Bachelor’s degree in Bioinformatics

Multivariate Calculus

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Functions of Two Variables

We have seen functions of one variable. But in many cases, a value depends not on just one variable but on two or more.

Example: The resistance *R* of blood flowing through an artery depends on the radius *r* and length *L* of the artery. In fact, one of Poiseuille's laws states that $R = CL/r^4$, where *C* is a positive constant determined by the viscosity of the blood. We say that *R* is a function of *r* and *L*, and we write

$$R(r, L) = C\frac{L}{r^4}$$

Definition A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Functions of Two Variables

We often write $z = f(x, y)$ to make explicit the value z taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**.

The domain is a subset of $\mathbb{R}^2$, the xy-plane.
Recall that the domain is the set of all possible inputs and the range is the set of all possible outputs.
If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

Example – *Wind-chill index*

In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index *W* is a subjective temperature that depends on the actual temperature *T* and the wind speed *v*. So *W* is a function of *T* and *v*, and we can write $W = f(T, v)$.

Table with values of *W* compiled by the National Weather Service of the US and the Meteorological Service of Canada.

Wind speed (km/h)

Actual temperature (°C) T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Wind-chill index as a function of air temperature and wind speed

Example – *Wind-chill index* cont’d

For instance, the table shows that if the temperature is –5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about –15°C with no wind. So

$$f(-5, 50) = -15°\text{C}$$

Example

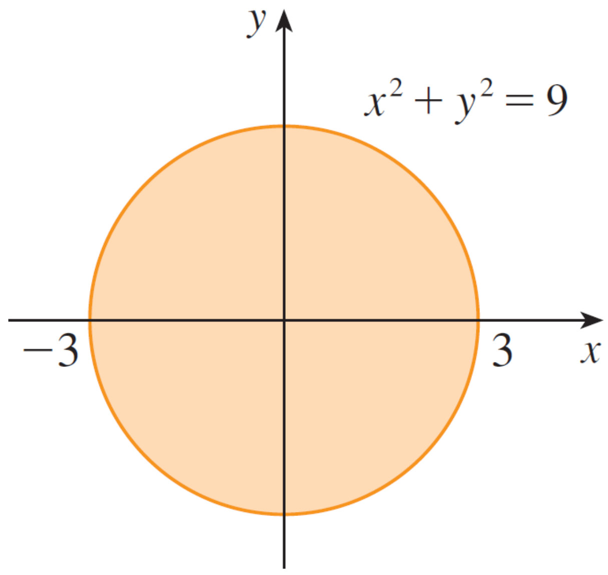
Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

The domain of *g* is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

which is the disk with center (0, 0) and radius 3



Example – *Solution*

cont'd

The range of *g* is

$$\left\{z \mid z = \sqrt{9 - x^2 - y^2},\ (x, y) \in D\right\}$$

Since *z* is a positive square root, $z \geq 0$. Also, because $9 - x^2 - y^2 \leq 9$, we have

$$\sqrt{9 - x^2 - y^2} \leqslant 3$$

So the range is

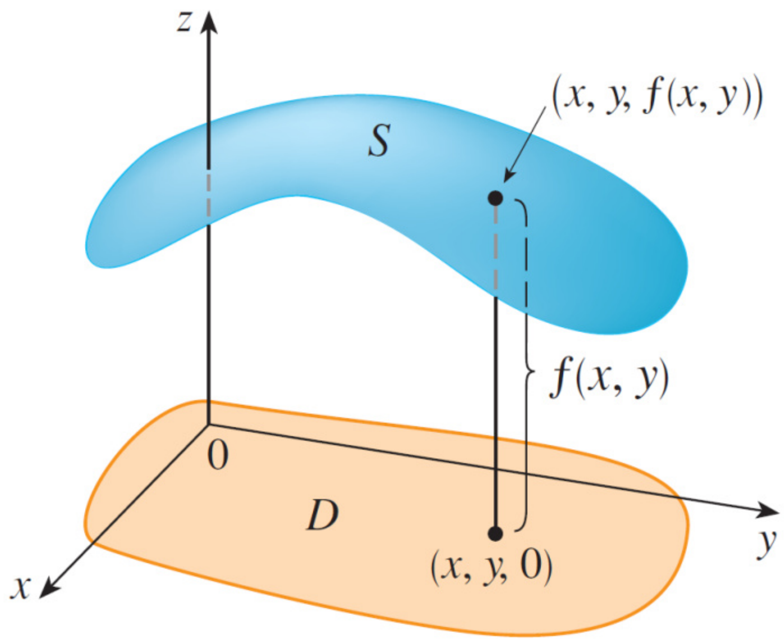
$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$

Graph of a function of 2 variables

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function *f* of one variable is a curve *C* with equation $y = f(x)$, so the graph of a function *f* of two variables is a surface *S* with equation $z = f(x, y)$

We can visualize the graph *S* of *f* as lying directly above or below its domain *D* in the *xy*-plane



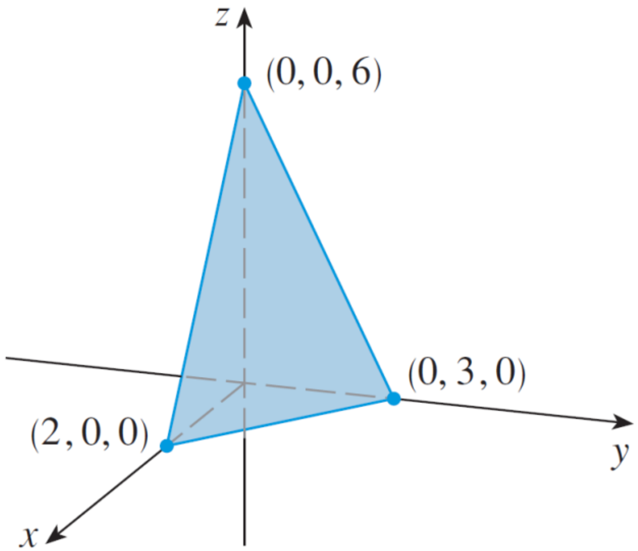
Example

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution:

The graph of *f* has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts.

Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the y-intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant



Graph of a function of 2 variables

That function is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a plane.

Example

Find the domain and range and sketch the graph of $h(x, y) = x^2 + y^2$.

Solution:

Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y), so the domain is $\mathbb{R}^2$, the entire xy-plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geq 0$ and $y^2 \geq 0$, so $h(x, y) \geq 0$ for all x and y.]

The graph of h has the equation $z = x^2 + y^2$. If we put $x = 0$, we get $z = y^2$, so the yz-plane intersects the surface in a parabola.

Example – *Solution*

cont'd

If we put *x* = *k* (a constant), we get $z = y^2 + k^2$. This means that if we slice the graph with any plane parallel to the *yz*-plane, we obtain a parabola that opens upward.

(These curves that we get by slicing a surface with a plane parallel to one of the coordinate planes are called **traces**.)

Similarly, if *y* = *k*, the trace is $z = x^2 + k^2$, which is again a parabola that opens upward. If we put *z* = *k*, we get the horizontal traces $x^2 + y^2 = k$, which we recognize as a family of circles.

Example – *Solution*

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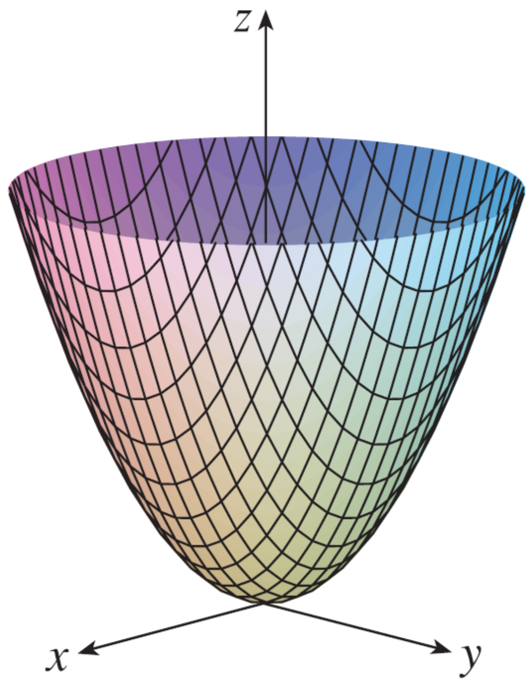
Knowing the shapes of the traces, we can sketch the graph of *f* . Because of the parabolic traces, the surface $z = x^2 + y^2$ is called a **paraboloid.**

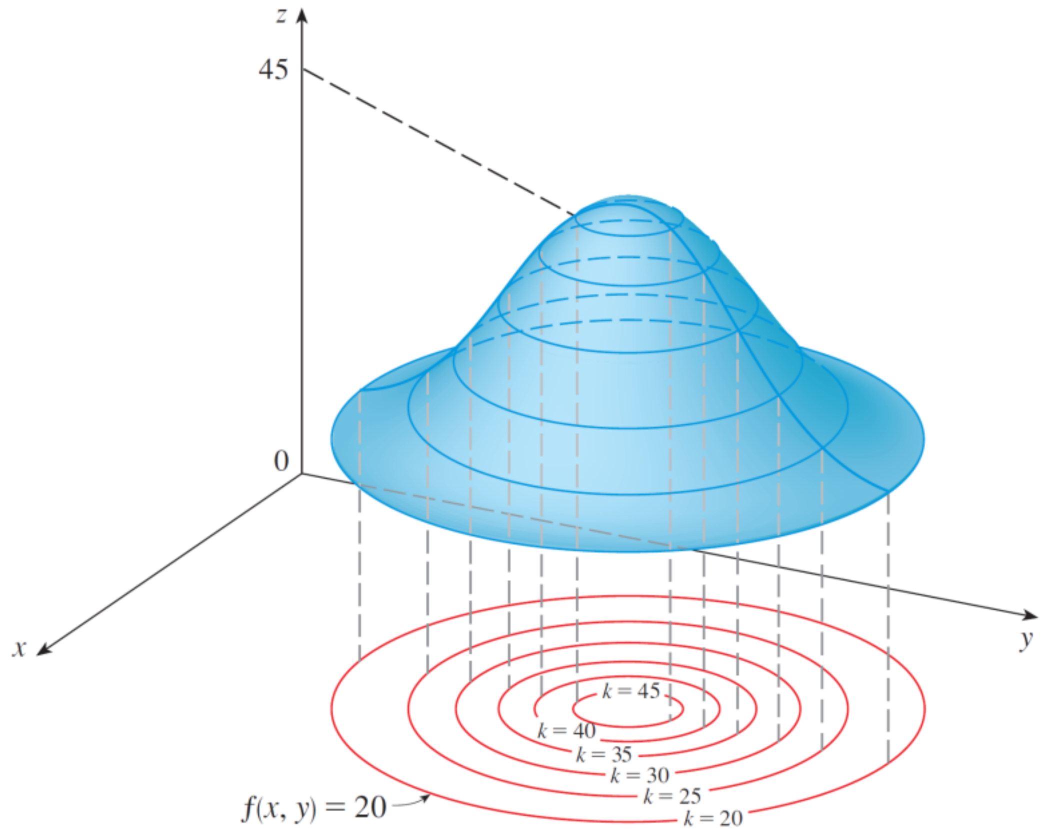


Level Curves

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain at which f takes on a given value k. In other words, it shows where the graph of f has height k

Example of level curves



Level Curves

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.

So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

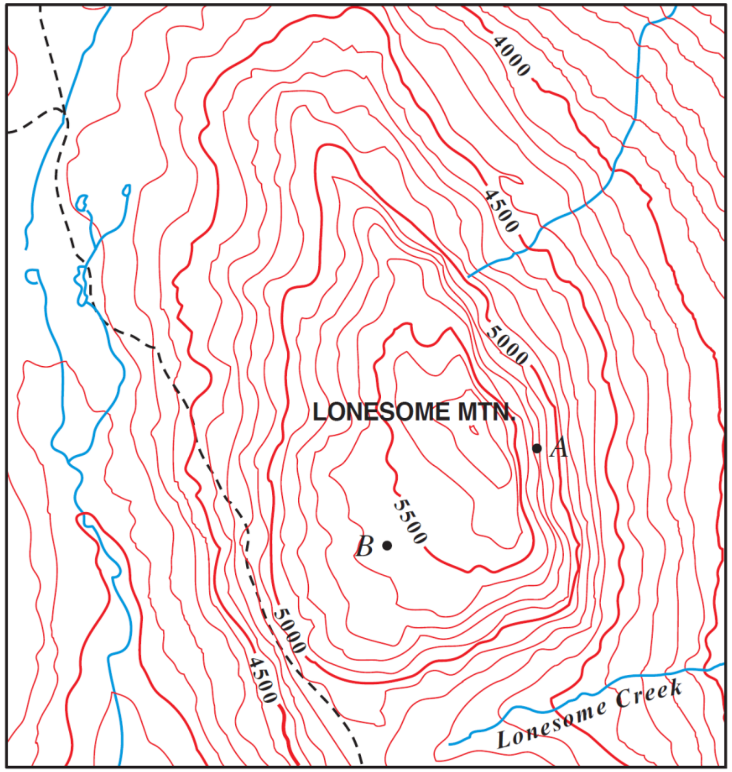
In other words, level curves **allow to represent on the plane** the 3-D surface of the graph of a function.

The surface is steep where the level curves are close together, and vice versa.

Example of Level Curves

Since they allow to represent a 3D surface on a 2D plane, level curves are very used in many fields.

One common example of level curves occurs in topographic maps of mountainous regions.

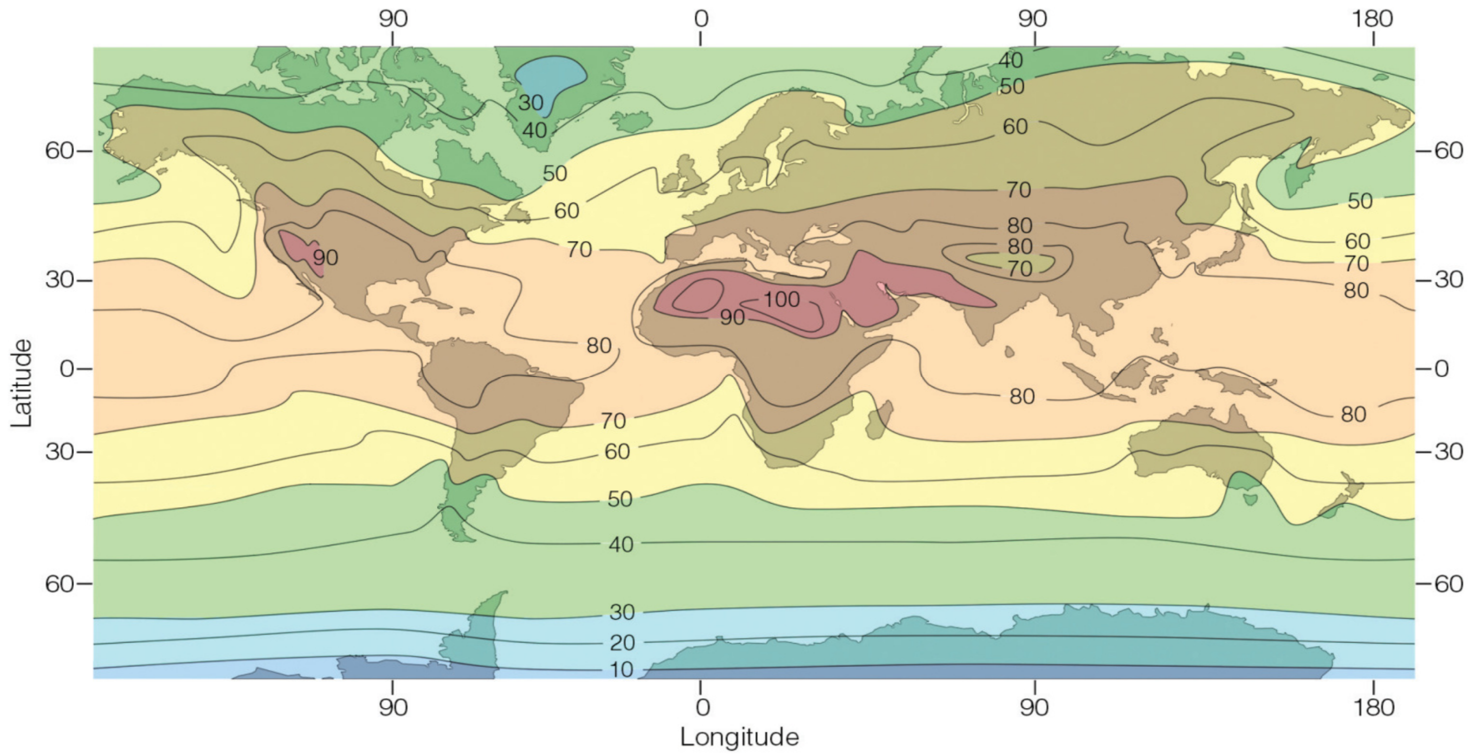


Here level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend.

Example of Level Curves

Another common example is the temperature at location (x, y), with longitude x and latitude y. Here the level curves are called **isothermals** and join locations with the same temperature.

A weather map of the world indicating average July temperatures. The isothermals are the curves that separate the colored bands.



Average air temperature near sea level in July (°F)

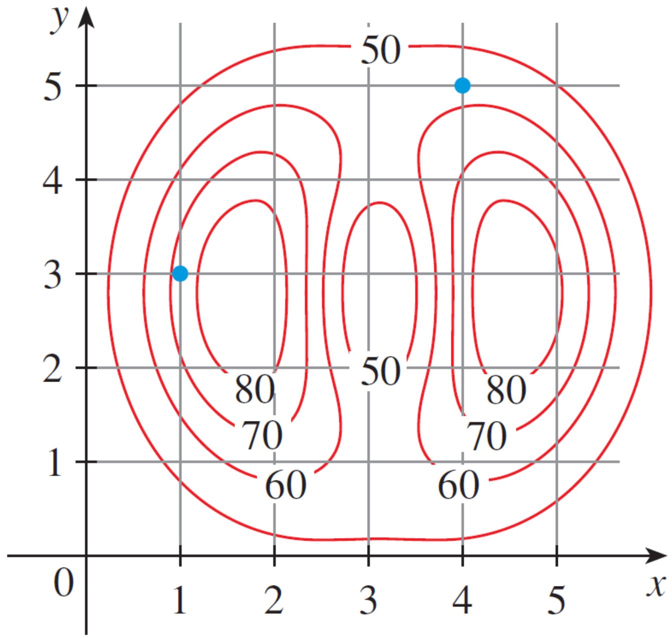
Example of Level Curves

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars** and join locations with the same pressure.

Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure, and are strongest where the isobars are tightly packed.

Example 8

A contour map for a function *f* is shown below. Use it to estimate the values of *f*(1, 3) and *f*(4, 5)



Solution:

The point (1, 3) lies partway between the level curves with *z*-values 70 and 80

We estimate that *f*(1,3) is about 73

Similarly, we estimate that *f*(4,5) is about 56

Example 9

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12.$

Solution:

The level curves are

$$6 - 3x - 2y = k \qquad \text{or} \qquad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope -3/2.

The four particular level curves with $k = -6, 0, 6,$ and 12 are $3x + 2y - 12 = 0,$ $3x + 2y - 6 = 0,$ $3x + 2y = 0,$ and $3x + 2y + 6 = 0.$

Example 9 – *Solution*

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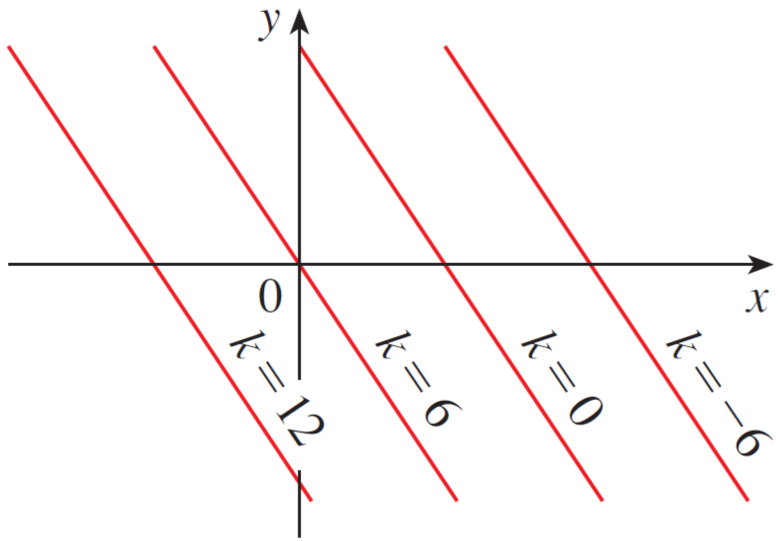
The level curves are equally spaced parallel lines because the graph of *f* is a plane



Contour map of
$f(x, y) = 6 - 3x - 2y$

Functions of Three Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write

$$T = f(x, y, t)$$

Functions of *n* Variables

A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Generalization of the case of 2 and 3 variables.

Limits for functions of 2 var.

(1) Definition We write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

and we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L if we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b). sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach.

Limits

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).
An important difference with respect to the case of one variable is that here the possible direction to approach the point (a, b) are infinitely many, not just two.

Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that

$$\lim_{(x, y)\to(a, b)} f(x, y) \text{ does not exist.}$$

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

Example 15

If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x, y) \to (0, 0)} f(x, y)$ exist?

Solution:

First let's approach (0, 0) along the *x*-axis. If *y* = 0, then *f*(*x*, 0) = 0/*x*^{2} = 0.

Therefore

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Example 15 – *Solution*

cont'd

Although we have obtained identical limits along the axes, that is still not enough to prove that the given limit is 0. Let's now approach (0, 0) along another line, say $y = x$. For all $x \neq 0$ we can write

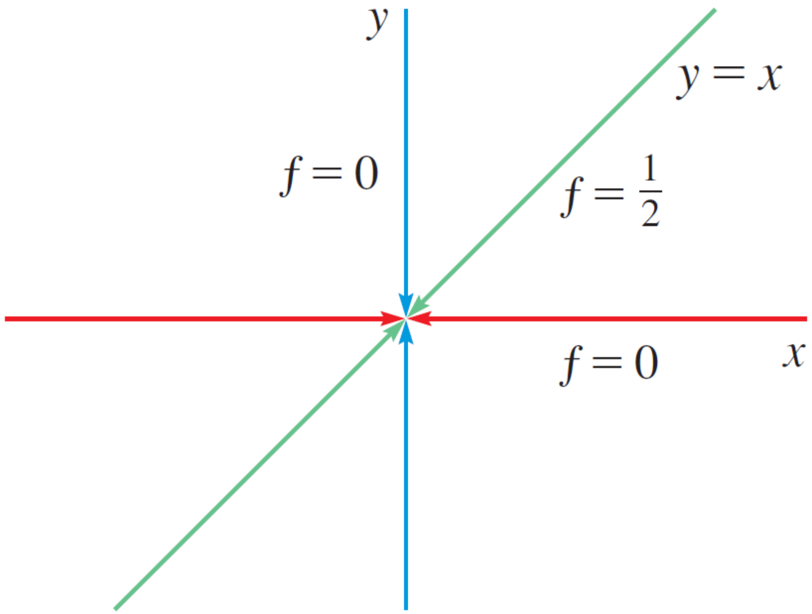
$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore

$$f(x, y) \to \tfrac{1}{2} \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along } y = x$$

Example 15 – *Solution*

cont'd



Since we obtain different limits along different paths, the given limit does not exist.

Limit Laws

The Limit Laws for functions of a single variable can be extended to functions of two variables:
the limit of a sum is the sum of the limits,
the limit of a product is the product of the limits, and so on.

In particular, the following equations are true.

(2) $$\lim_{(x,\, y) \to (a,\, b)} x = a \qquad \lim_{(x,\, y) \to (a,\, b)} y = b \qquad \lim_{(x,\, y) \to (a,\, b)} c = c$$

The Squeeze Theorem also holds.

Limits and Continuity

(3) Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

This means that a surface that is the graph of a continuous function has no hole or break.

Polynomials and rational funct.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers.

A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

A polynomial is continuous everywhere, a rational function is continuous at every point where the denominator is not 0

Example 17

Evaluate $\lim_{(x, y)\to(1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution:

Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x, y)\to(1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2$$

$$= 11$$

Limits for functions of 3 var.

The notation

$$\lim_{(x,\, y,\, z) \to (a,\, b,\, c)} f(x, y, z) = L$$

means that the values of *f*(*x*, *y*, *z*) approach the number *L* as the point (*x*, *y*, *z*) approaches the point (*a*, *b*, *c*) along any path in the domain of *f* . The function *f* is **continuous** at (*a*, *b*, *c*) if

$$\lim_{(x,\, y,\, z) \to (a,\, b,\, c)} f(x, y, z) = f(a, b, c)$$

Limits and Continuity

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$.

In other words, it is discontinuous on the sphere with center the origin and radius 1.

Partial Derivatives

In general, if *f* is a function of two variables *x* and *y*, suppose we let only *x* vary while keeping *y* fixed, say $y = b$, where *b* is a constant.

Then we are really considering a function of a single variable *x*, namely, $g(x) = f(x, b)$. If *g* has a derivative at *a*, then we call it the **partial derivative of *f* with respect to *x* at (*a*, *b*)** and denote it by $f_x(a, b)$. Thus

(1) $$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

Partial Derivatives

and so Equation 1 becomes

(2) $$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

(3) $$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

Partial Derivatives

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

(4) If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

An alternative notation for partial derivatives is similar to Leibniz notation but uses the symbol ∂ instead of d

Partial Derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:

Holding *y* constant and differentiating with respect to *x*, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding *x* constant and differentiating with respect to *y*, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretation of Partial Derivatives

To give a geometric interpretation of partial derivatives, consider the equation $z = f(x, y)$ representing a surface S (the graph of f).

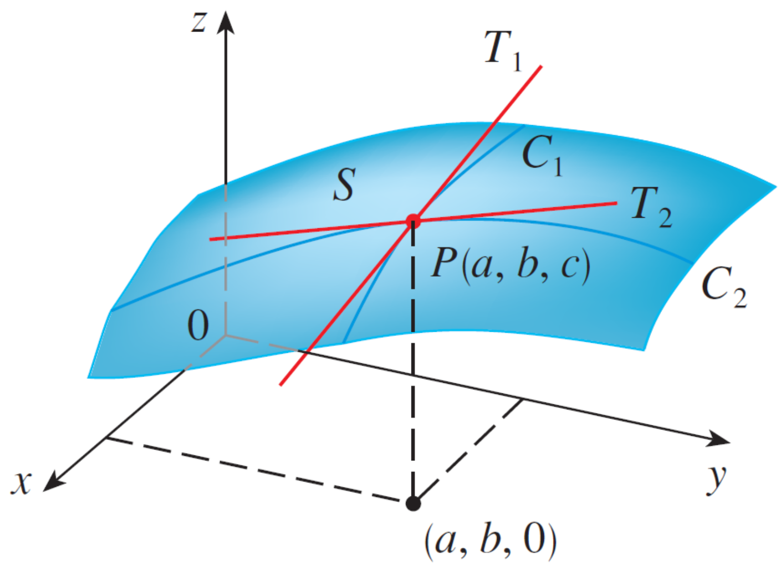
If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S.
In other words, C_1 is the trace of S in the plane $y = b$

Likewise, the vertical plane $x = a$ intersects S in a curve C_2

Interpretation of Partial Derivatives

Both of the curves C_1 and C_2 pass through the point P



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2

Interpretation of Partial Derivatives

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$

The curve C_2 is the graph of the function $h(y) = f(a, y)$, so the slope of its tangent T_2 at P is $h'(b) = f_y(a, b)$

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$

Example 2

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

We have

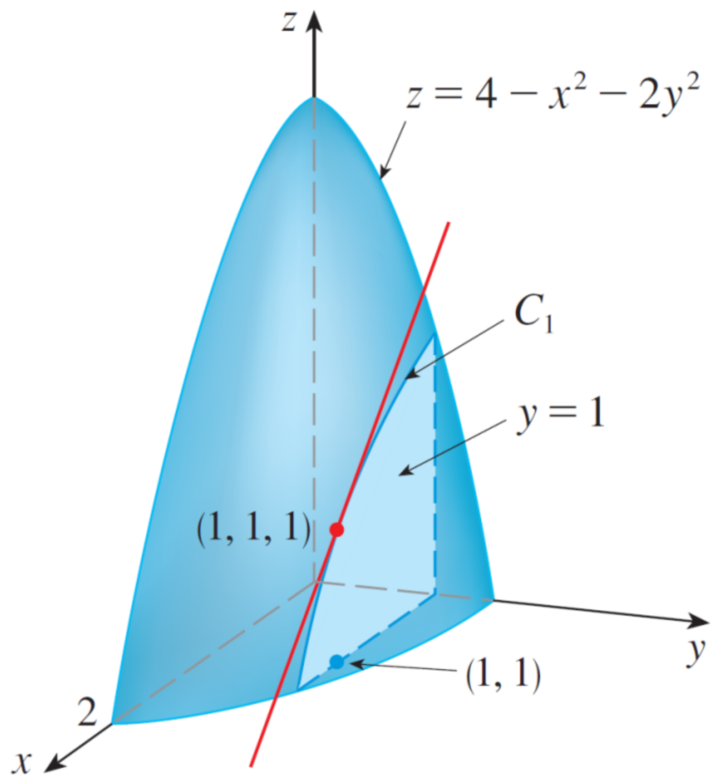
$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$.

Example 2 – *Solution*

This intersection is a curve called C_1



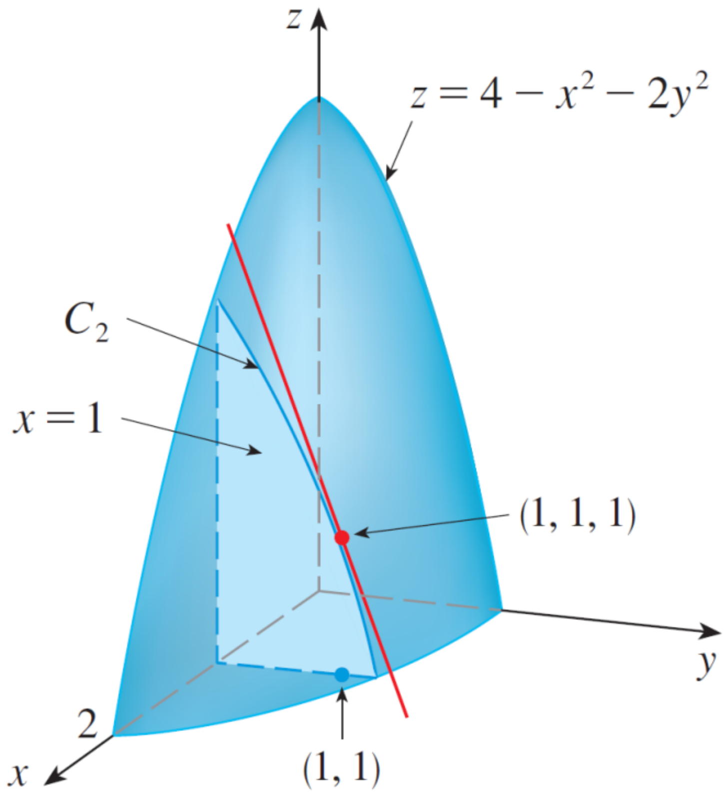
The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$

Example 2 – *Solution*

cont'd

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid $z = 4 - x^2 - 2y^2$ is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is

$f_y(1, 1) = -4$



Interpretation of Partial Derivatives

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed.

Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding *y* and *z* as constants and differentiating *f*(*x*, *y*, *z*) with respect to *x*.

If *w* = *f*(*x*, *y*, *z*), then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of *w* with respect to *x* when *y* and *z* are held fixed.

Functions of More Than Two Variables

But we can't interpret it geometrically because the graph of *f* lies in four-dimensional space.

In general, if *u* is a function of *n* variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the *i*-th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write this partial derivative as:

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = D_i f$$

Example 7

Find f_x, f_y, and f_z when $f(x, y, z) = e^{xy} \ln z$

Solution:

Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y can also be seen as functions of two variables.

So we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f

If $z = f(x, y)$, we use the notation given in the next slide:

Higher Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 9

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution:

In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Higher Derivatives

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x}$$

Example 10

Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Solution: note that it is a composite function, so the derivative with respect to *x* is

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

Partial Differential Equations

If a differential equation contains partial derivatives, then it is called *partial differential equation*.

For example, the following partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

is called the **wave equation** and describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Here the constant *a* depends on the density of the string and on the tension in the string.

Example 11

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

Solution:

$$u_x = \cos(x - at) \qquad u_t = -a\cos(x - at)$$

$$u_{xx} = -\sin(x - at) \qquad u_{tt} = -a^2\sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation.

Partial Differential Equations

Another partial differential equation that arises frequently in biology is the **diffusion equation**:

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2}$$

where *D* is a positive constant called the *diffusion constant*.

Tangent Planes

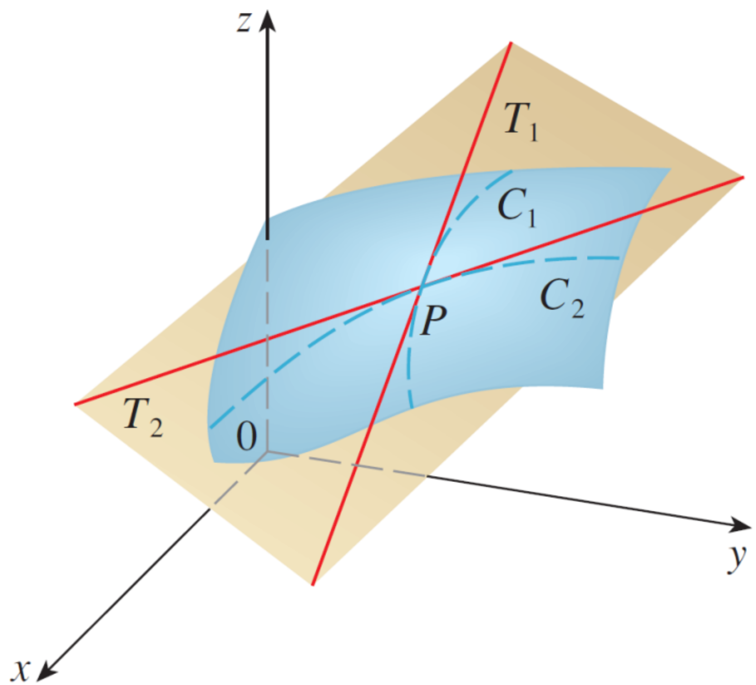
Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.

Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes

Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2



The tangent plane contains the tangent lines T_1 and T_2

Tangent Planes

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

(1) $$z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1

Tangent Planes

Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a. But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

(2) Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

Solution:

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at (1, 1, 3) as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

Linear Approximations

In general, we know from (2) that an equation of the tangent plane to the graph of a function *f* of two variables at the point (*a*, *b*, *f*(*a*, *b*)) is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely,

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{3}$$

is called the **linearization** of *f* at (*a*, *b*) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{4}$$

is called the **linear approximation** or the **tangent plane approximation** of *f* at (*a*, *b*).

Linear Approximations

(5) Definition The function $f(x, y)$ is **differentiable** at (a, b) if

$$\lim_{(x, y) \to (a, b)} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

Definition 5 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b).

In other words, the tangent plane approximates the graph of f well near the point of tangency.

(6) Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2

Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution:

The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 6. The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y = x + y$$

Example 2 – *Solution*

cont'd

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542.$$

Linear Approximations

Linear approximations and differentiability can be defined in a similar manner for functions of more than two variables.

A differentiable function is defined by an expression similar to the one in Definition 5. For functions of three variables the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

The Chain Rule

(2) The Chain Rule Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Solution:

The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t.

Example 1 – *Solution*

cont'd

We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$

The Chain Rule

For functions of three variables, where $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, the Chain Rule has an extra term:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 4

Find dw/dt if $w = xe^{y/z}$, where $x = 3t + 2$, $y = t^2$, and $z = t^3 - 1$.

Solution:

By the Chain Rule, we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= e^{y/z} \cdot 3 + xe^{y/z} \cdot \frac{1}{z} \cdot 2t + xe^{y/z}\left(-\frac{y}{z^2}\right) \cdot 3t^2$$

$$= e^{y/z}\left(3 + \frac{2xt}{z} - \frac{3xyt^2}{z^2}\right)$$

Implicit Differentiation

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Implicit Differentiation

But *dx*/*dx* = 1, so if $\partial F/\partial y \neq 0$ we solve for *dy*/*dx* and obtain

(4) $$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that *F*(*x*, *y*) = 0 defines *y* implicitly as a function of *x*.

Implicit Differentiation

The **Implicit Function Theorem** gives conditions under which this assumption is valid:
It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 4.

Example 5

Find y' if $x^3 + y^3 = 6xy$.

Solution:

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 4 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{3x^2 - 6y}{3y^2 - 6x}$$

$$= -\frac{x^2 - 2y}{y^2 - 2x}$$

Implicit Differentiation

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f.

If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation

But $$\frac{\partial}{\partial x}(x) = 1 \qquad \text{and} \qquad \frac{\partial}{\partial x}(y) = 0$$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0,$ we solve for $\partial z/\partial x$ and obtain the first formula in Equations 5. The formula for $\partial z/\partial y$ is obtained in a similar manner.

(5) $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Implicit Differentiation

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid:
If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by Equation 5.

Example – *Infectious disease outbreak size*

Epidemiologists often wish to predict the fraction of the population that will ultimately be infected when a disease begins to spread. Mathematical models have been used to do so. The Kermack-McKendrick model leads to the following equation

$$\rho e^{-qA} = 1 - A$$

where A is the fraction of the population ultimately infected, q is a measure of disease transmissibility, and ρ is a measure of the fraction of the population that is initially susceptible to infection.

How does the outbreak size A change with an increase in the transmissibility q?

Example – *Solution*

Let $F(\rho, q, A) = \rho e^{-qA} - 1 + A$. Then, from Equations 5, the rate of change of *A* with respect to *q* is

$$\frac{\partial A}{\partial q} = -\frac{F_q}{F_A}$$

$$= -\frac{-\rho A e^{-qA}}{-\rho q e^{-qA} + 1}$$

$$= \frac{\rho A}{e^{qA} - \rho q}$$

This is the rate of increase of the outbreak size as the transmissibility increases while ρ remains constant.

Directional Derivatives

We know that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

(1)

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y- directions, that is, in the directions of the unit vectors **i** = [1, 0] and **j** = [0, 1].

Directional Derivatives

(2) Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = [a, b]$ is

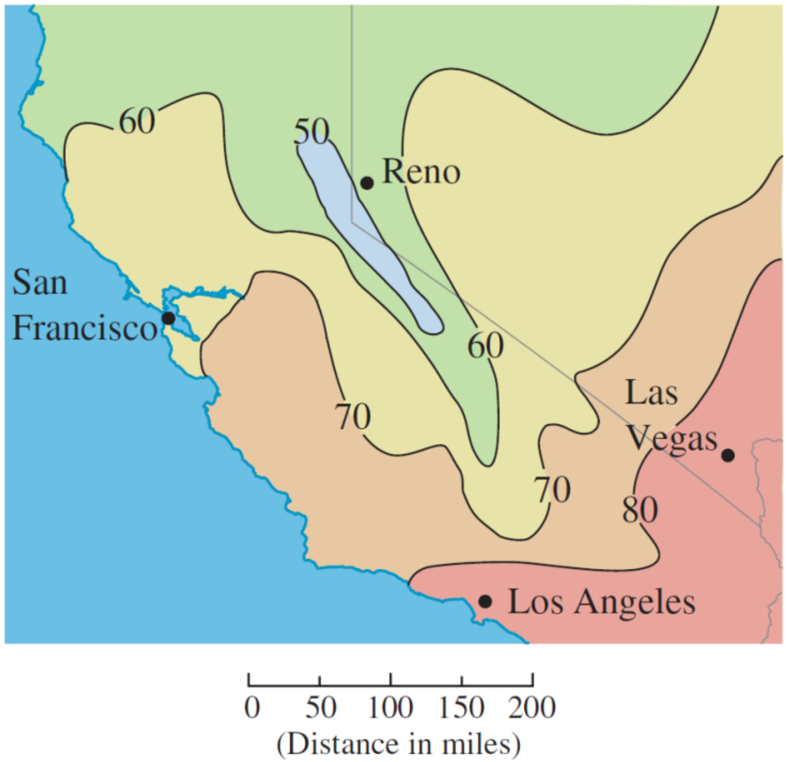
$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations 1 we see that if $\mathbf{u} = \mathbf{i} = [1, 0]$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = [0, 1]$, then $D_{\mathbf{j}} f = f_y$.

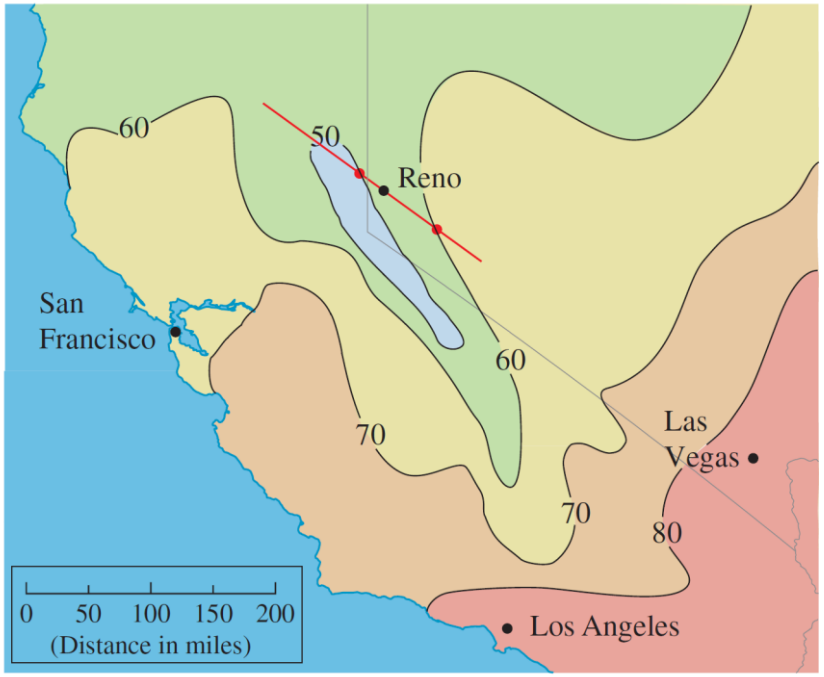
Example 1

Use the weather map in figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.



Example 1 – *Solution*

The unit vector directed toward the southeast is $\mathbf{u} = \left[1/\sqrt{2}, -1/\sqrt{2}\right]$, but we won't need to use this expression. We start by drawing a line through Reno toward the southeast



Example 1 – *Solution*

cont'd

We approximate the directional derivative $D_{\mathbf{u}}T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$.

The temperature at the point southeast of Reno is $T = 60°F$ and the temperature at the point northwest of Reno is $T = 50°F$.

The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}}T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13°\text{F/mi}$$

Directional Derivatives

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

(3) Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = [a, b]$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$$

If the unit vector **u** makes an angle θ with the positive x-axis, then we can write **u** = [cos θ, sin θ] and the formula in Theorem 3 becomes

(6) $$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Example 2

Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and **u** is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1, 2)$?

Solution:

Formula 6 gives

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$

Example 2 – *Solution*

cont'd

$$= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}\,(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right]$$

$$= \frac{13 - 3\sqrt{3}}{2}$$

The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\textbf{(7)} \qquad \begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\,a + f_y(x, y)\,b \\ &= [\,f_x(x, y), f_y(x, y)\,] \cdot [\,a, b\,] \\ &= [\,f_x(x, y), f_y(x, y)\,] \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well.

The Gradient Vector

So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f , which is read "del f ").

(8) Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = [\, f_x(x, y), f_y(x, y) \,]$$

Example 3

If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = [f_x, f_y] = [\cos x + ye^{xy}, xe^{xy}]$$

and $$\nabla f(0, 1) = [2, 0]$$

The Gradient Vector

With this notation for the gradient vector, we can rewrite the expression (7) for the directional derivative of a differentiable function as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \tag{9}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

Example 4

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point (2, –1) in the direction of the vector **v** = [2, 5].

Solution:

We first compute the gradient vector at (2, –1):

$$\nabla f(x, y) = [2xy^3, 3x^2y^2 - 4]$$

$$\nabla f(2, -1) = [-4, 8]$$

Note that **v** is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left[\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right]$$

Example 4 – *Solution* cont'd

Therefore, by Equation 9, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u}$$

$$= [-4, 8] \cdot \left[\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right]$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}}$$

$$= \frac{32}{\sqrt{29}}$$

Maximizing the Directional Derivative

(10) Theorem If f is a differentiable function and (a, b) is in the domain of f, then the maximum value of the directional derivative $D_{\mathbf{u}} f(a, b)$ is $|\nabla f(a, b)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(a, b)$.

Example 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution:

(a) We first compute the gradient vector:

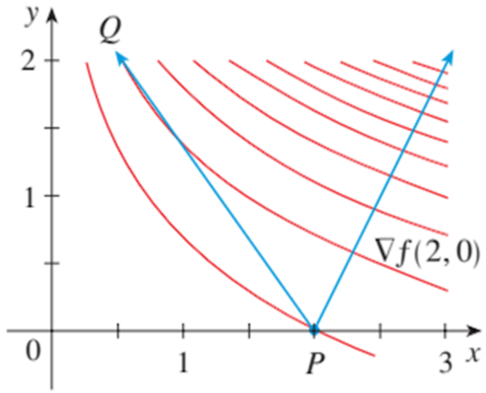
$$\nabla f(x, y) = [f_x, f_y] = [e^y, xe^y]$$

$$\nabla f(2, 0) = [1, 2]$$

Example 6 – *Solution*

cont'd

To go from point $P(2, 0)$ to Q(1/2, 2) we must go in the direction of the vector [-1.5, 2]



The unit vector in the direction of $\overrightarrow{PQ} = [-1.5, 2]$ is $\mathbf{u} = \left[-\frac{3}{5}, \frac{4}{5}\right]$,
($\sqrt{(-1.5)^2 + 2^2} = \sqrt{6.25} = 2.5$, so the unit vector is [-1.5/2.5, 2/2.5])
so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u}$$

$$= [1, 2] \cdot \left[-\tfrac{3}{5}, \tfrac{4}{5}\right]$$

$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

Example 6 – *Solution*

cont'd

(b) According to Theorem 10, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = [1, 2]$.

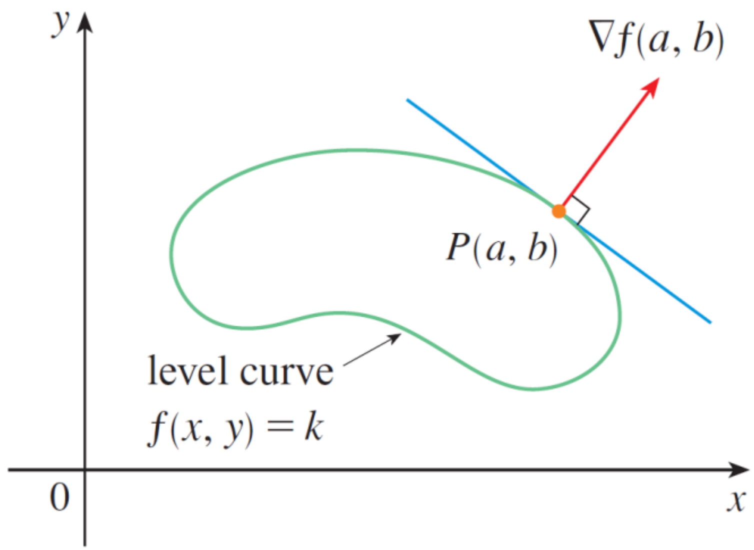
The maximum rate of change is

$$|\nabla f(2, 0)| = |[1, 2]|$$

$$= \sqrt{5}$$

Maximizing the Directional Derivative

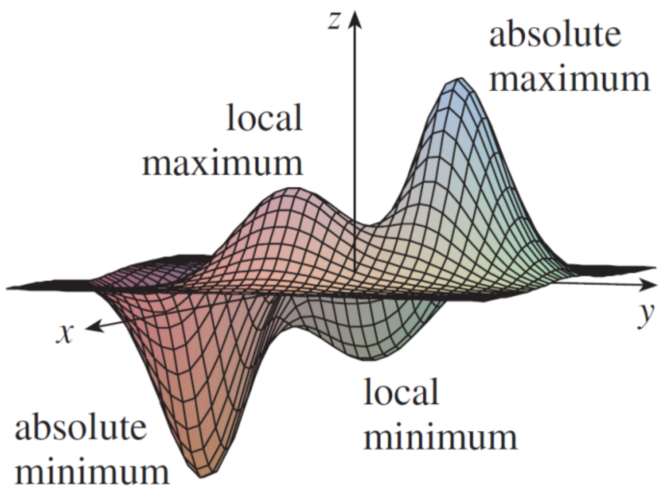
Another geometric aspect of the gradient vector is illustrated here:



The gradient vector $\nabla f(a, b)$ is perpendicular to the level curve $f(x, y) = k$ that passes through the point (a, b).

Maximum and Minimum Values

Look at the hills and valleys in the graph of this *f*



There are two points where $f(a, b)$ has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values $f(x, y)$

Maximum and Minimum Values

The larger of these two values is the *absolute maximum*. Likewise, *f* has two *local minima*, where *f*(*a*, *b*) is smaller than nearby values. The smaller of these two values is the *absolute minimum*.

(1) Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for all points (*x*, *y*) in the domain of *f*, then *f* has an **absolute maximum** (or **absolute minimum**) at (*a*, *b*). An absolute maximum or minimum is also called a **global** maximum or minimum

Maximum and Minimum Values

(2) Fermat's Theorem for Functions of Two Variables If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f

However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is (1, 3). We can rewrite the f as $x^2 -2x +1 \; + y^2 -6y +9 \; +4$ where we recognize the squares of binomials, so we rewrite again f as

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

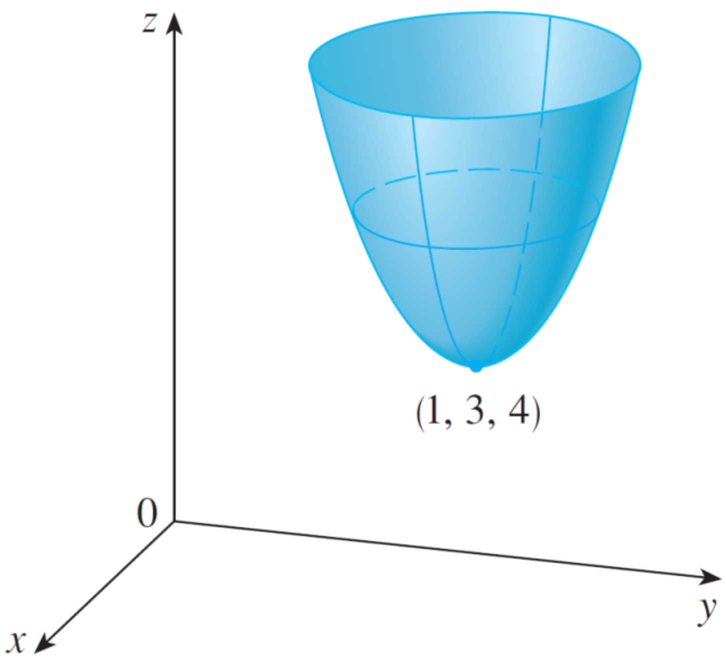
Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y.

Example 1

cont'd

Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f

This can be confirmed geometrically from the graph of f, which is the paraboloid with vertex (1, 3, 4)



$$z = x^2 + y^2 - 2x - 6y + 14$$

Maximum and Minimum Values

(3) Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Note 1

In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

Maximum and Minimum Values

Note 2

If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f

Note 3

The formula for D comes from the fact that it is the determinant of the matrix of second partial derivatives, called Hessian matrix:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Example 3

Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution:

We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad\qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

Example 3 – *Solution*

cont'd

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are (0, 0), (1, 1), and (–1, –1).

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Example 3 – *Solution*

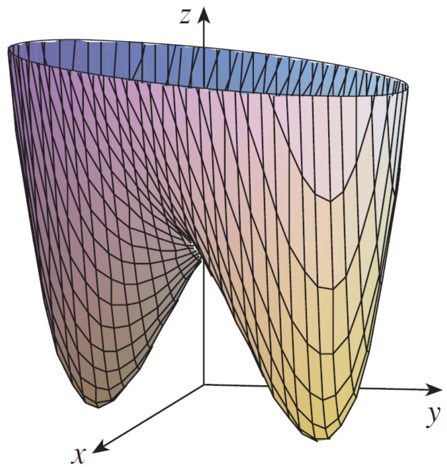
cont'd

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$.

Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum.

Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is:



$$z = x^4 + y^4 - 4xy + 1$$

Absolute Maximum and Minimum Values

As a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent.

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

Absolute Maximum and Minimum Values

(4) Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

(5) To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

Solution:

Since *f* is a polynomial, it is continuous on the closed, bounded rectangle *D*, so Theorem 4 tells us there is both an absolute maximum and an absolute minimum.

According to step 1 in (5), we first find the critical points.

Example 7 – *Solution*

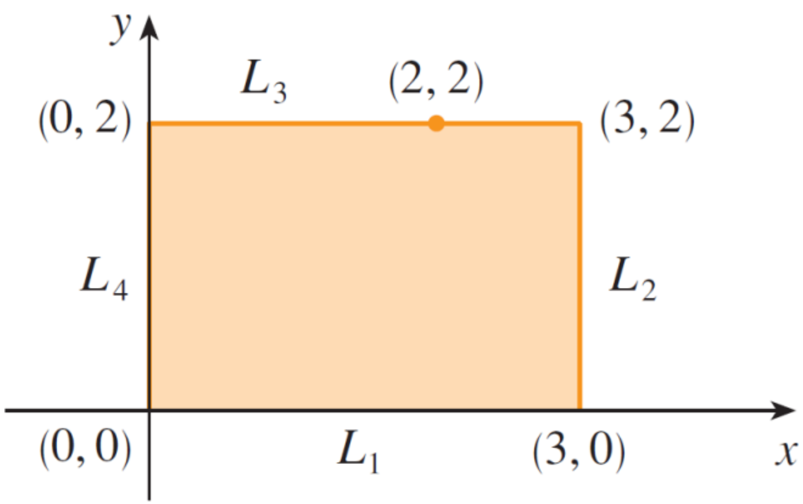
cont'd

These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is (1, 1), and the value is $f(1, 1) = 1$

In step 2 we look at the values of *f* on the boundary of *D*, which consists of the four line segments L_1, L_2, L_3, L_4 shown



Example 7 – *Solution*

cont'd

On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

Example 7 – *Solution*

cont'd

By observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Example 7 – *Solution*

cont'd

In step 3 we compare these values with the $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$

$$f(x, y) = x^2 - 2xy + 2y$$