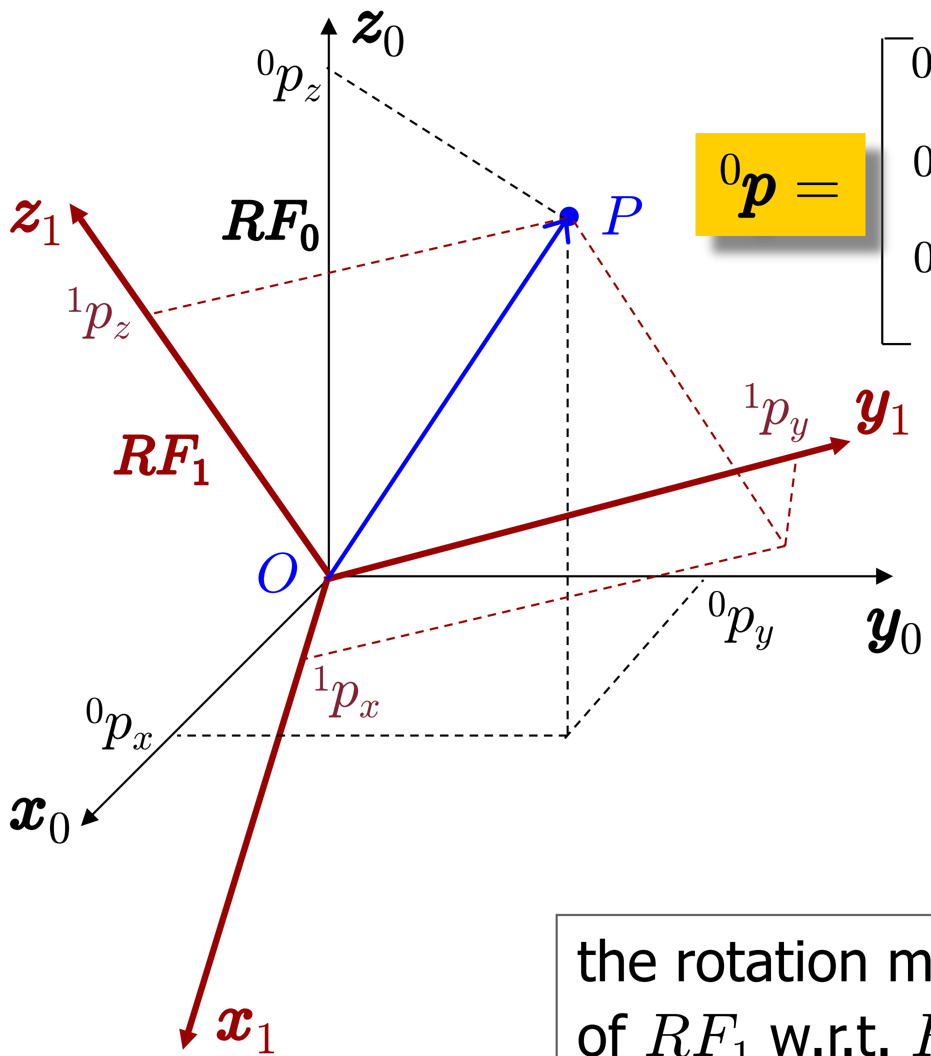
orientation of RF_i w.r.t. RF_k

orientation of RF_j w.r.t. RF_i

orientation of RF_j w.r.t. RF_k

NOTE: in general, the **product** of rotation matrices does **not** commute!

Change of coordinates



$$
\boxed{{}^0\boldsymbol{p} =} \begin{bmatrix} {}^0p_x \\ {}^0p_y \\ {}^0p_z \end{bmatrix} = {}^0p_x\,\underset{\uparrow\,(1\ \ 0\ \ 0)^T}{{}^0\boldsymbol{x}_0} + {}^0p_y\,\underset{\uparrow\,(0\ \ 1\ \ 0)^T}{{}^0\boldsymbol{y}_0} + {}^0p_z\,\underset{\uparrow\,(0\ \ 0\ \ 1)^T}{{}^0\boldsymbol{z}_0}
$$

$$
= {}^1p_x\,{}^0\boldsymbol{x}_1 + {}^1p_y\,{}^0\boldsymbol{y}_1 + {}^1p_z\,{}^0\boldsymbol{z}_1
$$

$$
= \begin{bmatrix} {}^0\boldsymbol{x}_1 & {}^0\boldsymbol{y}_1 & {}^0\boldsymbol{z}_1 \end{bmatrix} \begin{bmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \end{bmatrix}
$$

$$
\boxed{= {}^0R_1\,{}^1\boldsymbol{p}}
$$

the rotation matrix ${}^0R_1$ (i.e., the orientation of RF_1 w.r.t. RF_0) represents also the change of coordinates of a vector from RF_1 to RF_0

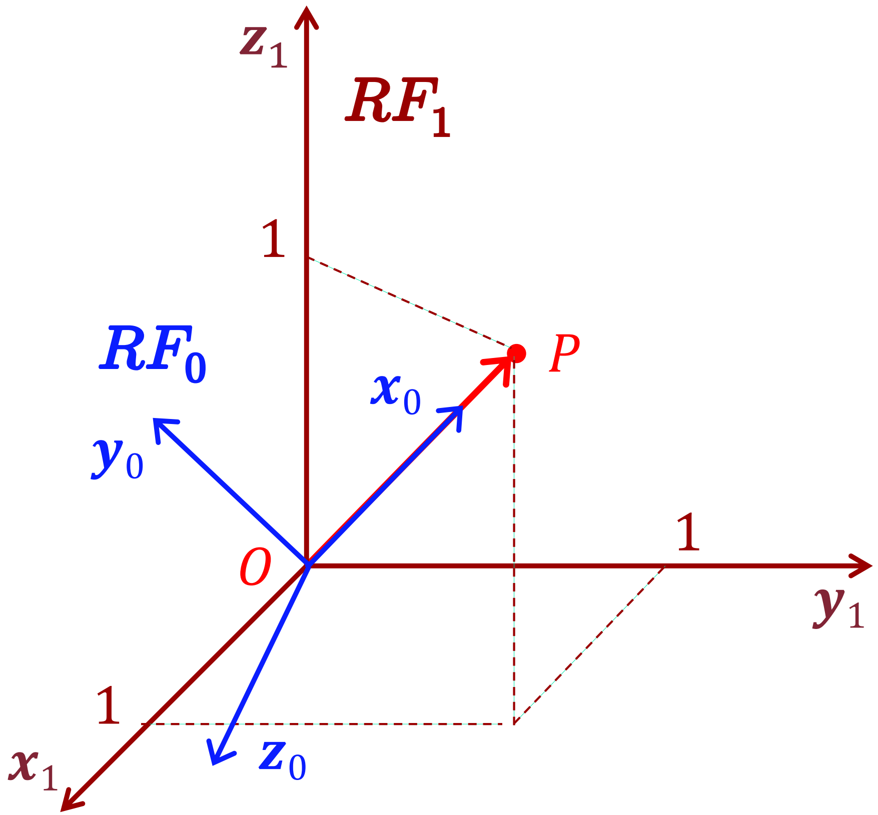
Change of coordinates

$$^1\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$^0R_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$^0\boldsymbol{p} = {}^0R_1\,{}^1\boldsymbol{p} = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}$$

$$\|\,\boldsymbol{p}\,\| = \|\,{}^0\boldsymbol{p}\,\| = \|\,{}^1\boldsymbol{p}\,\| = \sqrt{3}$$

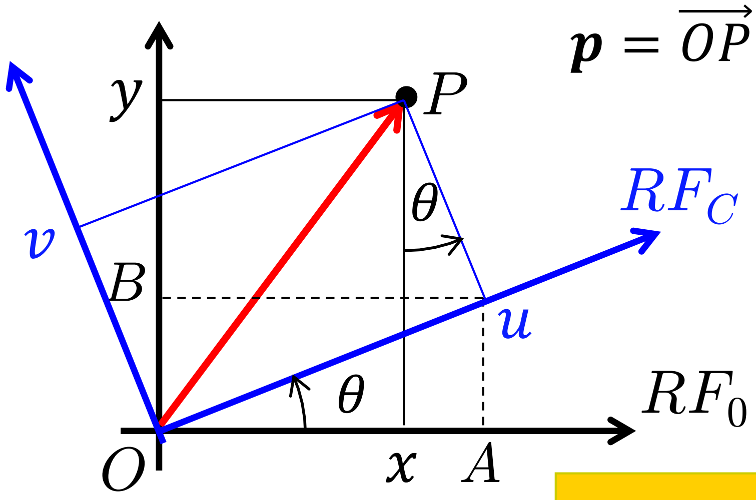


... and where is $\boldsymbol{RF_0}$?

- $\boldsymbol{x}_0$ is aligned with $\boldsymbol{p} = \overrightarrow{OP}$
- $\boldsymbol{z}_0$ is orthogonal to $\boldsymbol{y}_1$ ($\boldsymbol{z}_0^T\,\boldsymbol{y}_1 = 0$) and is positive on $\boldsymbol{x}_1$ ($\boldsymbol{z}_0^T\,\boldsymbol{x}_1 = 1/\sqrt{2}$)
- $\boldsymbol{y}_0$ completes a right-handed frame

Orientation of frames in a plane

(elementary rotation around z-axis)



$$\boldsymbol{p} = \overrightarrow{OP}$$

$$x = OA - xA = u\cos\theta - v\sin\theta$$
$$y = OB + By = u\sin\theta + v\cos\theta$$
$$z = w$$

or...

$$\begin{array}{ccccccc} & & {}^0\boldsymbol{x}_C & {}^0\boldsymbol{y}_C & {}^0\boldsymbol{z}_C & & {}^C\boldsymbol{p} \\ & & \downarrow & \downarrow & \downarrow & & \downarrow \end{array}$$

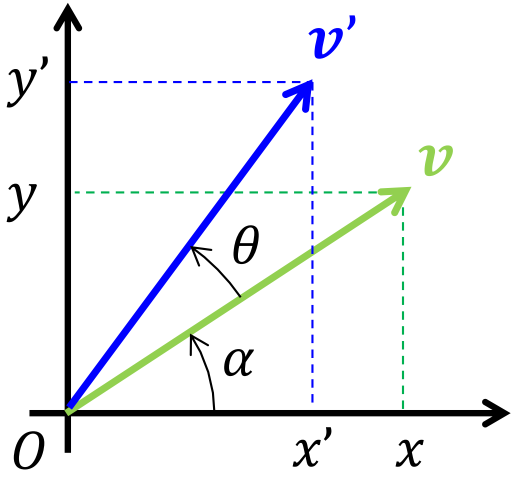
$${}^0\boldsymbol{p} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_z(\theta) \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\boxed{R_z(-\theta) = R_z^T(\theta)}$$

similarly:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation of a vector around z



$$x = \|\boldsymbol{v}\| \cos\alpha$$
$$y = \|\boldsymbol{v}\| \sin\alpha$$

$$x' = \|\boldsymbol{v}\| \cos(\alpha+\theta) = \|\boldsymbol{v}\|(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$
$$= x\cos\theta - y\sin\theta$$
$$y' = \|\boldsymbol{v}\| \sin(\alpha+\theta) = \|\boldsymbol{v}\|(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$
$$= x\sin\theta + y\cos\theta$$
$$z' = z$$

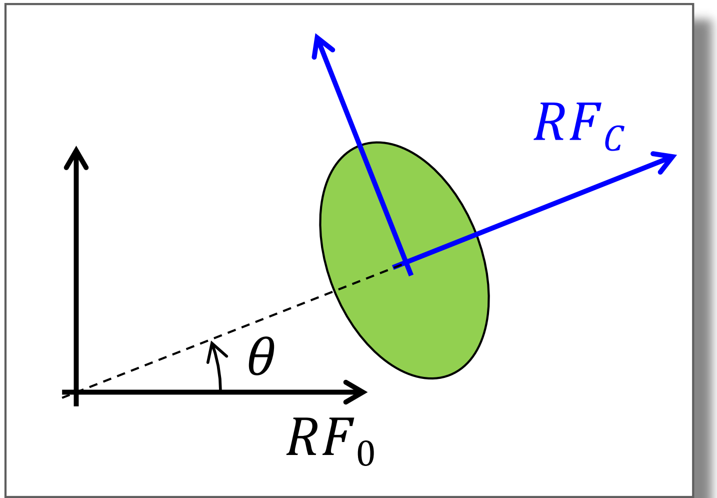
or...

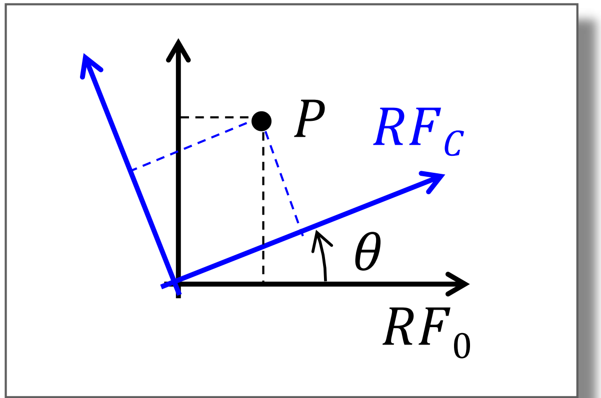
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

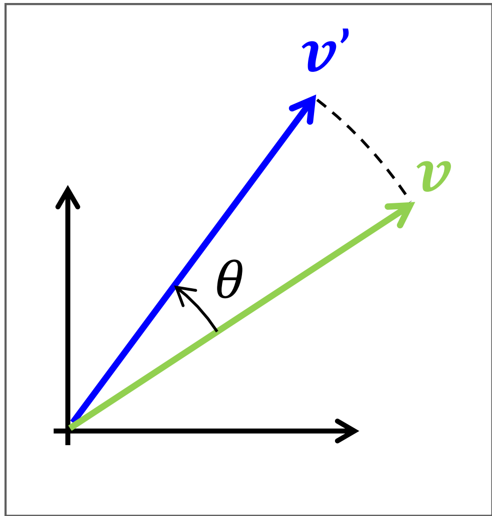
... same as before!

Equivalent interpretations of a rotation matrix

the same rotation matrix (e.g., $R_z(\theta)$) may represent







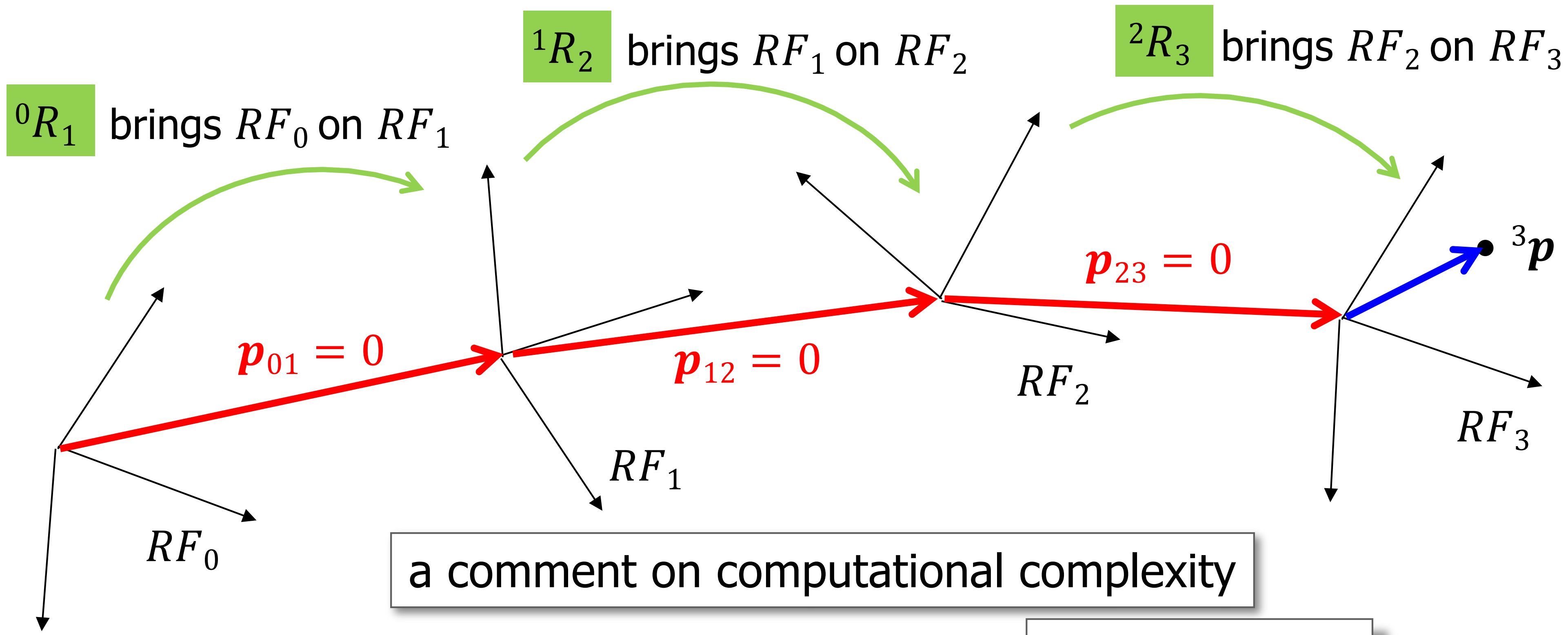
the orientation of a rigid body with respect to a reference frame RF_0
e.g., $[{}^0\boldsymbol{x}_c\ {}^0\boldsymbol{y}_c\ {}^0\boldsymbol{z}_c] = R_z(\theta)$

the change of coordinates from RF_C to RF_0
e.g., ${}^0\boldsymbol{p} = R_z(\theta)\ {}^C\boldsymbol{p}$

the rotation operator on vectors
e.g., $\boldsymbol{v}' = R_z(\theta)\ \boldsymbol{v}$

the rotation matrix ${}^0R_C$ is an operator superposing frame RF_0 to frame RF_C

Composition of rotations



a comment on computational complexity

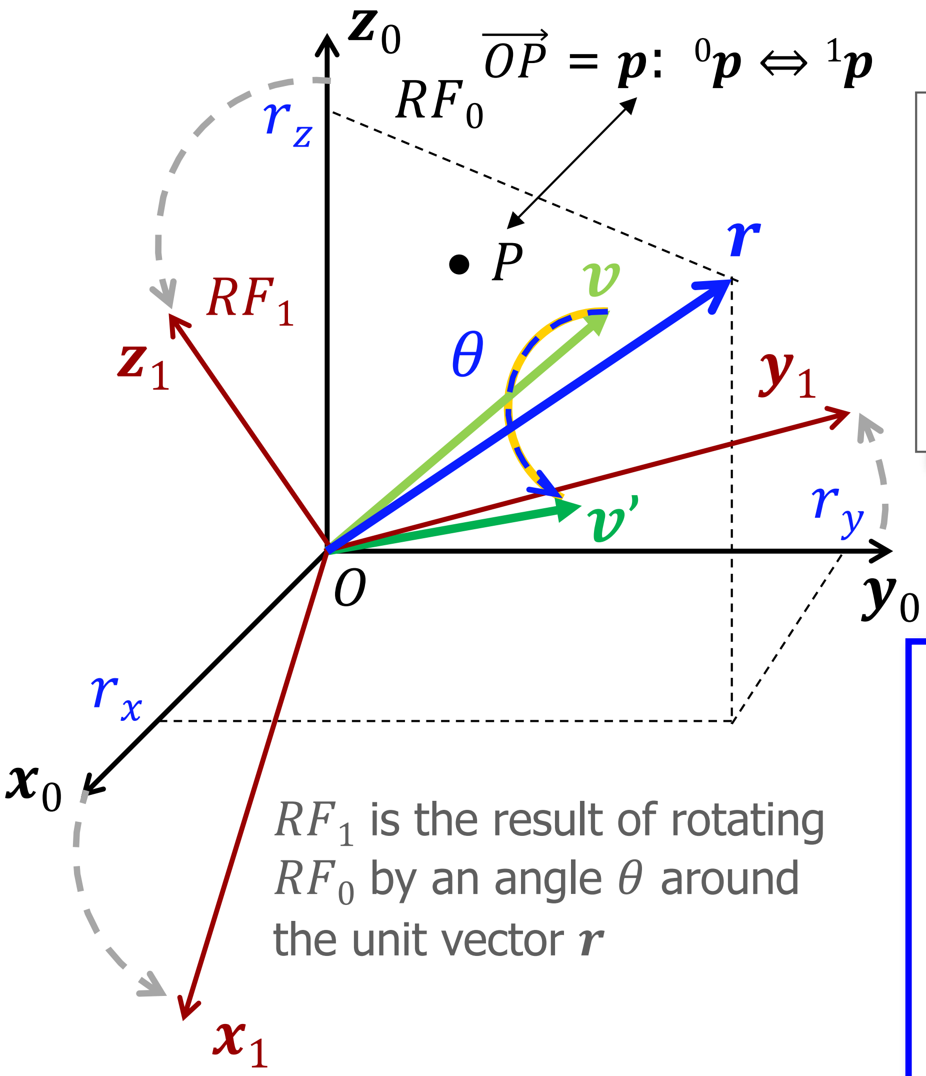
$${}^0\boldsymbol{p} = ({}^0R_1\,{}^1R_2\,{}^2R_3)\,{}^3\boldsymbol{p} = {}^0R_3\,{}^3\boldsymbol{p}$$

63 products
42 summations

$${}^0\boldsymbol{p} = {}^0R_1\,\underbrace{({}^1R_2\,\underbrace{({}^2R_3\,{}^3\boldsymbol{p})}_{{}^2\boldsymbol{p}})}_{{}^1\boldsymbol{p}}$$

27 products
18 summations

Axis/angle representation



RF_1 is the result of rotating RF_0 by an angle θ around the unit vector $\boldsymbol{r}$

DATA

- axis $\boldsymbol{r}$ (unit vector in $\mathbb{R}^3$, $\|\boldsymbol{r}\| = 1$)
- angle θ, positive counterclockwise (as seen from an "observer" oriented like $\boldsymbol{r}$ with the head placed on the arrow, looking down to her/his feet)

DIRECT PROBLEM

parametrized by the given data!

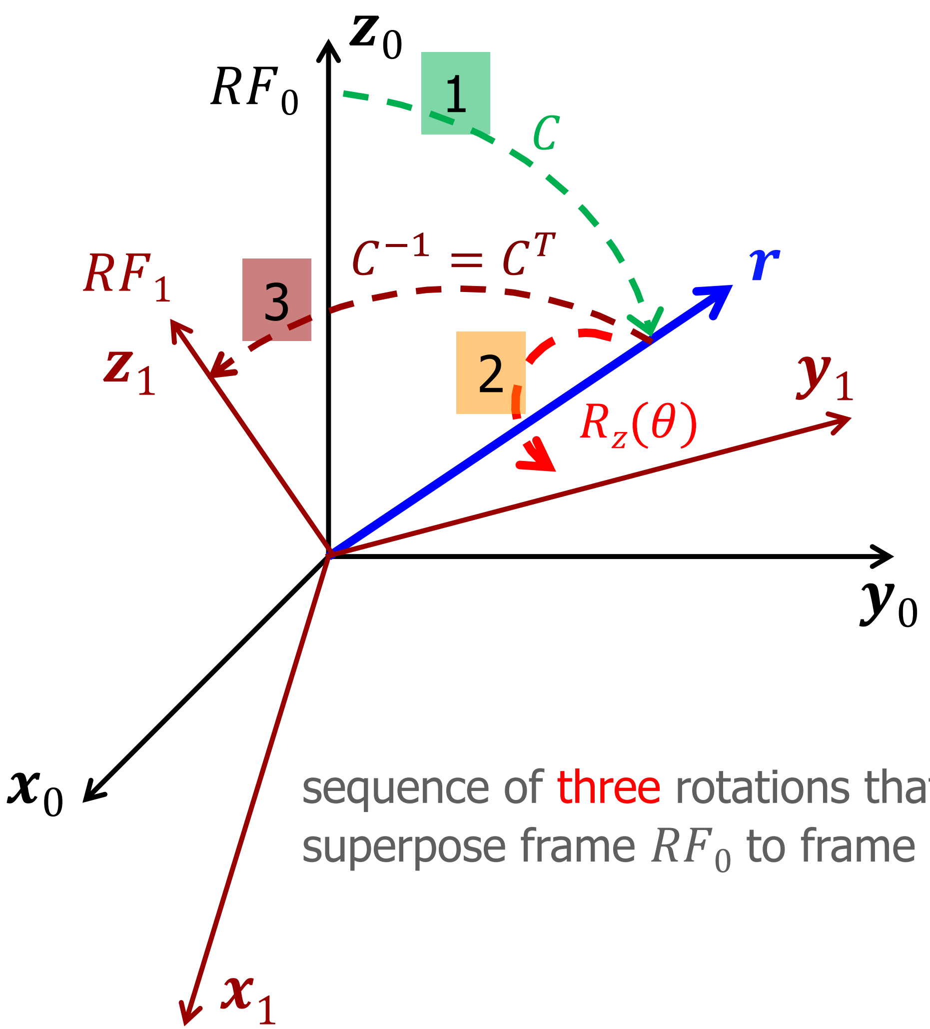
find a rotation matrix $R(\theta, \boldsymbol{r})$ such that

$$R(\theta, \boldsymbol{r}) = [{}^0\boldsymbol{x}_1 \; {}^0\boldsymbol{y}_1 \; {}^0\boldsymbol{z}_1]$$

$${}^0\boldsymbol{p} = R(\theta, \boldsymbol{r})\, {}^1\boldsymbol{p}$$

$${}^0\boldsymbol{v}' = R(\theta, \boldsymbol{r})\, {}^0\boldsymbol{v}$$

Axis/angle: Direct problem



sequence of three rotations that superpose frame RF_0 to frame RF_1

$$R(\theta, \boldsymbol{r}) = C\, R_z(\theta)\, C^T$$

sequence of three rotations
(one of which is elementary)

$$C = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix}$$

after the first rotation the z-axis coincides with $\boldsymbol{r}$

$\boldsymbol{n}$ and $\boldsymbol{s}$ are orthogonal unit vectors such that $\boldsymbol{n} \times \boldsymbol{s} = \boldsymbol{r}$

Inner and outer products

whiteboard…

- (inner) row by column products between two 3×3 (orthonormal) matrices

$$C^T C = \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

- dyadic expansion of a $n{\times}n$ generic matrix

$$\boldsymbol{e}_i = [0 \quad \dots \quad 1 \quad \dots \quad 0]^T, \qquad i = 1, \dots, n \qquad \Longrightarrow \qquad A = \sum_{i,j=1}^{n} a_{ij} \boldsymbol{e}_i \boldsymbol{e}_j^T \; (= I\, A\, I^T)$$

- product of three $n{\times}n$ matrices using dyadic form

$$B = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \dots & \boldsymbol{b}_{n-1} & \boldsymbol{b}_n \end{bmatrix} \quad \Longrightarrow \quad B\, A\, B^T = \sum_{i,j=1}^{n} a_{ij} \boldsymbol{b}_i \boldsymbol{b}_j^T$$

- (outer) column by row products between two 3×3 matrices

$$C C^T = I \Longrightarrow C C^T = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix}$$

$$= \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

Skew-symmetric matrices

whiteboard…

- properties of a skew-symmetric matrix
  - a square matrix S is skew-symmetric iff $S^T = -S$

    $\Leftrightarrow\ s_{ij} = -s_{ji}\ \Rightarrow\ s_{ii} = 0$ (zeros on the diagonal)
  - any square matrix A can be decomposed into its symmetric and skew-symmetric parts

    $$A = \frac{A+A^T}{2} + \frac{A-A^T}{2} = A_{symm} + A_{skew}$$
  - in quadratic forms the skew-symmetric part vanishes (only the symmetric part matters)

    $$x^T A\, x = \tfrac{1}{2}[x^T A\, x + (x^T A\, x)^T] = \tfrac{1}{2}[x^T A\, x + x^T A^T x] = x^T \frac{A+A^T}{2}\, x = x^T A_{symm}\, x$$
  - canonical form of a 3×3 skew-symmetric matrix

    also called vee map $\vee$

    $$\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\Longrightarrow\; S(\boldsymbol{v}) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \;\overset{v = S^{\vee}}{\Longrightarrow}\; \boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- expression of the vector product between two vectors $\in \mathbb{R}^3$

  Sarrus rule for determinant of $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ n_x & n_y & n_z \\ s_x & s_y & s_z \end{bmatrix}$

  $$\boldsymbol{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix},\ \boldsymbol{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \;\Longrightarrow\; \boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \boldsymbol{n} \times \boldsymbol{s} = \begin{bmatrix} n_y s_z - s_y n_z \\ n_z s_x - s_z n_x \\ n_x s_y - s_x n_y \end{bmatrix} = S(\boldsymbol{n})\, \boldsymbol{s}$$

  $$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = S(\boldsymbol{v}_1)\boldsymbol{v}_2 = -\,\boldsymbol{v}_2 \times \boldsymbol{v}_1 = -\,S(\boldsymbol{v}_2)\boldsymbol{v}_1 = S^T(\boldsymbol{v}_2)\boldsymbol{v}_1$$

Axis/angle: Direct problem

solution

$$R(\theta,\boldsymbol{r}) = C\,R_z(\theta)\,C^T$$

$$R(\theta,\boldsymbol{r}) = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix}$$

$$= \boldsymbol{r}\boldsymbol{r}^T + (\boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T)\,c\theta + (\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T)\,s\theta$$

taking into account

$$CC^T = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

$$\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$$

depends only on $\boldsymbol{r}$ and θ !

$$R(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)\,c\theta + S(\boldsymbol{r})\,s\theta$$

Final expression of $R(\theta, \boldsymbol{r})$

developing computations…

$$R(\theta, \boldsymbol{r}) =$$

$$\begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)-r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

note that

sum of the diagonal elements of a matrix

$$\mathrm{trace}\, R(\theta, \boldsymbol{r}) = 1 + 2\ \cos\theta$$

$$R(\theta, \boldsymbol{r}) = R(-\theta, -\boldsymbol{r}) = R^T(-\theta, \boldsymbol{r})$$

Axis/angle: a simple example

$$R(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)\, c\theta + S(\boldsymbol{r})\, s\theta$$

$$\boldsymbol{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{z}_0$$

$$R(\theta,\boldsymbol{r}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} c\theta + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s\theta$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_z(\theta)$$

Properties of $R(\theta, \boldsymbol{r})$

1. $R(\theta, \boldsymbol{r})\boldsymbol{r} = \boldsymbol{r}$ ($\boldsymbol{r}$ is the invariant axis in this rotation)
2. when $\boldsymbol{r}$ is one of the coordinate axes, R boils down to one of the known elementary rotation matrices
3. $(\theta, \boldsymbol{r}) \to R$ is not an injective map: $R(\theta, \boldsymbol{r}) = R(-\theta, -\boldsymbol{r})$
4. $\det R = +1 = \prod_i \lambda_i$ (eigenvalues)
5. $\text{trace}\, R = \text{trace}\, \boldsymbol{r}\boldsymbol{r}^T + \text{trace}\,(I - \boldsymbol{r}\,\boldsymbol{r}^T)c\theta = 1 + 2c\theta = \sum_i \lambda_i$

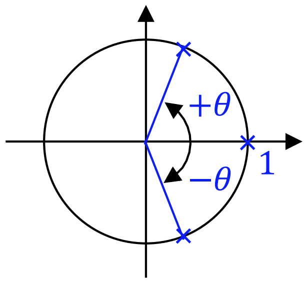
identities in green hold for any matrix!

1\. $\Rightarrow \lambda_1 = 1$

4\. & 5. $\Rightarrow \lambda_2 + \lambda_3 = 2\, c\theta \;\Rightarrow\; \lambda^2 - 2\, c\theta\, \lambda + 1 = 0$

$$\Rightarrow \lambda_{2,3} = c\theta \pm \sqrt{c^2\theta - 1} = c\theta \pm i\, s\theta = e^{\pm i\theta}$$

all eigenvalues λ have unitary module ($\Leftarrow R$ orthonormal)



Axis/angle: Inverse problem

GIVEN a rotation matrix $R = \{R_{ij}\}$,
FIND a unit vector $\boldsymbol{r}$ and an angle θ such that

$$R = \boldsymbol{r}\,\boldsymbol{r}^T + (I - \boldsymbol{r}\,\boldsymbol{r}^T)\cos\theta + S(\boldsymbol{r})\sin\theta = R(\theta, \boldsymbol{r})$$

note first that trace $R = R_{11} + R_{22} + R_{33} = 1 + 2\cos\theta$; so, one could solve

$$\theta = \arccos\frac{R_{11} + R_{22} + R_{33} - 1}{2}$$

but

- this formula provides only values in $[0, \pi]$ (thus, never negative angles θ)
- loss of numerical accuracy for $\theta \to 0$ (sensitivity of $\cos\theta$ is low around 0)
- also, we better use more of the input data..

Axis/angle: Inverse problem

solution

from the data $\longleftrightarrow$ from $R(\theta, \boldsymbol{r})$

$$R - R^T = \begin{bmatrix} 0 & R_{12} - R_{21} & R_{13} - R_{31} \\ R_{21} - R_{12} & 0 & R_{23} - R_{32} \\ R_{31} - R_{13} & R_{32} - R_{23} & 0 \end{bmatrix} = 2 \sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

it follows

$$\|\boldsymbol{r}\| = 1 \;\Rightarrow\; \sin\theta = \pm \frac{1}{2}\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2} \quad (*)$$

thus

(**)

$$\theta = \text{atan2}\left\{\pm\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1\right\}$$

see the slide with its definition!

$$\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

can be used only if

$$\boxed{\sin\theta \neq 0}$$

test is made on (*) using the data $\{R_{ij}\}$

Singular cases

(use when $\sin\theta = 0$)

- if $\theta = 0$ from (**), there is no solution for $\boldsymbol{r}$ (rotation axis undefined)
- if $\theta = \pm\pi$ from (**), then set $\sin\theta = 0, \cos\theta = -1$ and solve

$$\Rightarrow \quad R = 2\boldsymbol{r}\,\boldsymbol{r}^T - I$$

$$\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \pm\sqrt{(R_{11}+1)/2} \\ \pm\sqrt{(R_{22}+1)/2} \\ \pm\sqrt{(R_{33}+1)/2} \end{bmatrix} \quad \text{with} \quad \boxed{\begin{aligned} r_x r_y &= R_{12}/2 \\ r_x r_z &= R_{13}/2 \\ r_y r_z &= R_{23}/2 \end{aligned}} \Leftrightarrow$$

used to resolve sign ambiguities $\Rightarrow$ two solutions of opposite sign

homework: write a code that determines the two solutions $(\theta, \boldsymbol{r})$

$$\text{for } R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Unit quaternion

- to eliminate non-uniqueness and singular cases of the axis/angle $(\theta, \boldsymbol{r})$ representation, the unit quaternion can be used

$$Q = \{\underbrace{\eta}_{\text{a scalar}}, \underbrace{\boldsymbol{\epsilon}}_{\text{3-dim vector}}\} = \{\cos(\theta/2), \sin(\theta/2)\,\boldsymbol{r}\}$$

- $\eta^2 + \|\boldsymbol{\epsilon}\|^2 = 1$ (thus, "unit ...")
- $(\theta, \boldsymbol{r})$ and $(-\theta, -\boldsymbol{r})$ are associated to the same quaternion Q
- the rotation matrix R associated to a given quaternion Q is

$$R(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

- no rotation is $Q = \{1, \mathbf{0}\}$, while the inverse rotation is $Q = \{\eta, -\boldsymbol{\epsilon}\}$
- unit quaternions are composed with special rules

$$Q_1 * Q_2 = \{\eta_1\eta_2 - \boldsymbol{\epsilon}_1^T\boldsymbol{\epsilon}_2, \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2\}$$