Autonomous and Mobile Robotics

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Humanoid Robots 2:
Dynamic Modeling

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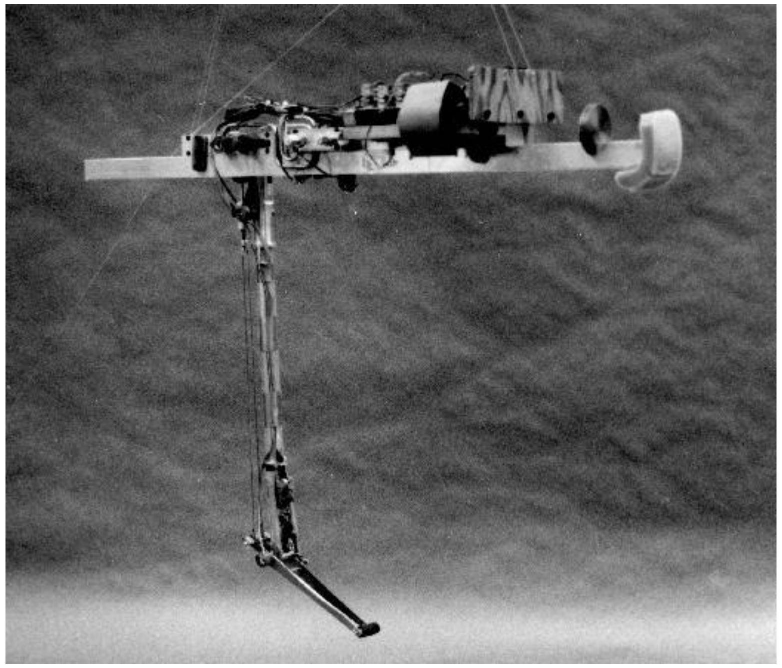
like a manipulator?

can we consider this as a part (leg) of a legged robot?

NO: this manipulator cannot fall because its base is clamped to the ground

this is a one-legged robot: Monopod from MIT



robot configuration

- to fully describe a humanoid robot it is not sufficient to indicate the joint angles

- the complete configuration is an element of $SE(3) \times \mathbb{R}^N$:
  - a **position** and **orientation** in 3D space
  - the internal configuration specified as **joint angles**

- N+6 generalized coordinates for a robot with N joints

$$\boldsymbol{q} = (\boldsymbol{x}_0, \boldsymbol{\theta}_0, \hat{\boldsymbol{q}})$$

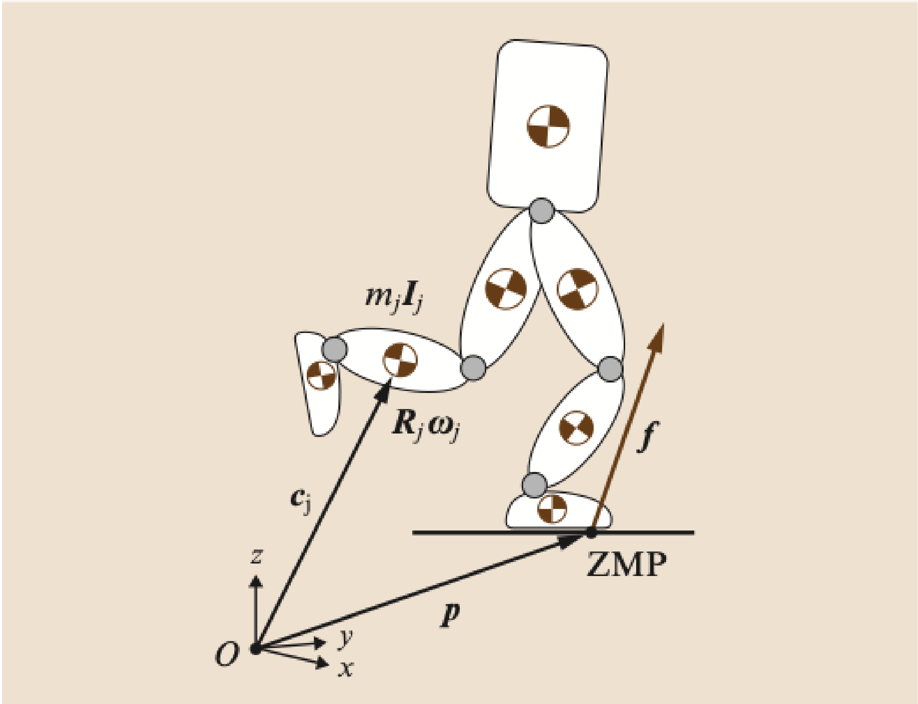
3 + 3 + N

robot configuration

$$\boldsymbol{q} = (\boldsymbol{x}_0, \boldsymbol{\theta}_0, \hat{\boldsymbol{q}})$$

- here, $\boldsymbol{x}_0$ and $\boldsymbol{\theta}_0$ represent the position and orientation of a robot link called the **floating base**

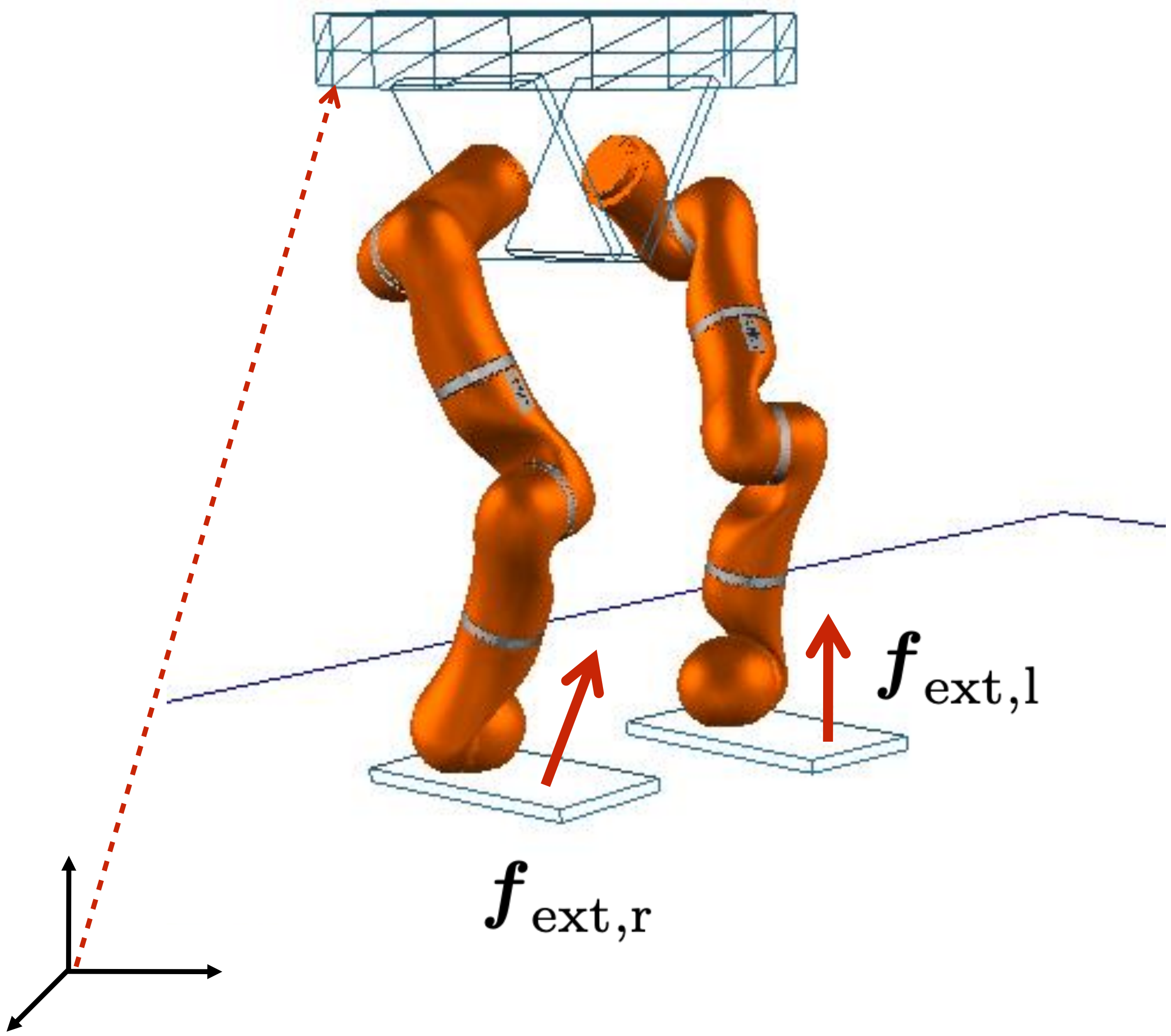
- any link can be chosen, although it is more common to chose the torso or one of the feet

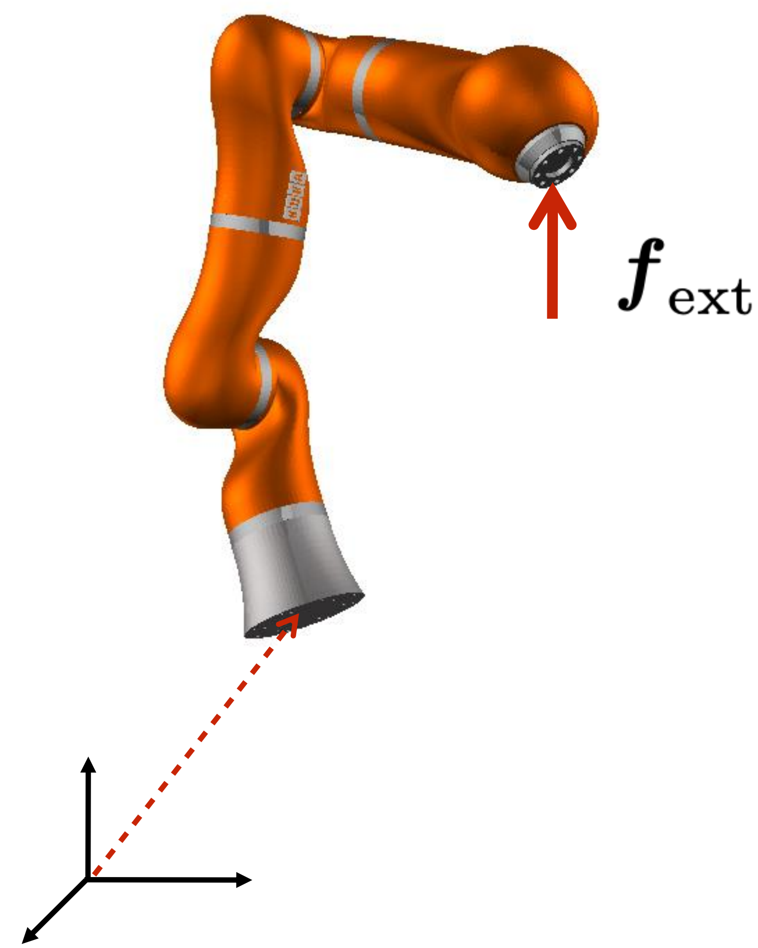


- $\hat{\boldsymbol{q}}$ is the vector of joint angles: if the floating base is fixed, a humanoid robot is exactly described like a manipulator

floating base model

the resulting model is a **floating base multi-body** system





Lagrangian dynamics

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{Q}\boldsymbol{\tau} + \sum_i \boldsymbol{J}_i^T \boldsymbol{f}_i$$

- with these generalized coordinates, we can write the **Lagrange equations** to derive a dynamic model for the robot

- $\boldsymbol{B}$ is the inertia matrix
- $\boldsymbol{N}$ collects Coriolis + centrifugal + gravity terms
- $\boldsymbol{Q}$ maps joint torques to generalized coordinates
- $\boldsymbol{J}_i$ is the Jacobian of the i-th contact point
- $\boldsymbol{f}_i$ is the i-th external force

Lagrangian dynamics

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{Q}\boldsymbol{\tau} + \sum_i \boldsymbol{J}_i^T \boldsymbol{f}_i$$

- external forces might be

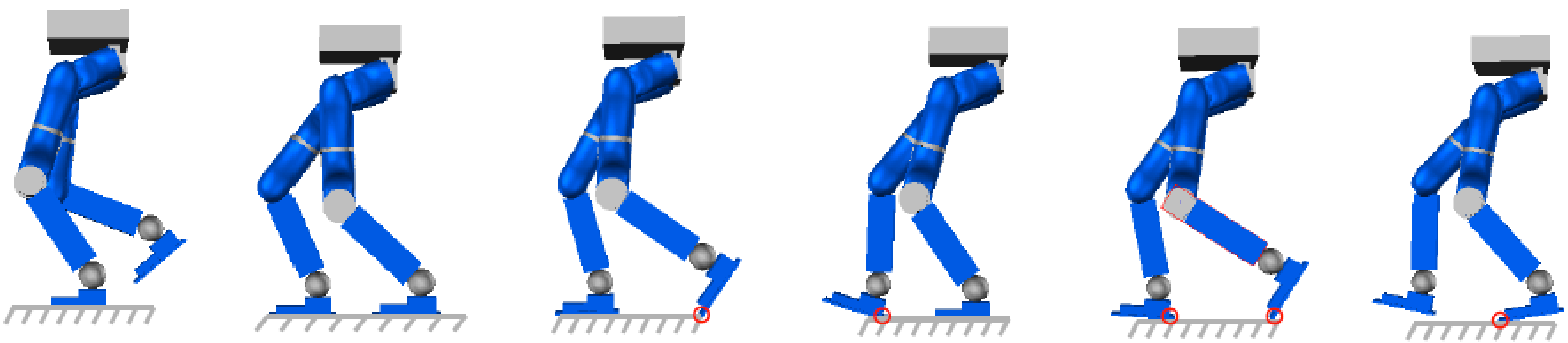
  - **applied forces**, like someone pushing on the robot arm

  - **reaction forces**, for example given by the robot interacting with the ground surface

contacts

- reaction contact forces are a fundamental part of this model!

- since the robot is **underactuated**, it is necessary to utilize these forces in order to move the robot in the environment

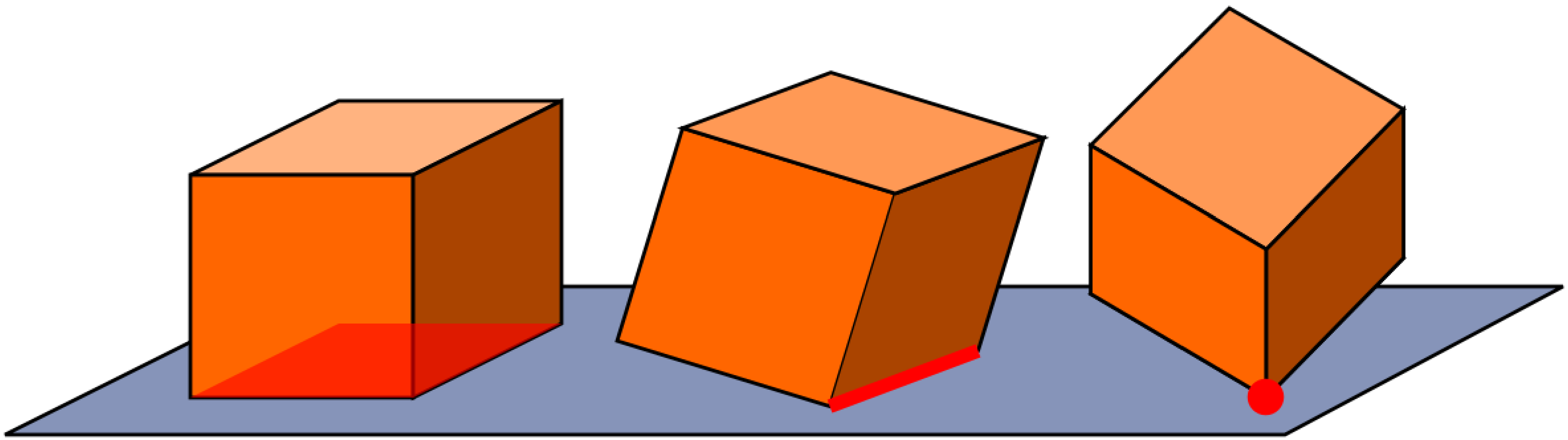


- from now on we will focus on reaction contact forces and ignore the presence of other kinds of external forces

contacts

- contact forces are applied as a pressure distribution over a contact surface, which is in general a 2D region, but it can in some instances become a line, or even a point

- this can happen if the contacting body is **tilting**

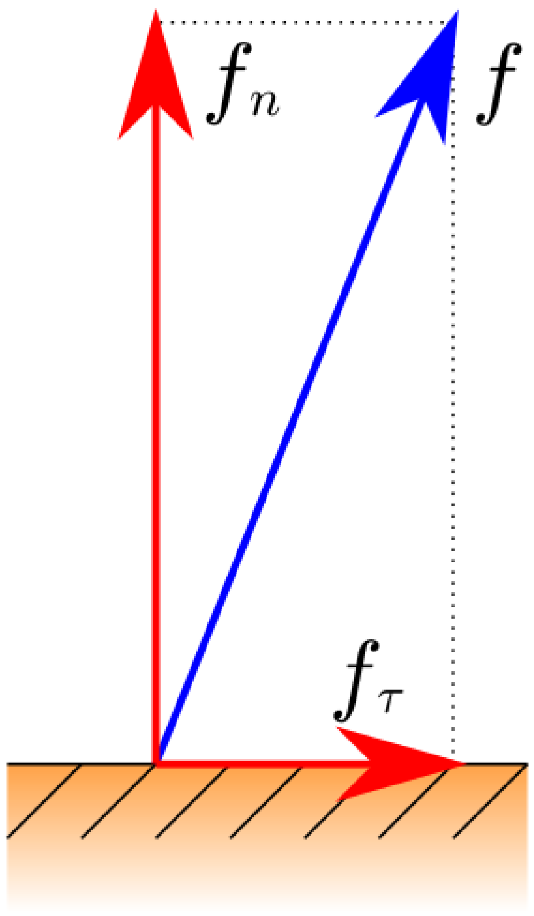


contacts

- contact forces are constituted by two components: **normal** forces f_n and **tangential** forces f_τ

- the normal force expresses the action of a non-holonomic constraint, by ensuring that the two contacting bodies don't compenetrate each other

- the non-holonomic character comes from the fact that this is an inequality constraint: normal contact forces are **unilateral**



contacts

- tangential forces express the action of **friction**: this can be either **static** or **dynamic** friction depending on whether there is **sliding**
- **Coulomb friction model**: the tangential force is proportional to the magnitude of the normal force

$$|f_\tau| \leq \mu_s |f_n| \qquad\qquad |f_\tau| = \mu_d |f_n|$$

static friction — dynamic friction

- the direction of the tangential force is such to counteract the motion

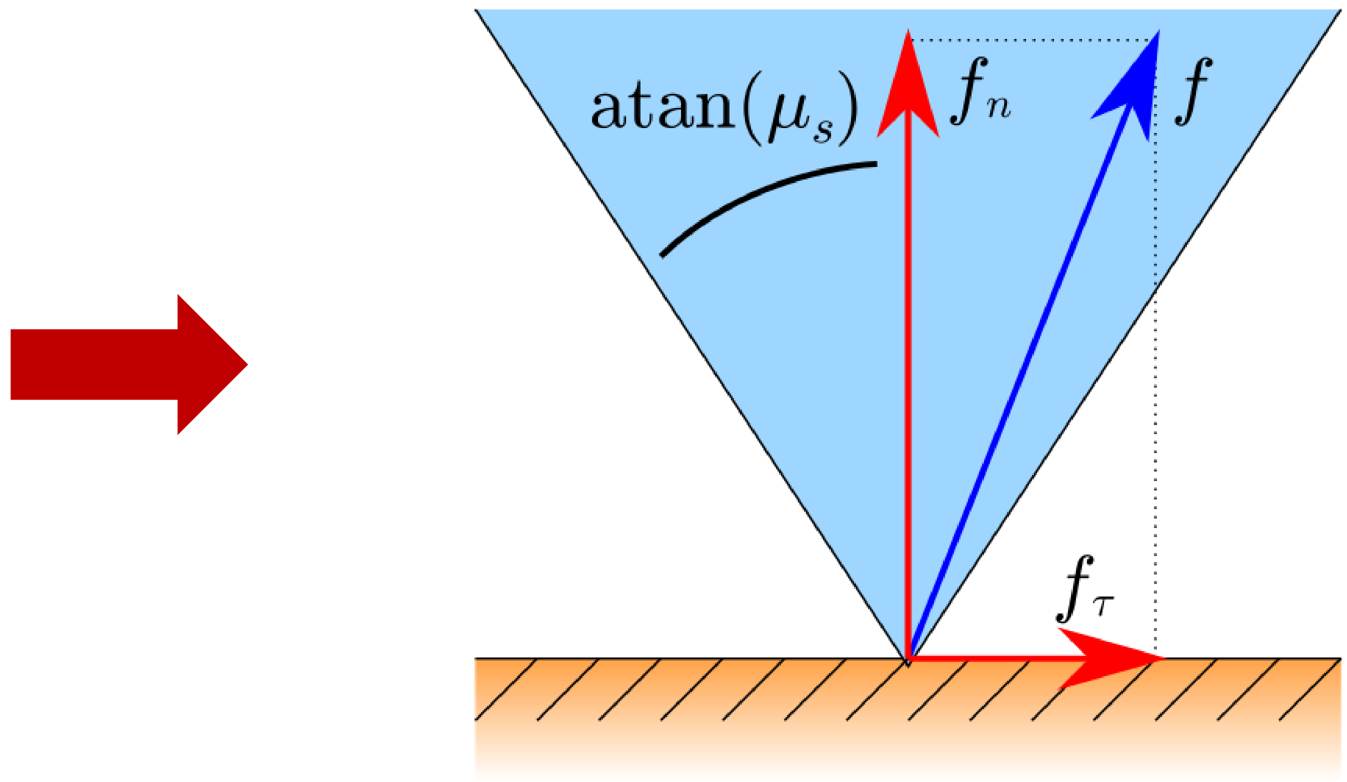
contacts

$$|f_\tau| \leq \mu_s |f_n|$$

- static friction is maintained if the tangential force does not exceed a maximum value proportional to the normal force

- it can be geometrically interpreted as the total contact force being inside a **friction cone**, with its apex at the contact point

$$\frac{|f_\tau|}{|f_n|} \leq \mu_s$$

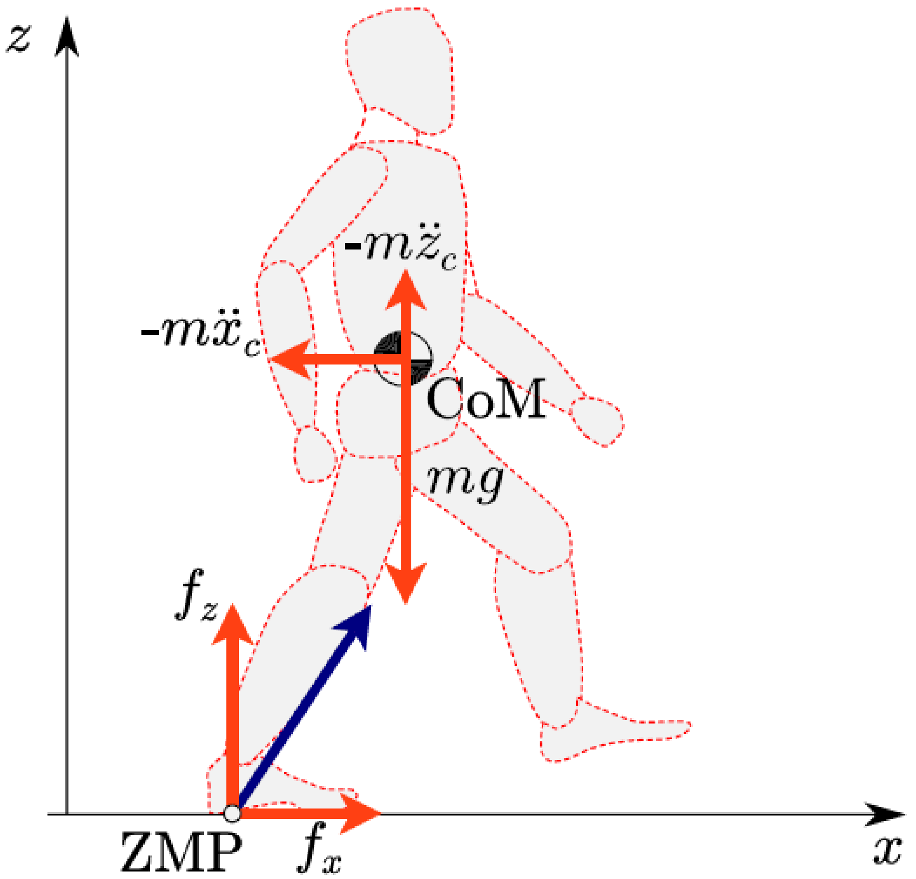


reduced-order models

- in order to simplify the control problem, we want to determine conditions for which there is neither tipping nor sliding

- for this, the full model is not required: a **reduced-order model** will work

- to derive such a model ,we will take a step back, and consider the **Newton-Euler equations** on the entire robot



Newton-Euler equations

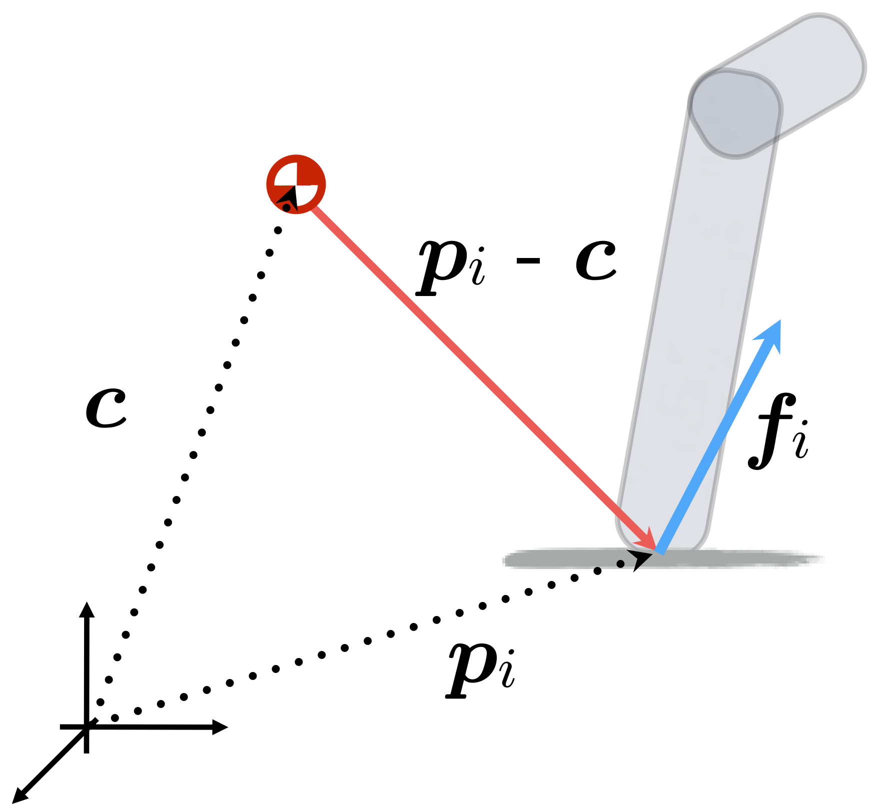
- the Newton-Euler equations are obtained by balancing **forces** and **moments** acting on the robot as a whole

- in statics, we would simply require that the balance is zero in both cases

- for a dynamic system, the force balance is equal to the variation of **linear momentum** and the moment balance is equal to the variation of **angular momentum**

Newton-Euler equations

recall: the **moment of a force** (or torque) is a measure of its tendency to cause a body to rotate about a specific point or axis



$$(\boldsymbol{p}_i - \boldsymbol{c}) \times \boldsymbol{f}_i$$

moment generated by the contact force $\boldsymbol{f}_i$ around the CoM

angular momentum around the CoM:
sum of the angular momentum of each robot link

$$\boldsymbol{L} = \sum_k (\boldsymbol{x}_k - \boldsymbol{c}) \times m_k \dot{\boldsymbol{x}}_k + I_k \boldsymbol{\omega}_k$$

$\boldsymbol{\omega}_k$: angular velocity of the k-th link

Newton-Euler equations

first Newton-Euler equation:
variation of **linear momentum** = **force balance**

$$M\ddot{\boldsymbol{c}} = \sum_i \boldsymbol{f}_i - M\boldsymbol{g}$$

$\boldsymbol{c}$: CoM position
M : total mass of the system

hence: we need contact forces to move the CoM in a direction **different from that of gravity**!

Newton-Euler equations

second Newton-Euler equation:
variation of **angular momentum** = **moment balance**

$$(\boldsymbol{c}-\boldsymbol{o}) \times M\ddot{\boldsymbol{c}} + \dot{\boldsymbol{L}} = M(\boldsymbol{c}-\boldsymbol{o}) \times \boldsymbol{g} + \sum_i (\boldsymbol{p}_i - \boldsymbol{o}) \times \boldsymbol{f}_i$$

moments are computed wrt to a generic point $\boldsymbol{o}$

$\boldsymbol{p}_i$: position of the contact point of force $\boldsymbol{f}_i$
$\boldsymbol{L}$: angular momentum of the robot wrt its CoM

Zero Moment Point

in the equation of moment balance

$$(\boldsymbol{c}-\boldsymbol{o}) \times M\ddot{\boldsymbol{c}} + \dot{\boldsymbol{L}} = M(\boldsymbol{c}-\boldsymbol{o}) \times \boldsymbol{g} + \sum_i (\boldsymbol{p}_i - \boldsymbol{o}) \times \boldsymbol{f}_i$$

choose the point $\boldsymbol{o}$ so that $\sum_i (\boldsymbol{p}_i - \boldsymbol{o}) \times \boldsymbol{f}_i$ is zero

$$M(\boldsymbol{c}-\boldsymbol{z}) \times (\ddot{\boldsymbol{c}} + \boldsymbol{g}) + \dot{\boldsymbol{L}} = 0$$

by $\boldsymbol{z}$ we denote the **Zero Moment Point (ZMP)**: the point wrt to which the **moment of the contact forces** is zero

or equivalently

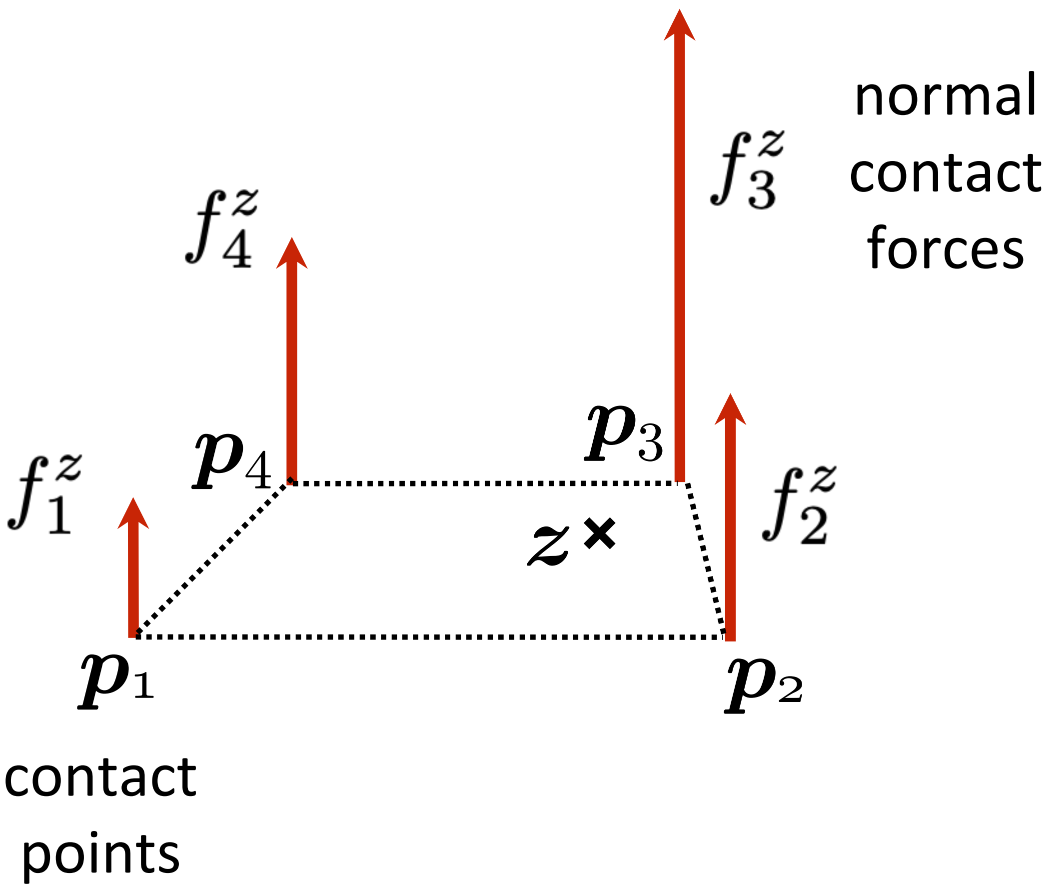
the **point of application** of the resultant **ground reaction force**

where is the ZMP?

consider a set of contact forces $\boldsymbol{f}_i$, each applied at a point $\boldsymbol{p}_i$

the ZMP can be found by summing the position vector of each contact point, weighted by the normal contact forces

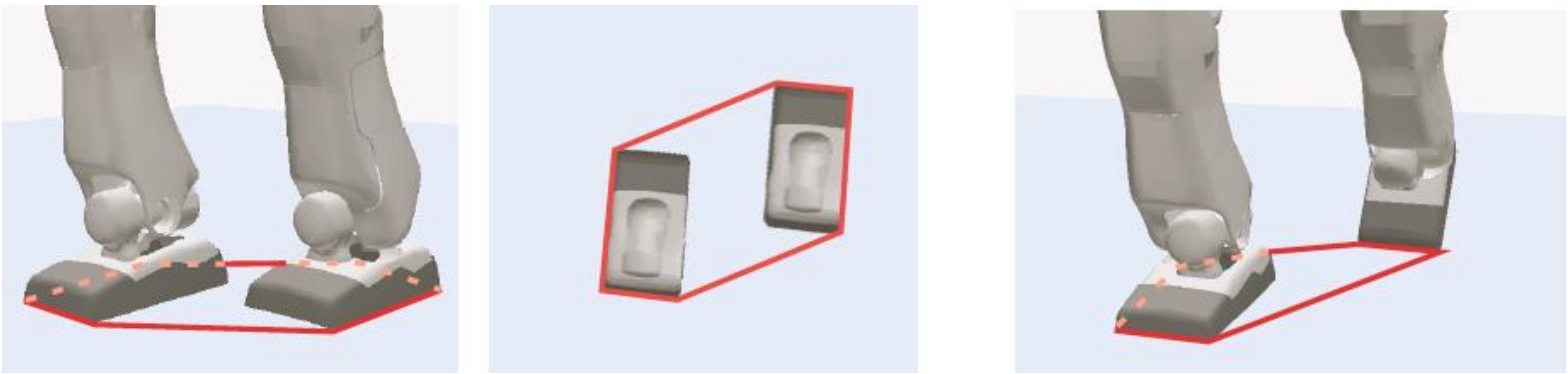
$$\boldsymbol{z} = \frac{\sum_i \boldsymbol{p}_i f_i^z}{\sum_i f_i^z}$$



where is the ZMP?

- the region where the ZMP can be is the region of all points that can be expressed as a **convex combination** of the contact points

- this region is called the **support polygon**: it is the **convex hull** of the contact surfaces between robot and ground



(recall that the convex hull of a set of points is the smallest convex region that contains all said points)

sufficient condition for balance

- the ZMP encodes information about the resultant of the **ground reaction forces**

unilaterality of contact forces

⇕

ZMP inside the support polygon

- this can be interpreted as a **non-tilting condition**, that will ensure that the foot is well planted on the ground

Newton-Euler equations

let’s go back to the Newton-Euler equations, and try to express the relation between CoM and ZMP

force balance

$$M\ddot{\boldsymbol{c}} = \sum_i \boldsymbol{f}_i - M\boldsymbol{g}$$

moment balance

$$M(\boldsymbol{c} - \boldsymbol{z}) \times (\ddot{\boldsymbol{c}} - \boldsymbol{g}) + \dot{\boldsymbol{L}} = 0$$

moment balance equation

in particular let's focus on the moment balance equation

$$M(\boldsymbol{c}-\boldsymbol{z}) \times (\ddot{\boldsymbol{c}}-\boldsymbol{g}) + \dot{\boldsymbol{L}} = 0$$

if we project this equation along the x and y axes, we obtain

$$M c^z \ddot{c}^x - M(c^x - z^x)(\ddot{c}^z + g^z) - \dot{L}_y = 0$$
$$M c^z \ddot{c}^y - M(c^y - z^y)(\ddot{c}^z + g^z) + \dot{L}_x = 0$$

where we have made a hypothesis of **flat horizontal ground**

nonlinear reduced-order model

both components can be written as a single equations

($\boldsymbol{S}$ selects the appropriate angular momentum derivative) $\boldsymbol{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\boldsymbol{c}^{x,y} - \frac{c^z}{\ddot{c}^z + g^z}\ddot{\boldsymbol{c}}^{x,y} + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)} = \boldsymbol{z}^{x,y}$$

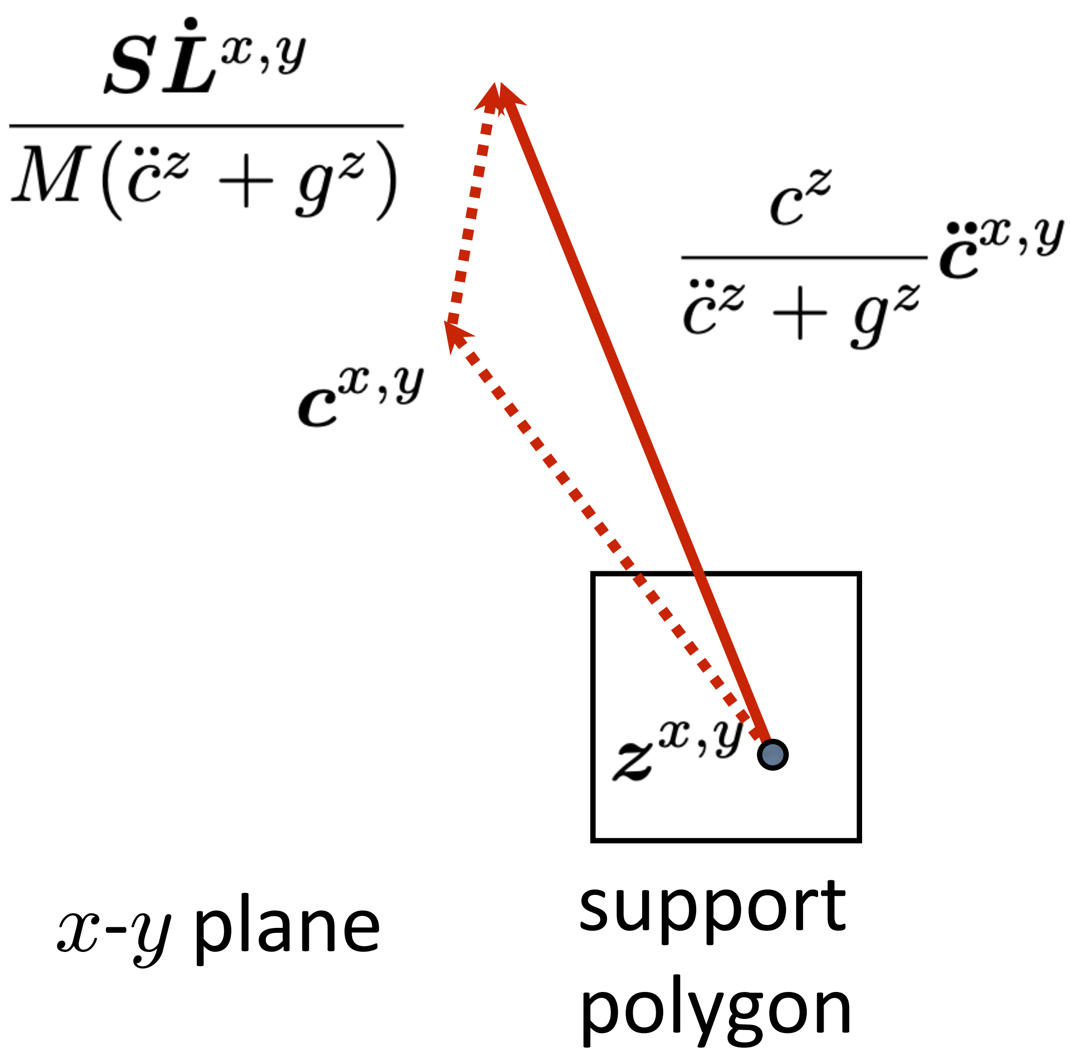
↓ rewritten as

$$\frac{c^z}{\ddot{c}^z + g^z}\ddot{\boldsymbol{c}}^{x,y} = \boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y} + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)}$$

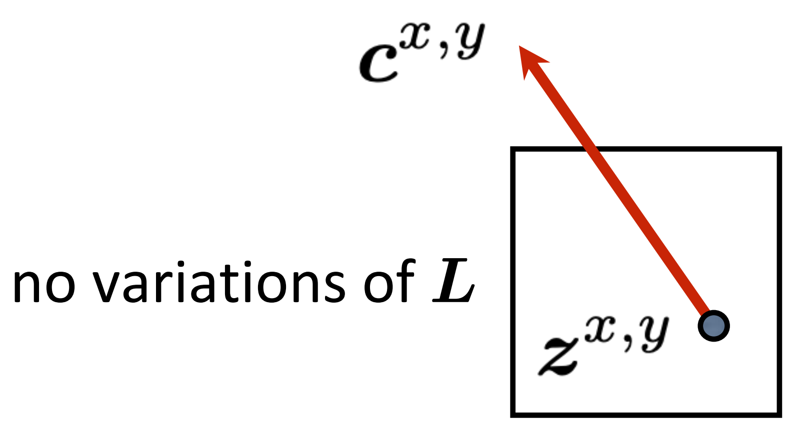
we can analyze the effect of the various terms on the CoM horizontal acceleration

nonlinear reduced-order model

$$\frac{c^z}{\ddot{c}^z + g^z}\ddot{\boldsymbol{c}}^{x,y} = \boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y} + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)}$$



aside from the effect of internal angular momentum variations, the CoM horizontal acceleration is the result of a **force pushing the CoM away** from the CoP



linear inverted pendulum

we already assumed flat horizontal ground, let us now add two more assumptions:

- **CoM height** is constant ⟶ $c^z = \text{constant}$
- derivative of **angular momentum** $\dot{\boldsymbol{L}}^{x,y}$ is negligible

$$\frac{c^z}{\ddot{c}^z + g^z}\ddot{\boldsymbol{c}}^{x,y} = \boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y} + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)}$$

$$\ddot{\boldsymbol{c}}^{x,y} = \frac{g^z}{c^z}\left(\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}\right)$$

Linear Inverted Pendulum (LIP)

linear inverted pendulum interpretation

2 independent equations

$$\ddot{\boldsymbol{c}}^{x,y} = \frac{g^z}{c^z}\left(\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}\right)$$

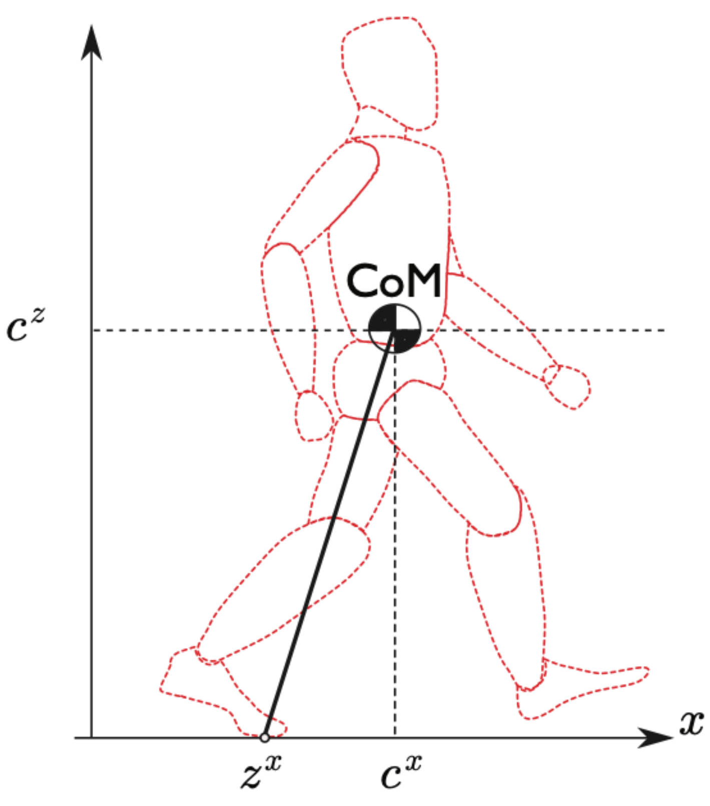
how the CoM moves in

$$\ddot{c}^x = \frac{g^z}{c^z}\left(c^x - z^x\right)$$

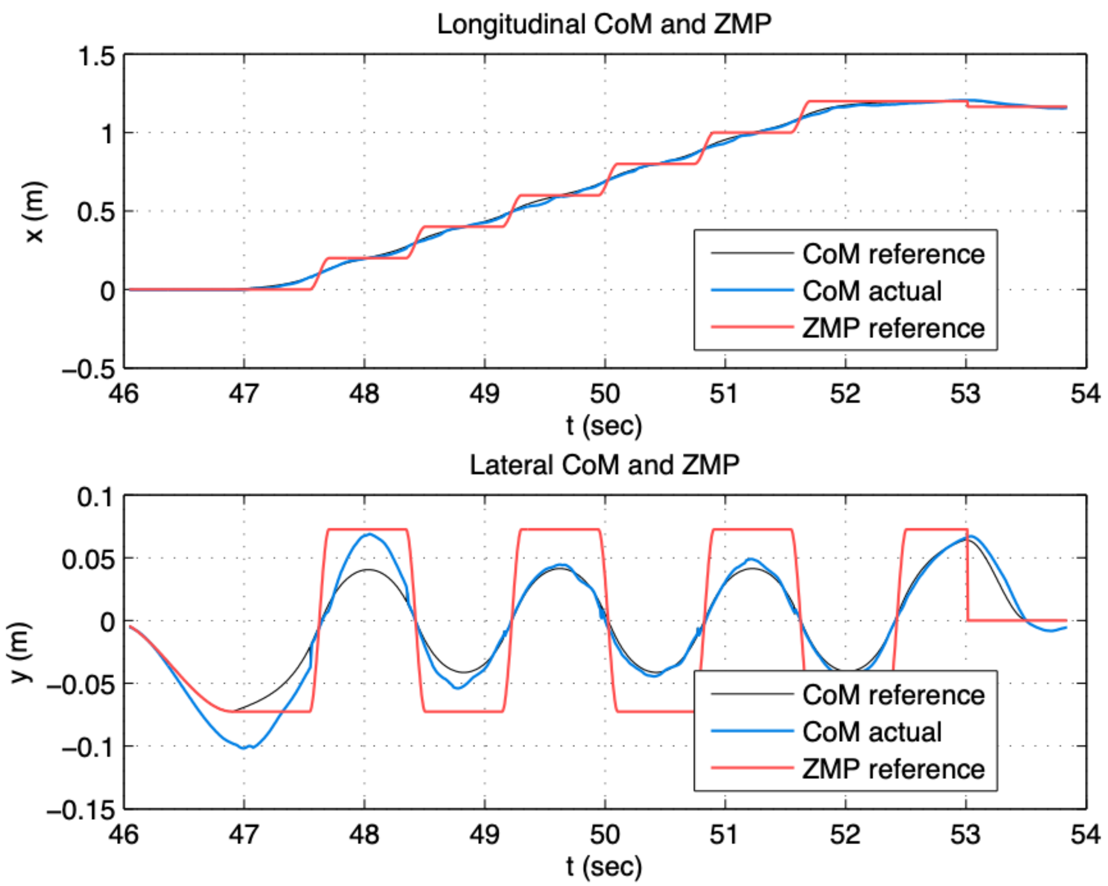
longitudinal direction (sagittal plane)

$$\ddot{c}^y = \frac{g^z}{c^z}\left(c^y - z^y\right)$$

lateral direction

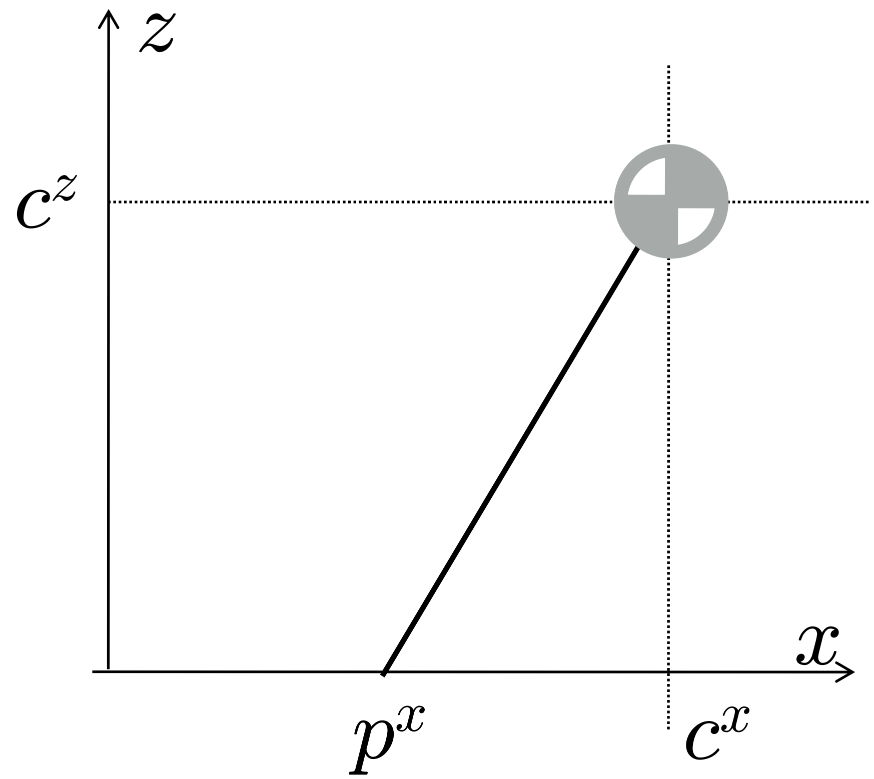


typical behaviors



linear inverted pendulum interpretation

- **point foot**
  the simplest interpretation of the LIP is that of a telescoping (so to remain at a constant height) massless leg in contact with the ground at p^x



we can interpret the LIP equation as a moment balance around p^x

$$M\ddot{c}^x c^z - Mg(c^x - p^x) = 0$$

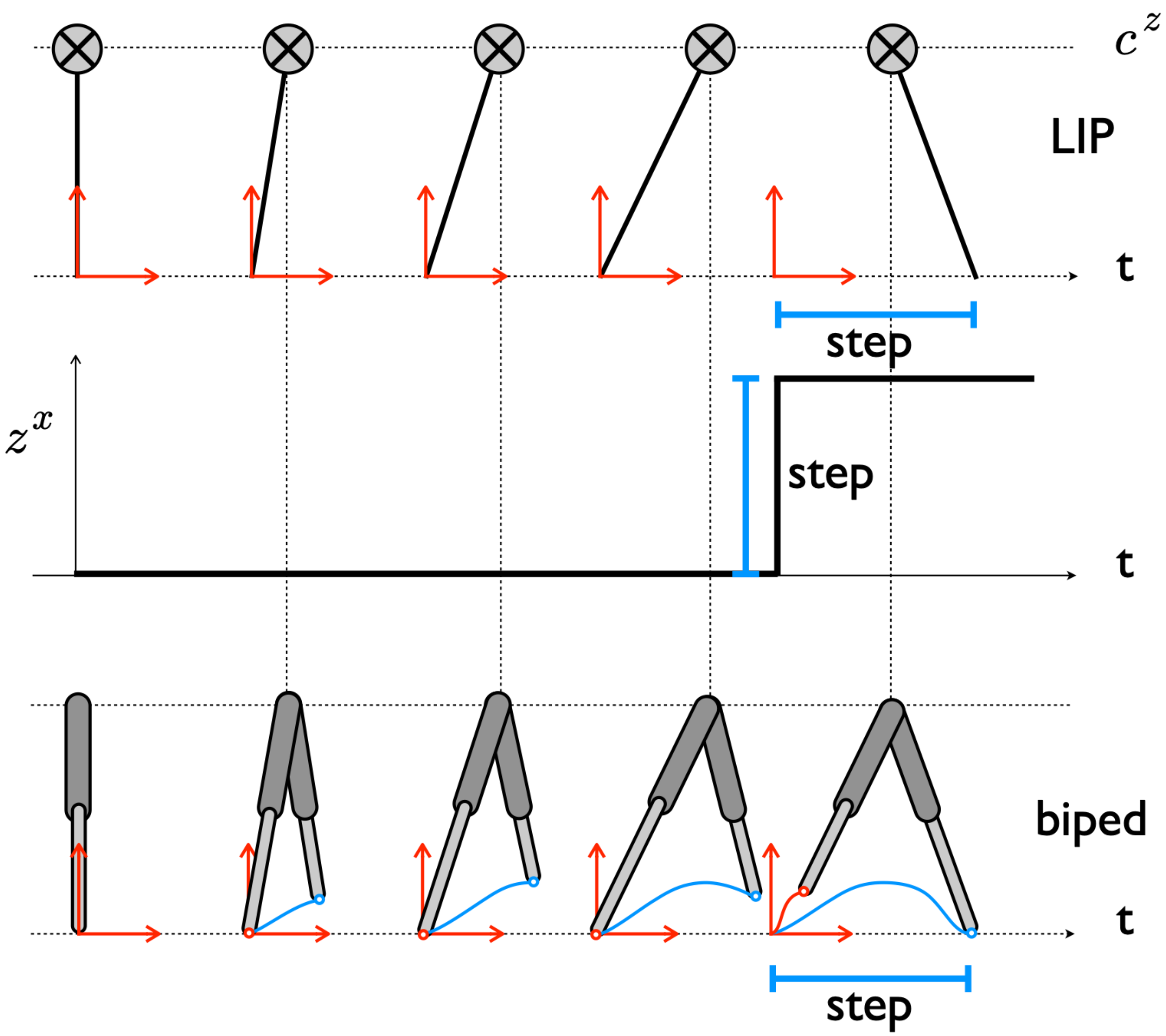
$$\ddot{c}^x = \frac{g^z}{c^z}(c^x - p^x)$$

in this case the ZMP z^x coincides with the point of contact p^x of the fictitious leg

$$p^x = z^x$$

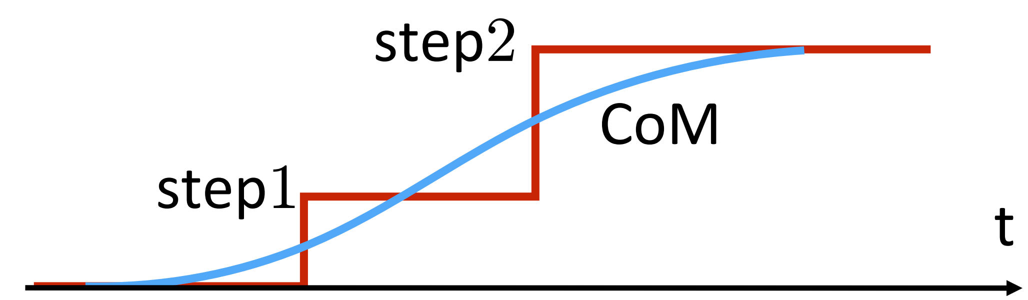
linear inverted pendulum interpretation

- point foot



typical footsteps and CoM

$p^x = z^x$ — c^x —

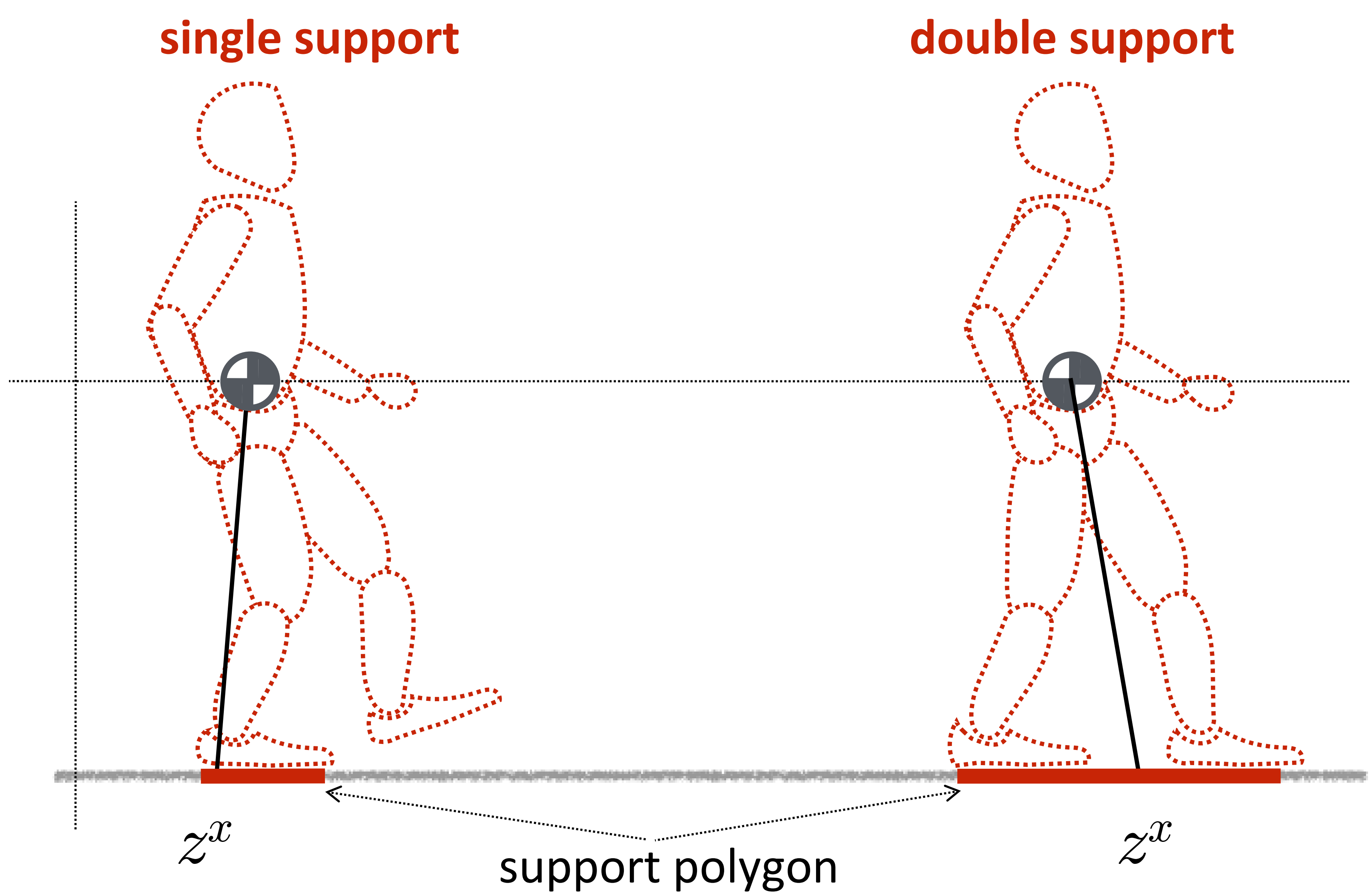


may also be seen as a compass biped with only one leg touching the ground at the same time

linear inverted pendulum interpretation

- **finite-sized foot**

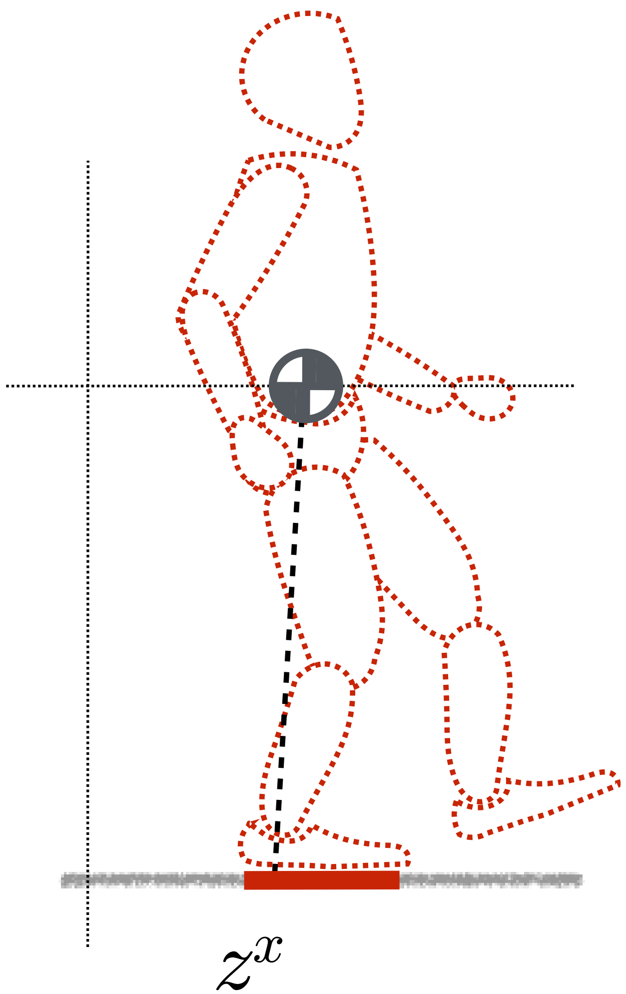
  since z^x represents the ZMP location, there is no difficulty in extending the interpretation of the LIP considering both single and double support phases with a finite foot dimension



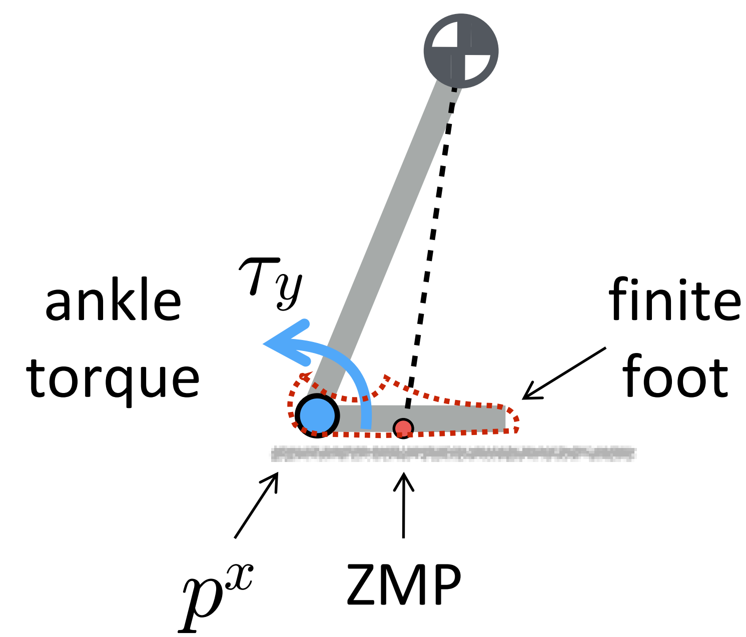
linear inverted pendulum interpretation

- **finite-sized foot**

  we can see the single support phase from the stance foot point of view i.e. with the dynamics of the rest of the humanoid represented by an equivalent fictitious leg. A way to keep the CoM balanced is using an equivalent ankle torque (the real joint torques are such that an equivalent ankle torque is applied)



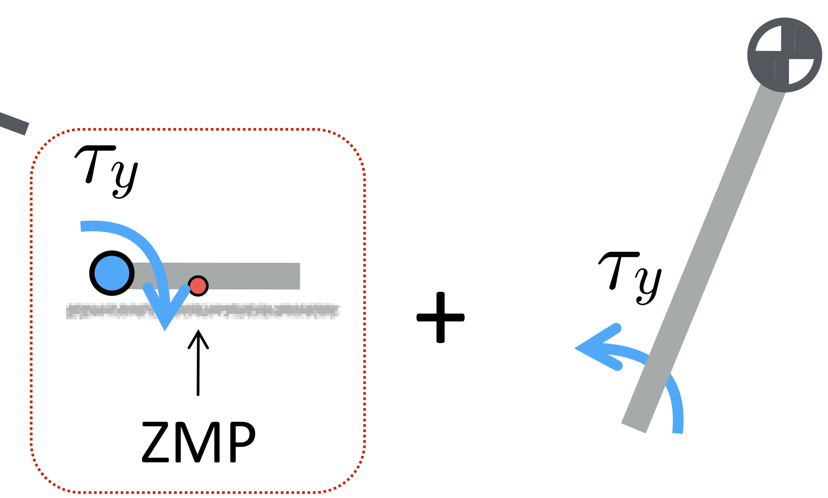
finite foot with equivalent massless leg plus ankle torque →



moment w.r.t. $z^x = 0$

$$Mg(z^x - p^x) - \tau_y = 0$$

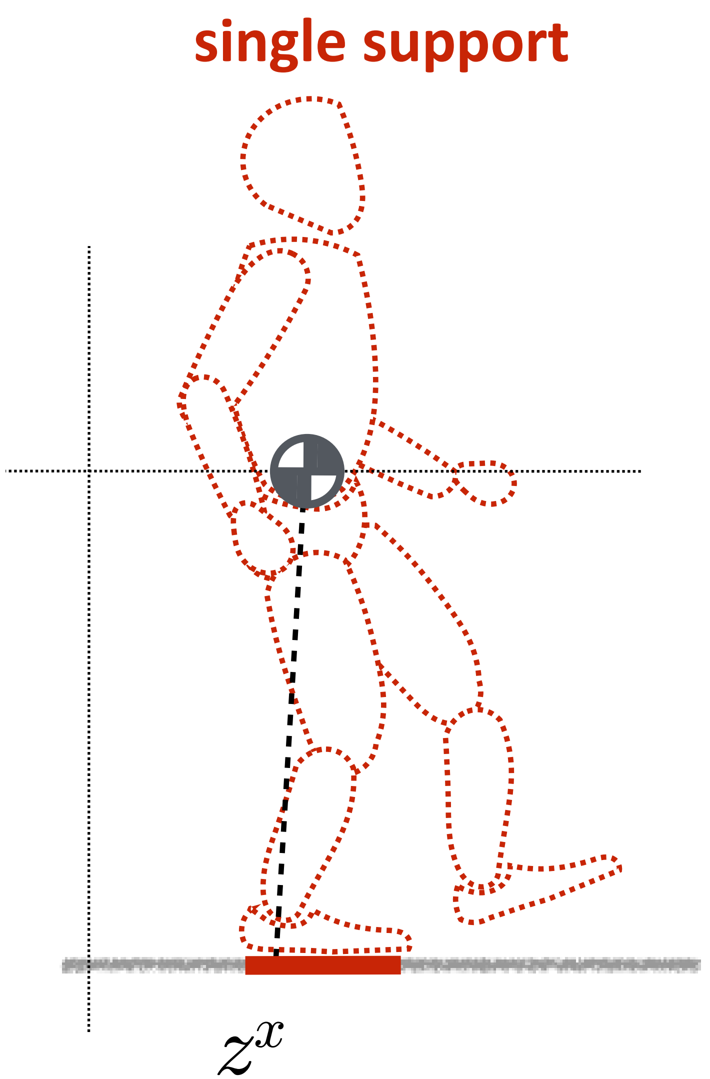
$$z^x = p^x + \frac{\tau_y}{Mg}$$



linear inverted pendulum interpretation

- **finite-sized foot**

  note: it is possible to move the ZMP through the ankle torque τ_y without stepping

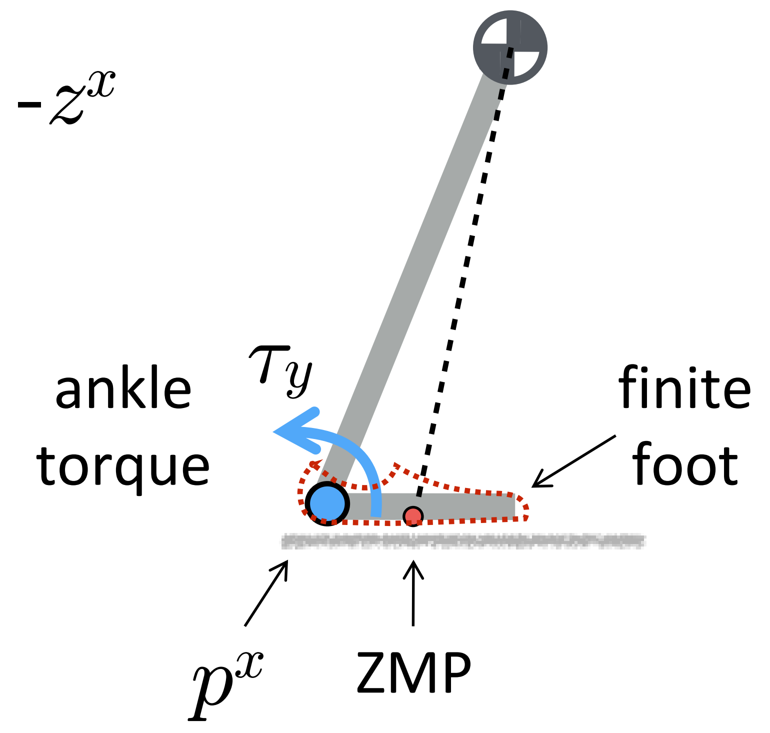


$$M\ddot{c}^x c^z - Mg(c^x - p^x) + \tau_y = 0$$

$$\ddot{c}^x = \frac{g^z}{c^z}\left(c^x - p^x - \frac{\tau_y}{Mg}\right)$$

– ZMP = – z^x

finite foot with equivalent massless leg plus ankle torque →



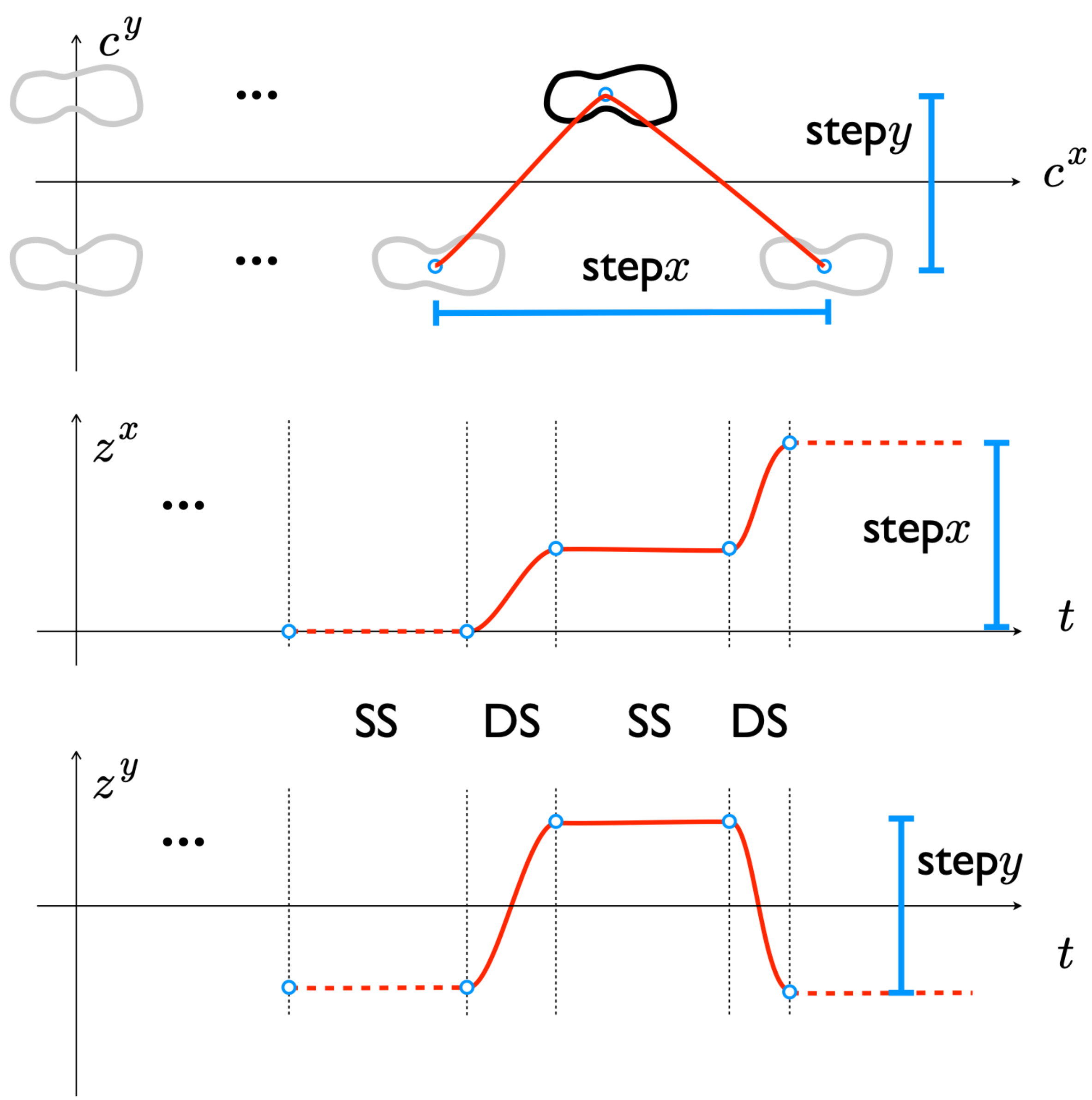
linear inverted pendulum interpretation

- **finite-sized foot**

with $z^x = p^x + \dfrac{\tau_y}{Mg}$

$$\ddot{c}^x = \frac{g^z}{c^z}(c^x - z^x)$$

longitudinal direction



typical footsteps with single and double support:
for example, in the first single support (- -) the left foot is swinging;
as soon as the right foot touches the ground the double support starts (—) and the ZMP moves from the left to the right foot
(longitudinal and lateral motions)

SS: single support
DS: double support

linear inverted pendulum: basic scope

$$\ddot{\boldsymbol{c}}^{x,y} = \frac{g^z}{c^z}\left(\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}\right)$$

- although extremely simplified, the LIP equation describes in first approximation the time evolution of the CoM trajectory
- it defines a **differential relationship** between the CoM trajectory and the ZMP time evolution
- a suitable ZMP trajectory can be chosen such to satisfy **dynamic balance** by avoiding tilting
- the associated CoM trajectory can then be **tracked** with a complete robot model