Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2020/2021

Solution of Problem 1

(a) FALSE. Such constraint is always holonomic, because it can be integrated as $\boldsymbol{A}^T\boldsymbol{q} = \boldsymbol{c}$.

(b) TRUE. With $n = 2$ generalized coordinates and $k = 1$ constraint, the kinematic model contains a single ($n - k = 1$) input vector field $\boldsymbol{g}$. Since $[\boldsymbol{g}, \boldsymbol{g}] = \boldsymbol{0}$ always, the dimension of the accessibility distribution remains 1 (one also says that the distribution is *involutive*). This means that the robot is not controllable and the constraint can be always integrated.

(c) FALSE. It is $(SO(2))^3$.

(d) TRUE. A unicycle can follow any geometric path in x, y by simply stopping and reorienting at angular points (the reorientation has no visible effect on the Cartesian path). This is impossible for a car-like robot, that cannot change its orientation on the spot.

(e) FALSE. Thanks to the redundancy, a given end-effector trajectory can be realized by an infinity of joint trajectories. State reconstructability is then impossible.

Solution of Problem 2

A unicycle can be put in $(2,3)$ chained form

$$\begin{aligned}\dot{z}_1 &= v_1\\ \dot{z}_2 &= v_2\\ z_3 &= z_2 v_1\end{aligned}$$

using the following coordinate transformation

$$\begin{aligned}z_1 &= \theta\\ z_2 &= x\cos\theta + y\sin\theta\\ z_3 &= x\sin\theta - y\cos\theta\end{aligned}$$

and input transformation

$$\begin{aligned}v_1 &= \omega\\ v_2 &= v - z_3\,\omega\end{aligned}$$

Using the chained form transformation to design the controller requires to express the control problem in the chained form coordinates z_1, z_2, z_3; then find a control law in the chained form inputs v_1, v_2; and finally map it back to the unicycle coordinates x, y, θ and inputs v, ω.

(1) The requirement of reaching the x axis with orientation parallel to the y axis means that the set-point is $y_d = 0$, $\theta_d = \pi/2$ (the final value of x is irrelevant). Plugging these values in the coordinate transformation, we deduce that the set-point for the chained form is $z_{1d} = \pi/2$, $z_{2d} = 0$ (the final value of z_3 is irrelevant). Since the dynamics of both z_1 and z_2 is represented by simple integrators, it is sufficient to set

$$\begin{aligned}v_1 &= k_1(\pi/2 - z_1)\\ v_2 &= -k_2 z_2\end{aligned}$$

with $k_1, k_2 > 0$, to drive both variables exponentially to the desired set-point. This control law can be written in terms of the original inputs and coordinates as

$$\begin{aligned}v &= v_2 + z_3 v_1 = -k_2(x\cos\theta + y\sin\theta) + k_1(x\sin\theta - y\cos\theta)(\pi/2 - \theta)\\ \omega &= v_1 = k_1(\pi/2 - \theta)\end{aligned}$$

(2) The requirement of reaching the y axis with orientation parallel to the x axis means that the set-point is $x_d = 0$, $\theta_d = 0$ (the final value of y is irrelevant). Plugging these values in the coordinate transformation, we deduce that the new set-point is $z_{1d} = 0$, $z_{2d} = 0$ (the final value of z_3 is irrelevant). The controller is then exactly the same as above with 0 in place of any occurrence of $\pi/2$.

Solution of Problem 3

The kinematic model of the mobile manipulator is readily obtained by augmenting the model of a differential-drive platform (generalized coordinates x, y, θ) with two integrators representing the velocity-controlled 2R arm (relative joint coordinates q_1, q_2):

$$\begin{aligned}
\dot{x} &= v\cos\theta \\
\dot{y} &= v\sin\theta \\
\dot{\theta} &= \omega \\
\dot{q}_1 &= u_1 \\
\dot{q}_2 &= u_2
\end{aligned}$$

where v, ω are a well-known function of the wheel angular velocities ω_L, ω_R, and u_1, u_2 are the joint velocities.

Using Euler integration, a discrete-time motion model is written as

$$\begin{aligned}
x_{k+1} &= x_k + T_s\, v_k \cos\theta_k \\
y_{k+1} &= y_k + T_s\, v_k \sin\theta_k \\
\theta_{k+1} &= \theta_k + T_s\, \omega_k \\
q_{1,k+1} &= q_{1,k} + T_s\, u_{1,k} \\
q_{2,k+1} &= q_{2,k} + T_s\, u_{2,k}
\end{aligned}$$

where T_s is the sampling interval.

The current wheel encoder measurements $\Delta\phi_L$ and $\Delta\phi_R$ are used to reconstruct the actual values of v_k and ω_k as follows. First, we compute the traveled length and orientation change

$$\Delta s = \frac{r}{2}(\Delta\phi_L + \Delta\phi_R) \qquad \Delta\theta = \frac{r}{d}(\Delta\phi_R - \Delta\phi_L)$$

where r is the wheel radius and d is the distance between the two wheels. Then, we set

$$v_k = \frac{\Delta s}{T_s} \qquad \omega_k = \frac{\Delta\theta}{T_s}$$

As for the actual values of $u_{1,k}$ and $u_{2,k}$, they are directly provided by the joint tachometers.

For the measurement model, the vision system returns the position of the end-effector, which is expressed as a function of the system state variables as follows:

$$\boldsymbol{h}_k = \begin{pmatrix} x_k + b\cos\theta_k + \ell_1\cos(\theta_k + q_{1,k}) + \ell_2\cos(\theta_k + q_{1,k} + q_{2,k}) \\ y_k + b\sin\theta_k + \ell_1\sin(\theta_k + q_{1,k}) + \ell_2\sin(\theta_k + q_{1,k} + q_{2,k}) \end{pmatrix}$$

The rest of the solution is straightforward: linearize the motion and measurement models and then write the EKF equations.

In the block scheme, the wheel encoders and joint tachometers will be used in the prediction stage of the filter, while only the vision system will be used for the correction stage.