Elective in Robotics 2014/2015

Analysis and Control of Multi-Robot Systems

Elements of Port-Hamiltonian Modeling

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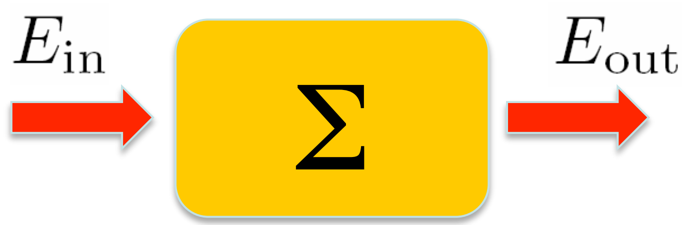
Introduction to Port-Hamiltonian Systems

- Port-Hamiltonian Systems (PHS): strong link with passivity

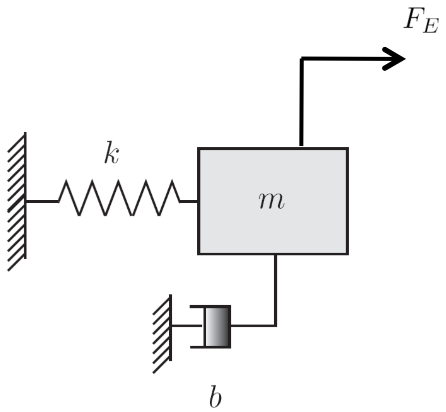


$$E_{\text{out}} \leq E_{\text{in}}$$

- Passivity:
  - I/O characterization
  - "Constraint" on the I/O energy flow
  - Many desirable properties
    - Stability of free-evolution
    - Stability of zero-dynamics
    - Easy stabilization with static output-feedback
    - Modularity: passivity is preserved under proper compositions

- However, no insights on the structure of a passive system
- PHS: focus on the structure behind passive systems

Mass-spring-damper vs. PHS



- Review of the mass-spring-damper example

$$m\ddot{x} + b\dot{x} + kx = f$$

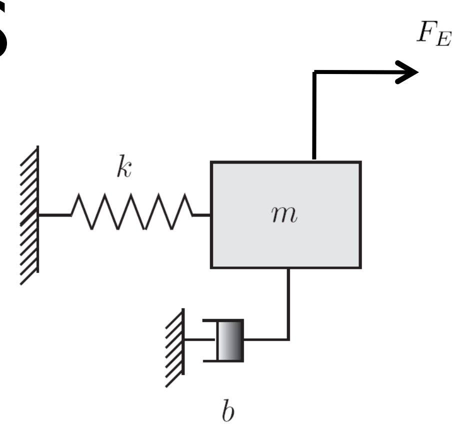
- This system was shown to be passive w.r.t. the pair (u, y) with $u = f$, $y = \dot{x}$, and as storage function the total energy (kinetic + potential)

$$V = E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Indeed, it is $\dot{V} = f\dot{x} - b\dot{x}^2 = yu - by^2 \leq yu$

- But why is it passive? We must investigate its internal structure...

Mass-spring-damper vs. PHS



- The spring-mass system is made of 2 components (2 states)
  - Assume for now no damping $b = 0$
  - Mass = kinetic energy $K = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m}$, $p = m\dot{x}$
  - Spring = elastic energy $V = \frac{1}{2}kx^2$

Linear momentum

- Let us consider the 2 components separately

$$\mathcal{K}: \begin{cases} \dot{p} &= f_p \\ v_p &= \frac{\partial K}{\partial p} = \frac{p}{m} (= \dot{x}) \end{cases}$$

Kinetic energy storing

$$\mathcal{V}: \begin{cases} \dot{x} &= v_x \\ f_x &= \frac{\partial V}{\partial x} = kx \end{cases}$$

Potential energy storing

- Note that these (elementary) systems are the "integrators with nonlinear outputs" we have seen before
- We know they are passive w.r.t. (v_p, f_p) and (v_x, f_x), respectively

Mass-spring-damper vs. PHS

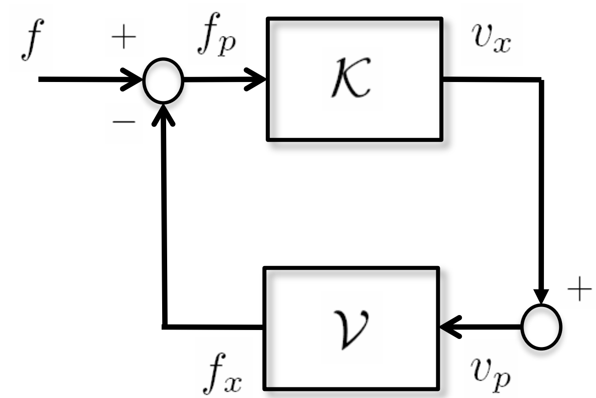
$$\mathcal{K}: \begin{cases} \dot{p} & = & f_p \\ v_p & = & \dfrac{\partial K}{\partial p} = \dfrac{p}{m} (= \dot{x}) \end{cases}$$

Kinetic energy storing

$$\mathcal{V}: \begin{cases} \dot{x} & = & v_x \\ f_x & = & \dfrac{\partial V}{\partial x} = kx \end{cases}$$

Potential energy storing

- Let us interconnect them in "feedback" $v_x = v_p,\, f_p = -f_x + f$



- The resulting system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \qquad (\blacksquare)$$

where $H(x, p) = K(p) + V(x)$ is the total energy (Hamiltonian)

- **Prove that (■) is equivalent to** $m\ddot{x} + kx = f$

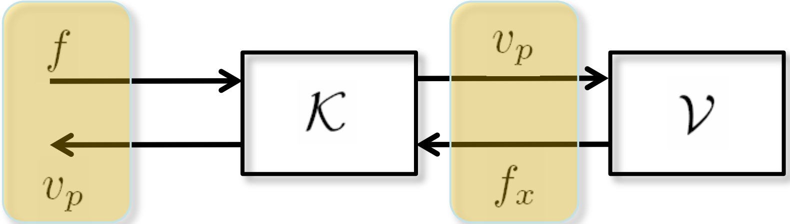
Mass-spring-damper vs. PHS

- How does the energy balance look like?

$$\dot{H} = \underbrace{\begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix}}_{\equiv 0} + \begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f = \frac{\partial H^T}{\partial p} f = f^T v_p$$

Skew-symmetric

- We find again the passivity condition w.r.t. the pair $(f,\, v_p)$
- The subsystems $\mathcal{K}$ and $\mathcal{V}$ exchange energy in a power-preserving way - no energy is created/destroyed



- The subsystem $\mathcal{K}$ exchanges energy with the "external world" through the pair $(f,\, v_p)$
- Total energy H can vary only because of the power flowing through $(f,\, v_p)$

Mass-spring-damper vs. PHS

- What if a damping term $b > 0$ is present in the system?
- By interconnecting $\mathcal{K}$ and $\mathcal{V}$ as before (feedback interconnection), we get

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \qquad (\blacksquare)$$

Skew-symmetric **Positive semi-def.**

- **Prove that (■) is equivalent to** $m\ddot{x} + b\dot{x} + kx = f$
- The energy balance now reads

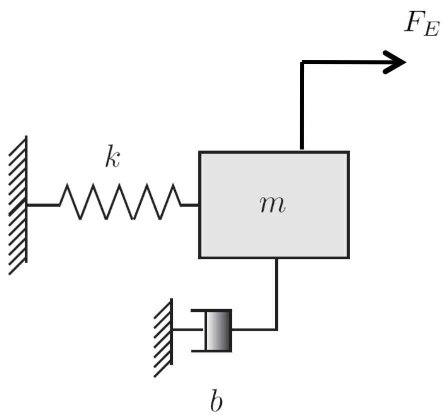
$$\dot{H} = \underbrace{-\begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix}}_{\leq 0} + \begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \leq \frac{\partial H^T}{\partial p} f = f^T v_p$$

Mass-spring-damper vs. PHS

$$\dot{H} = \underbrace{-\begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix}}_{\leq 0} + \begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \leq \frac{\partial H^T}{\partial p} f = f^T v_p$$

- Again the passivity condition w.r.t. the pair $(f,\, v_p)$
- Total energy H can now
  - vary only because of the power flowing through $(f,\, v_p)$
  - decrease because of internal dissipation
- But still, power-preserving exchange of energy between $\mathcal{K}$ and $\mathcal{V}$

Mass-spring-damper vs. PHS



- Summarizing, this particular passive system is made of:
  - Two atomic energy storing elements $\mathcal{K}$ and $\mathcal{V}$
  - A power-preserving interconnection among $\mathcal{K}$ and $\mathcal{V}$
  - An energy dissipation element b
  - A pair $(f,\ v_p)$ to exchange energy with the "external world"

- <u>Why passivity of the complete system?</u>

- $\mathcal{K}$ and $\mathcal{V}$ are passive (and "irreducible")
  - Their power-preserving interconnection is a feedback interconnection (thus, preserves passivity)
  - The element b dissipates energy
  - Therefore, any increase of the total energy H is due to the power flowing through $(f,\ v_p)$. For this reason, this pair is also called **<u>power-port</u>**

- How general are these results?

Introduction to Port-Hamiltonian Systems

- In the linear time-invariant case $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ (■)

passivity implies existence of a storage function $H(x) = \frac{1}{2}x^T Q x,$ $Q = Q^T \geq 0$

such that $A^T Q + QA \leq 0$ and $C = B^T Q$

- If $\ker Q \subset \ker A$ (always true if $Q > 0$)

then (■) can be rewritten as

$$\begin{cases} \dot{x} = (J - R)Qx + Bu, & J = -J^T, \quad R = R^T \geq 0 \\ y = B^T Q x \end{cases}$$

and energy balance $\dot{H} = -x^T QRQx + x^T QBu \leq y^T u$

- $H(x)$ is called the Hamiltonian function

Introduction to Port-Hamiltonian Systems

- Similarly, most nonlinear passive system can be rewritten as

$$\begin{cases} \dot{x} = [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u, \quad J(x) = -J^T(x),\ R(x) \geq 0 \\ y = g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

with $H(x) \geq 0$ being the Hamiltonian function (storage function) and

$$\dot{H} = -\frac{\partial H^T}{\partial x}R(x)\frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x}g(x)u \leq y^T u$$

showing the passivity condition

- Roles:

$H(x)$ represents the energy stored by the system

$R(x)$ represents the internal dissipation in the system

$J(x)$ represents an internal power-preserving interconnection among different components

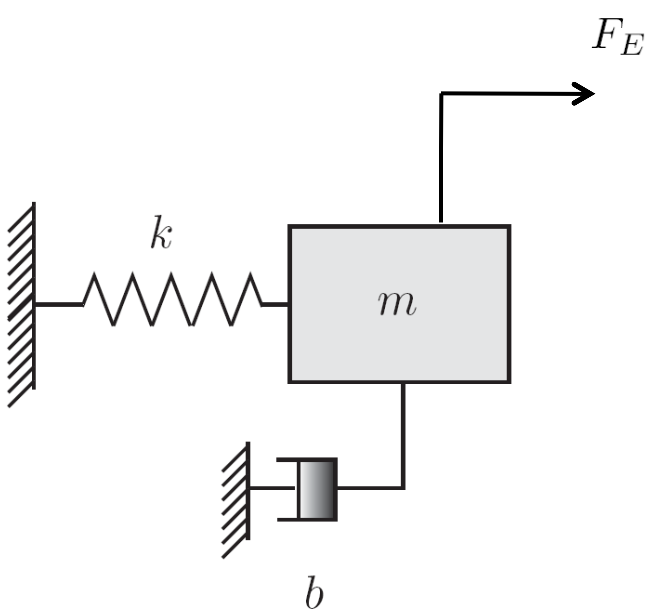
$(u,\ y)$ represents a "power-port", allowing energy exchange (in/out) with the external world

Introduction to Port-Hamiltonian Systems

- In the mass-spring-damper case, the generic Port-Hamiltonian formulation

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u, \quad J(x) = -J^T(x),\ R(x) \geq 0 \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

specializes into $J(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ R(x) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix},\ g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Introduction to Port-Hamiltonian Systems

- In the (more abstract) example we have seen during the Passivity lectures, we showed that

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1^3 + u \end{cases}$$

is a passive system with passive output $y = x_2$ and Storage function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \geq 0$$

- Can it be recast in PHS form with $H(x) = V(x)$ being the Hamiltonian?

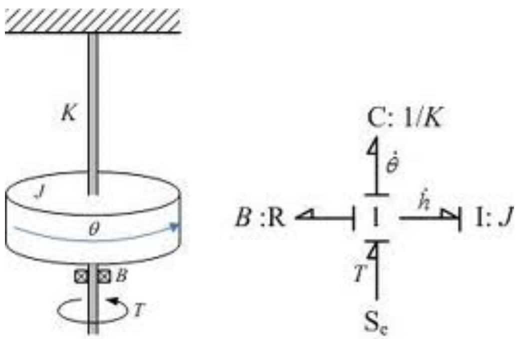
- Yes:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dfrac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y & = & \begin{bmatrix} 0 & 1 \end{bmatrix} \dfrac{\partial H}{\partial x} \end{cases}$$

Introduction to Port-Hamiltonian Systems

- What is then Port-Hamiltonian modeling?

- It is a cross-domain energy-based modeling philosophy, generalizing Bond Graphs
  - Historically, network modeling of lumped-parameter physical systems (e.g., circuit theory)



- Main insights: all the physical domains deal, in a way or another, with the concept of Energy storage and Energy flows
  - Electrical
  - Hydraulical
  - Mechanical
  - Thermodynamical

- Dynamical behavior comes from the exchange of energy
- The "energy paths" (power flows) define the internal model structure

Introduction to Port-Hamiltonian Systems

- Port-Hamiltonian modeling

- Most (**passive**) physical systems can be modeled as a set of simpler subsystems (modularity!) that either:
  - Store energy
  - Dissipate energy
  - Exchange energy (internally or with the external world) through power ports

- Role of energy and the interconnections between subsystems provide the basis for various control techniques

- Easily address complex nonlinear systems, especially when related to real "physical" ones

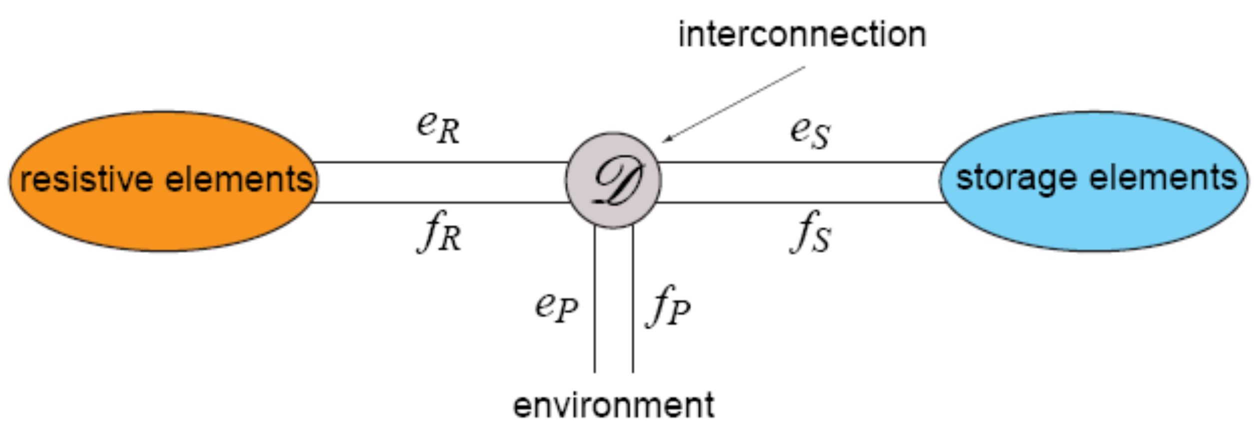
Introduction to Port-Hamiltonian Systems

- Port-Hamiltonian systems can be formally defined in an abstract way

- Everything revolves about the concepts of
  - **Power ports** (medium to exchange energy)
  - **Dirac structures** (“pattern” of energy flow)
  - **Hamiltonian** (storage of energy)

- We will now give a (**very brief and informal**) introduction of these concepts

- Big guys in the field:
  - Arjan van der Schaft
  - Romeo Ortega
  - Bernard Maschke
  - Mark W. Spong
  - Stefano Stramigioli
  - Alessandro Astolfi
  - and many more (maybe one of you in the future?)



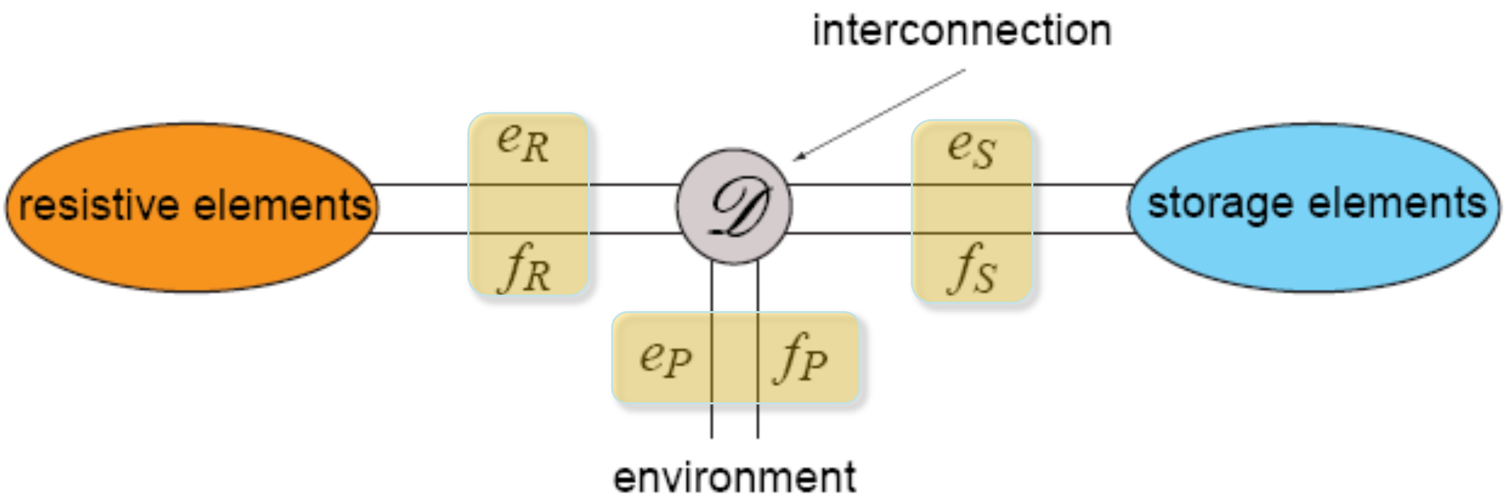
Introduction to Port-Hamiltonian Systems

- A power port is a pair of variables $(e, \ f)$ called "effort" and "flow" that mediates a power exchange (energy flow) among 2 physical components

Physical domain	**Flow** f	**Effort** e
electric	Current	Voltage
magnetic	Voltage	Current
Potential (mechanics)	Velocity	Force
Kinetic (mechanics)	Force	Velocity
Potential (hydraulic)	Volume flow	Pressure
Kinetic (hydraulics)	pressure	Volume flow
chemical	Molar flow	Chemical potential
thermal	Entropy flow	temperature

Introduction to Port-Hamiltonian Systems

- A generic port-Hamiltonian model is then
  - A set of energy storage elements (with their power ports $(e_S,\ f_S)$)
  - A set of resistive elements (with their power ports $(e_R,\ f_R)$)
  - A set of open power-ports (with their power ports $(e_P,\ f_P)$)
  - An internal **power-preserving interconnection** $\mathcal{D}$, called Dirac structure



- An explicit example of a "Dirac structure" is the power-preserving interconnection represented by the skew-symmetric matrix $J(x)$

General Mechanical System

- Any mechanical system (also constrained) described by the Euler-Lagrange equations can be recast in a Port-Hamiltonian form

- Start with a set of generalized coordinates $q = [q_1^T \ \ldots \ q_n^T]^T$

- Define the Lagrangian $L = K(q,\, \dot{q}) - V(q)$ with $K(q,\, \dot{q}) = \dfrac{1}{2}\dot{q}^T M(q)\dot{q}$

being the kinetic energy, $V(q)$ the potential energy, and $M(q) > 0$ the positive definite Inertia matrix

- Apply a change of coordinates $(q,\, \dot{q}) \rightarrow (q,\, p)$ where $p = M(q)\dot{q}$ are usually called "generalized momenta"

- The kinetic energy in the new coordinates is $K(q,\, p) = \dfrac{1}{2}p^T M^{-1}(q)p$

General Mechanical System

- Define the Hamiltonian (total energy) of the system as

$$H(q,\, p) = K(q,\, p) + V(q)$$

- The Euler-Lagrange equations for the system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q,\, \dot{q})\right) - \frac{\partial L}{\partial q}(q,\, \dot{q}) = \tau \qquad (\blacksquare)$$

- Since $p = \dfrac{\partial L}{\partial \dot{q}} = \dfrac{\partial K}{\partial \dot{q}}$ we can rewrite (■) as

$$\begin{cases} \dot{q} &= \dfrac{\partial H}{\partial p} \\ \dot{p} &= -\dfrac{\partial H}{\partial q} + \tau \end{cases} \Rightarrow \left\{ \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau \right.$$

General Mechanical System

- It follows that $\dot{H} = \frac{\partial H^T}{\partial p}\tau = \dot{q}^T\tau$

- If $H(q,\, p)$ (i.e., $V(q)$) is bounded from below, the system is passive w.r.t. the power port $(\dot{q},\, \tau)$

- Similarly, a mechanical system with collocated inputs and outputs (also underactuated) is generally described by

$$
\begin{cases}
\dot{q} &= \dfrac{\partial H}{\partial p} \\
\dot{p} &= -\dfrac{\partial H}{\partial q} + B(q)u \\
y &= B^T(q)\dfrac{\partial H}{\partial p} \qquad (= B^T(q)\dot{q})
\end{cases}
$$

- Again, passivity w.r.t. $(y,\, u)$

Modularity

- As one can expect, the “proper” interconnection of a number of Port-Hamiltonian Systems

$$(\mathcal{M}_i,\, \mathcal{D}_i,\, H_i),\ i = 1 \ldots k$$

through a Dirac structure $\mathcal{D}_I$ is again a Port-Hamiltonian System $(\mathcal{M},\, \mathcal{D},\, H)$ with

- Hamiltonian $H = H_1 + \ldots H_k$
- State manifold $\mathcal{M} = \mathcal{M}_1 \times \ldots \mathcal{M}_k$
- Dirac structure $\mathcal{D} = \mathcal{D}_1, \ldots, \mathcal{D}_k,\, \mathcal{D}_I$
- This allows for **modularity and scalability**

Modularity

- Example: given two Port-Hamiltonian System

$$\begin{cases} \dot{x}_1 &= (J_1(x_1) - R_1(x_1))\dfrac{\partial H_1}{\partial x_1} + g_1(x_1)u_1 \\ y_1 &= g_1^T(x_1)\dfrac{\partial H_1}{\partial x_1} \end{cases} \qquad \begin{cases} \dot{x}_2 &= (J_2(x_2) - R_2(x_2))\dfrac{\partial H_2}{\partial x_2} + g_2(x_2)u_2 \\ y_2 &= g_2^T(x_2)\dfrac{\partial H_2}{\partial x_2} \end{cases}$$

- Define an interconnecting Dirac structure $\mathcal{D}_I$ as (**for example**)

$$u_1 = y_2,\ u_2 = -y_1$$

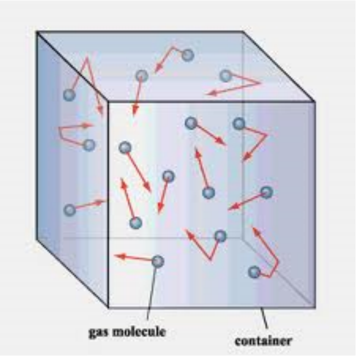
- The composed system is again Port-Hamiltonian

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} J_1(x_1) & g_1(x_1)g_2^T(x_2) \\ -g_2(x_2)g_1^T(x_1) & J_2(x_2) \end{bmatrix} - \begin{bmatrix} R_1(x_1) & 0 \\ 0 & R_2(x_2) \end{bmatrix} \right) \begin{bmatrix} \dfrac{\partial H_1}{\partial x_1} \\ \dfrac{\partial H_2}{\partial x_2} \end{bmatrix}$$

with Hamiltonian function $H(x_1,\, x_2) = H_1(x_1) + H_2(x_2)$

Further generalizations

- Much more could be said on Port-Hamiltonian System....
- Can model distributed parameters physical systems (wherever energy plays a role)
  - Transmission line
  - Flexible beams
  - Wave equations
  - Gas/fluid dynamics



- Are modular (re-usability)
  - Network structure (.... -> multi-agent)
- Are flexible
  - State-dependent (time-varying) interconnection structure $J(x)$

Summary

- PHS are a powerful way to model a very large class of physical systems
  - For instance, every physical system admitting an Energy concept (the whole physics?)

- In PHS, the emphasis in on the internal structure of a system. A PHS system is a network of
  - **Power ports**: medium to exchange energy
  - Elementary/irreducible **energy storing elements** endowed with their power ports
  - **Dissipating elements** endowed with their power ports
  - **“External world” power ports** for external interaction
  - **A power-preserving interconnection** structure (Dirac structure) among the internal power ports

- The total energy of a PHS is called Hamiltonian. If the Hamiltonian is bounded from below, a PHS is passive w.r.t. its external ports

- Proper compositions of PHS are PHS

Control of PHS

- How to control a Port-Hamiltonian System?

$$\begin{cases} \dot{x} = [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y = g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$$

- A PHS is still a dynamical system in the general form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

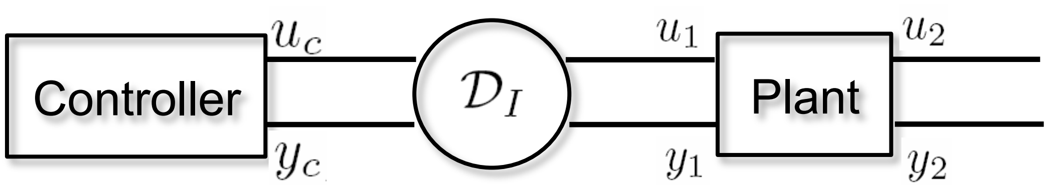
hence, one could use any of the available (nonlinear) control techniques

- However, in closed-loop, we want to retain and to exploit the PHS structure
  - PHS plant and controller
  - Power-preserving interconnection among them

Control of PHS

- The general idea is: assume a plant and controller in PHS form, and interconnected through a suitable $\mathcal{D}_I$

$$\begin{cases} \dot{x} = [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y = g^T(x) \dfrac{\partial H}{\partial x} \end{cases} \qquad \begin{cases} \dot{x}_c = (J_c(x_c) - R_c(x_c)) \dfrac{\partial H_c}{\partial x_c} + g_c(x_c)u_c \\ y_c = g_c^T(x_c) \dfrac{\partial H_c}{\partial x_c} \end{cases}$$



where we split the plant port $(u,\, y)$ into $(u_1,\, y_1)$ and $(u_2,\, y_2)$, and use $(u_1,\, y_1)$ for the interconnection with the controller port $(u_c,\, y_c)$

- In general, one can imagine two distinct control goals
  - Regulation to x^* or tracking of $x^*(t)$ for the plant state variables $x(t)$
  - Desired (closed-loop) behavior of the plant at the interaction port $(u_2,\, y_2)$
  - The latter is for instance the case of Impedance Control for robot manipulators

Energy Transfer Control

- Consider two PHS

$$\begin{cases} \dot{x}_1 = J_1(x_1)\dfrac{\partial H_1}{\partial x_1} + g_1(x_1)u_1 \\ y_1 = g_1^T(x_1)\dfrac{\partial H_1}{\partial x_1} \end{cases} \qquad \begin{cases} \dot{x}_2 = J_2(x_2)\dfrac{\partial H_2}{\partial x_2} + g_2(x_2)u_2 \\ y_2 = g_2^T(x_2)\dfrac{\partial H_2}{\partial x_2} \end{cases}$$

- And assume we want to transfer some amount of energy among them by keeping the total energy $H_1(x_1) + H_2(x_2)$ constant

- This can be done by interconnecting the two PHS as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha y_1(x_1)y_2^T(x_2) \\ \alpha y_2(x_2)y_1^T(x_1) & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \alpha \in \mathbb{R}$$

Skew-symmetric

- Note that this is an example of a state-modulated power preserving interconnection

$$J(x) = \begin{bmatrix} 0 & -\alpha y_1(x_1)y_2^T(x_2) \\ \alpha y_2(x_2)y_1^T(x_1) & 0 \end{bmatrix}$$

Energy Transfer Control

- Since the interconnection is power-preserving, it follows that the total Hamiltonian $H(x_1, x_2) = H(x_1) + H(x_2)$ stays constant, i.e.,

$$\dot{H}(x_1, x_2) = 0$$

- However, what happens to the individual energies?

- **Exercise: show that** $\dot{H}_1(x_1) = -\alpha \|y_1\|^2 \|y_2\|^2 \quad \dot{H}_2(x_2) = \alpha \|y_1\|^2 \|y_2\|^2$

- Thus, depending on the parameter α, energy is extracted/injected from system 1 to system 2 (no energy transfer with $\alpha = 0$)

- If $H_1(x_1)$ is lower-bounded, a finite amount of energy will be transferred to system 2. Indeed, at the minimum, $y_1 = 0 \implies \dot{H}_1 = 0$ and $\dot{H}_2 = 0$

- The same of course holds for $H_2(x_2)$

- We will use these ideas in some of the following developments

Energy Tanks

- Let us examine a concrete example of the Energy Transfer Control technique
- To this end, we introduce the **concept** of "**Energy Tank**"
- Assume the usual PHS

$$\begin{cases} \dot{x} &= [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$$

- We know it is passive w.r.t. (u, y) since

$$\dot{H} = -\frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x} g(x) u \leq y^T u$$

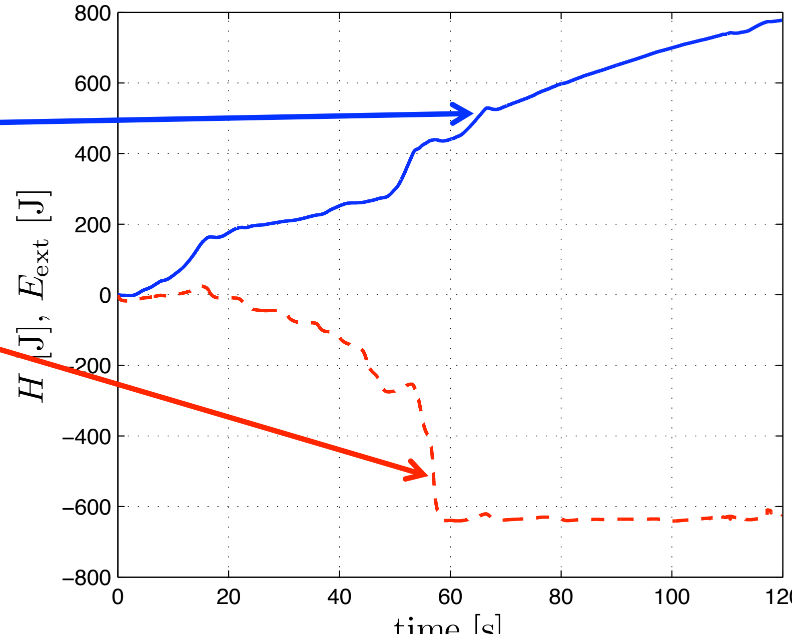
Energy Tanks

- In its integral form, the passivity condition reads

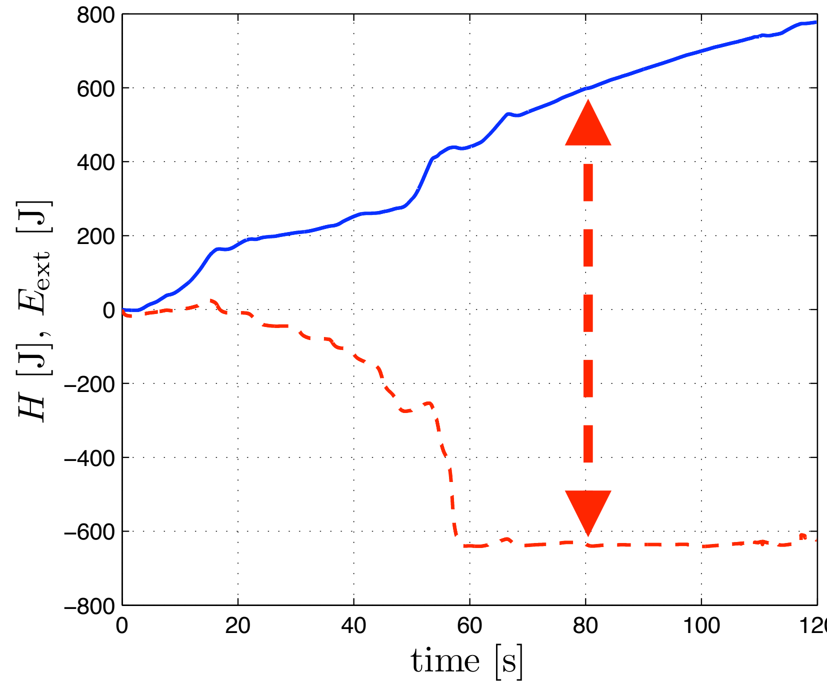
$$H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, \mathrm{d}\tau - \underbrace{\int_{t_0}^{t} \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \mathrm{d}\tau}_{\leq 0}$$

- Let $E_{\text{in}}(t) = H(t) - H(t_0)$ and $E_{\text{ext}}(t) = \int_{t_0}^{t} y^T u \, \mathrm{d}\tau$

- Over time, $E_{\text{in}}(t) \leq E_{\text{ext}}(t)$



Energy Tanks



- Why this gap over time between $E_{\text{ext}}(t)$ and $E_{\text{in}}(t)$?

- Because of the integral of the dissipation term

$$H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, \mathrm{d}\tau \underbrace{- \int_{t_0}^{t} \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \mathrm{d}\tau}_{\leq 0}$$

- However, we would be happy (from the passivity point of view) by just ensuring a lossless energy balance

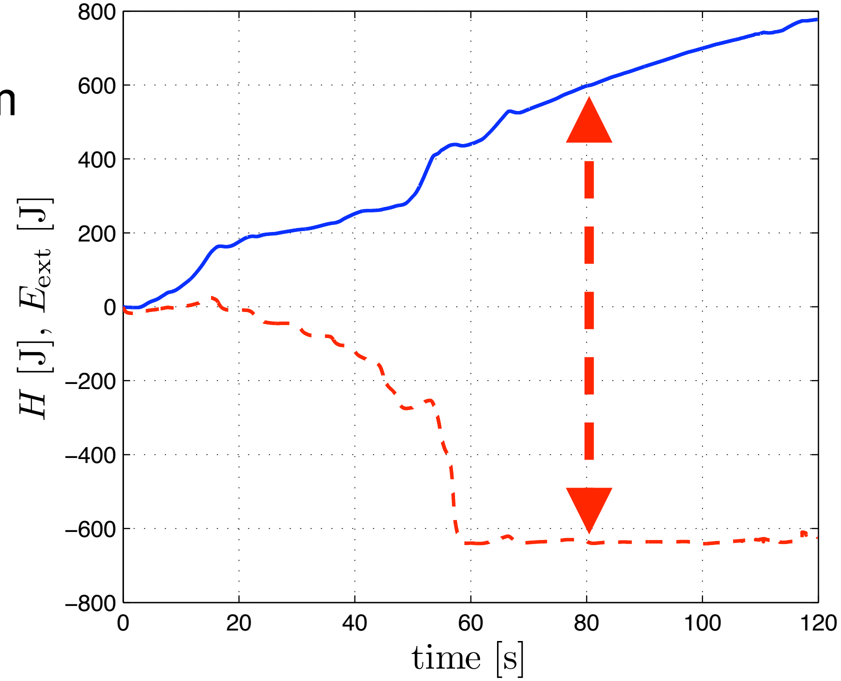
$$H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, \mathrm{d}t \iff E_{\text{in}}(t) = E_{\text{ext}}(t)$$

Energy Tanks

- Dissipation term: passivity margin of the system

- Imagine we could recover this “passivity gap”

- This recovered energy can be freely used for whatever goal **without violating the passivity constraint**



- This idea is at the basis of the Energy Tank machinery

- Energy Tank: an atomic energy storing element with state $x_t \in \mathbb{R}$ and energy function $T(x_t) = \frac{1}{2}x_t^2 \geq 0$

$$\begin{cases} \dot{x}_t = u_t \\ y_t = \dfrac{\partial T}{\partial x_t} (= x_t) \end{cases}$$

Energy Tanks

- We want to exploit the tank for:
  - storing back the natural dissipation of a PHS
  - allowing to use the stored energy for implementing some action on the PHS
  - this tank-based action will necessarily meet the passivity constraint

- How to achieve these goals? Let us consider again the PHS and Tank Energy element

$$\begin{cases} \dot{x} = [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y = g^T(x)\dfrac{\partial H}{\partial x} \end{cases} \qquad \begin{cases} \dot{x}_t = u_t \\ y_t = x_t \end{cases}$$

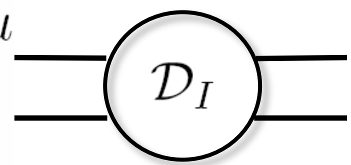
- Let $D(x) = \dfrac{\partial H^T}{\partial x} R(x) \dfrac{\partial H}{\partial x}$ represent the (scalar) dissipation rate of the PHS

- We start by choosing $u_t = \dfrac{1}{x_t} D(x) + \tilde{u}_t$ in the Tank dynamics

Energy Tanks

- The choice $u_t = \dfrac{1}{x_t} D(x) + \tilde{u}_t$ allows to store back the dissipated energy
- In fact, $\dot{T}(x_t) = x_t \left(\dfrac{1}{x_t} D(x) + \tilde{u}_t \right) = D(x) + x_t \tilde{u}_t$
- In order to exploit this stored energy to implement an action on the PHS system, we must design a suitable (power-preserving) interconnection among the PHS and Tank element

$$\begin{cases} \dot{x} &= [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x) u \\ y &= g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$$



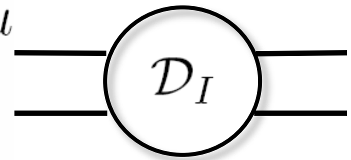
$$\begin{cases} \dot{x}_t &= \dfrac{1}{x_t} D(x) + \tilde{u}_t \\ y_t &= x_t \end{cases}$$

- We will make use of the ideas seen in the **Energy Transfer Control technique!**
- Implement the desired action as a "lossless energy transfer" between Tank and PHS
- **This action will always preserve passivity by construction**

Energy Tanks

- Assume we want to implement the action $w \in \mathbb{R}^m$ on the PHS ($m = \dim(u)$)

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$



$$\begin{cases} \dot{x}_t &= \dfrac{1}{x_t}D(x) + \tilde{u}_t \\ y_t &= x_t \end{cases}$$

- We then interconnect the PHS and the Tank element by means of this state-modulated power-preserving interconnection

$$\begin{bmatrix} u \\ \tilde{u}_t \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{w}{x_t} \\ -\dfrac{w^T}{x_t} & 0 \end{bmatrix} \begin{bmatrix} y \\ y_t \end{bmatrix}$$

- Since this coupling is skew-symmetric, no energy is created/lost during the transfer

Energy Tanks

- After this coupling the individual dynamics become

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)\left(\frac{w}{x_t}y_t\right) = [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)\boxed{w}$$

and

$$\dot{x}_t = \frac{1}{x_t}D(x) - \frac{w^T}{x_t}y = \frac{1}{x_t}D(x) - \frac{w^T}{x_t}g^T(x)\frac{\partial H}{\partial x}$$

- And altogether, a new PHS with Hamiltonian $\mathcal{H}(x,\, x_t) = H(x) + T(x_t)$

$$\begin{bmatrix}\dot{x}\\ \dot{x}_t\end{bmatrix} = \left(\begin{bmatrix}J(x) & \frac{w}{x_t}\\ -\frac{w^T}{x_t} & 0\end{bmatrix} - \begin{bmatrix}R(x) & 0\\ -\frac{1}{x_t}\frac{\partial \mathcal{H}^T}{\partial x}R(x) & 0\end{bmatrix}\right)\begin{bmatrix}\frac{\partial \mathcal{H}}{\partial x}\\ \frac{\partial \mathcal{H}}{\partial x_t}\end{bmatrix}$$

Skew-symmetric

Energy Tanks

- **Fact 1**: action w is correctly implemented on the original PHS

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)w$$

- **Fact 2**: the composite PHS is (altogether) a passive (lossless) system whatever the expression of w

- Proof: evaluating $\dot{\mathcal{H}}$ along the system trajectories, we obtain a lossless energy balance

$$\begin{bmatrix} \dot{x} \\ \dot{x}_t \end{bmatrix} = \left(\begin{bmatrix} J(x) & \frac{w}{x_t} \\ -\frac{w^T}{x_t} & 0 \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ -\frac{1}{x_t}\frac{\partial \mathcal{H}^T}{\partial x}R(x) & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial x} \\ \frac{\partial \mathcal{H}}{\partial x_t} \end{bmatrix}$$

$$\dot{\mathcal{H}} = -\frac{\partial \mathcal{H}^T}{\partial x}R(x)\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial \mathcal{H}^T}{\partial x_t}\frac{1}{x_t}\frac{\partial \mathcal{H}}{\partial x}R(x)\frac{\partial \mathcal{H}}{\partial x} = 0$$

Energy Tanks

- **Fact 3**: the machinery proposed so far **becomes singular** when $x_t = 0$

- What does the condition $x_t = 0$ represent?

- From the definition of the Tank energy function $T(x_t) = \frac{1}{2}x_t^2 \geq 0$ we have that $x_t = 0 \iff$ the **Tank energy is depleted**

- Therefore, this singularity represents the impossibility of passively perform the desired action w

- One can always imagine some (safety) switching parameter $\alpha(t)$ such that

$$\begin{cases} \alpha = 1 & \text{if} \quad T(x_t) \geq \epsilon > 0 \\ \alpha = 0 & \text{if} \quad T(x_t) < \epsilon \end{cases}$$

and implement $\alpha(t)w$ instead of w (i.e., implement w only if you can in a "passive way"). If cannot implement w, **wait for better times (the Tank gets replenished)**

Energy Tanks

- Note that the Tank dynamics is made of two terms

$$\dot{x}_t = \boxed{\frac{1}{x_t}D(x)} - \boxed{\frac{w^T}{x_t}g^T(x)\frac{\partial H}{\partial x}}$$

- The first term is always non-negative, and represents the "refilling" action due to the dissipation present in the PHS plant

- The second term can have any sign, also **negative**. It is then possible for the action w to **actually refill the tank**!

- Finally, note that **no condition is present** on $x_t(t_0)$! This can be chosen as any $x_t(t_0) > 0$

- In other words, complete freedom in choosing the initial amount of energy in the tank $T(x_t(t_0))$

- In fact, passivity ultimately is: **bounded amount of extractable energy**, but for whatever **initial energy in the system** (only needs to be **finite**)