Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2016/2017

Solution of Problem 1

1. The augmented configuration vector is $\boldsymbol{q} = (x \;\; y \;\; \theta \;\; \phi)^T$, with the usual meaning for x, y, θ. The configuration space is $I\!R^2 \times SO(2) \times SO(2)$ and has dimension $n = 4$.

2. Since the wheel rotation ϕ is a generalized coordinate, the *driving* angular velocity $\omega_\phi = \dot{\phi}$ can be directly taken as the first velocity input in place of v. The latter will then be obtained as $v = R\,\omega_\phi$, where R is the wheel radius. As usual, the second velocity input will be the *steering* angular velocity $\omega_\theta = \dot{\theta}$. This leads to the following kinematic model:

$$
\begin{aligned}
\dot{x} &= R\,\omega_\phi \cos\theta \\
\dot{y} &= R\,\omega_\phi \sin\theta \\
\dot{\theta} &= \omega_\theta \\
\dot{\phi} &= \omega_\phi
\end{aligned}
\qquad \text{i.e.,} \quad \dot{\boldsymbol{q}} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} \omega_\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \omega_\theta = \boldsymbol{g}_1(\boldsymbol{q})\,\omega_\phi + \boldsymbol{g}_2(\boldsymbol{q})\,\omega_\theta
$$

   The same model could have been obtained by the classical approach, i.e., writing down the kinematic constraints and solving for the generalized velocities $\dot{\boldsymbol{q}}$. In this case, there are two such constraints: one is pure rolling, and the other is the relationship $v = R\,\omega_\phi$, which can be indeed rewritten as a kinematic constraint as follows

$$
\sqrt{\dot{x}^2 + \dot{y}^2} - R\,\dot{\phi} = 0 \qquad (*)
$$

   However, this constraint is not linear in $\dot{\boldsymbol{q}}$; therefore, the usual procedure (find a basis for the null space of the constraint matrix) cannot be applied. One should first solve the pure rolling constraint (which is linear) and then use $(*)$ in the partial solution. This is equivalent to the direct augmentation procedure illustrated above.

   As an alternative, one may note that $v = v\,(\cos^2\theta + \sin^2\theta) = \dot{x}\cos\theta + \dot{y}\sin\theta$, so that $v = R\,\omega_\phi$ can be rewritten as

$$
\dot{x}\cos\theta + \dot{y}\sin\theta - R\,\dot{\phi} = 0 \qquad (**)
$$

   which is linear in $\dot{\boldsymbol{q}}$. Correspondingly, the constraint matrix accounting for pure rolling and $(**)$ becomes

$$
\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -R \end{pmatrix}
$$

   whose null space is spanned by the above vector fields $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$.

3. To study controllability, we use the accessibility rank condition. One easily obtains

$$
[\boldsymbol{g}_1, \boldsymbol{g}_2] = \begin{pmatrix} R\sin\theta \\ -R\cos\theta \\ 0 \\ 0 \end{pmatrix} \triangleq \boldsymbol{g}_3 \qquad [\boldsymbol{g}_1, \boldsymbol{g}_3] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad [\boldsymbol{g}_2, \boldsymbol{g}_3] = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 0 \end{pmatrix} \triangleq \boldsymbol{g}_4
$$

Controllability is then proven by noting that

$$\text{rank}\left(\boldsymbol{g}_1\,\boldsymbol{g}_2\,\boldsymbol{g}_3\,\boldsymbol{g}_4\right) = \text{rank}\left(\boldsymbol{g}_3\,\boldsymbol{g}_4\,\boldsymbol{g}_1\,\boldsymbol{g}_2\right) = \text{rank}\begin{pmatrix} R\sin\theta & R\cos\theta & R\cos\theta & 0 \\ -R\cos\theta & R\sin\theta & R\sin\theta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 4$$

because the determinant of the last matrix is $-R^2$ (the determinant of a block triangular matrix is the product of the determinants of the block on the diagonal).

4. To move the robot from $\boldsymbol{q}_s = (x_s, y_s, \theta_s, \phi_s)$ to $\boldsymbol{q}_g = (x_g, y_g, \theta_g, \phi_g)$, first take care of x, y, θ as follows: (1) rotate the wheel around the vertical axis until sagittal axis goes through (x_g, y_g); (2) drive the contact point in a straight line from (x_s, y_s) to (x_g, y_g); and (3) rotate the wheel around the vertical axis so as to achieve the desired orientation θ_g. The final step is to change the wheel angle from ϕ_3 (its value at the end of step 3) to ϕ_g. This may be obtained by (4) moving the contact point along a circle, so that x, y, θ will go back to x_g, y_g, θ_g. In particular, the wheel angle variation along the full circle must match $\Delta_\phi = \phi_g - \phi_3$; denoting by r the circle radius, this requires $2\,\pi\,r = R\,\Delta_\phi$, from which we get $r = R\,\Delta_\phi/2\pi$.

Solution of Problem 2

The (2,3) chained form is

$$\begin{aligned} z_1' &= \tilde{v}_1 \\ z_2' &= \tilde{v}_2 \\ z_3' &= z_2\,\tilde{v}_1 \end{aligned}$$

Note the use of the geometric version of the kinematic model (derivatives w.r.t. s and geometric inputs) because we are assigned a path planning problem. The problem may be solved by using either flat outputs or parameterized inputs; for a (2,3) chained form, the resulting path will be the same (see "Robotics: Modelling, Planning and Control", Problem 11.12). Let us choose the first route because it provides the required configuration space path without integration.

The flat outputs are z_1, z_3. Using the third and the first equation, the remaining state variable can be reconstructed as $z_2 = z_3'/z_1'$. We must therefore choose $z_1(s)$ and $z_3(s)$, $s \in [s_i, s_f]$, so as to satisfy their endpoint conditions

$$z_1(s_i) = z_3(s_i) = 0 \qquad z_1(s_f) = z_3(s_f) = 1$$

as well as the boundary conditions on z_2

$$\frac{z_3'(s_i)}{z_1'(s_i)} = 0 \qquad \frac{z_3'(s_f)}{z_1'(s_f)} = 1$$

Several choices are possible. For example, letting $s \in [0,1]$, we can use a 1st-order polynomial for z_1 and a 3rd-order polynomial for z_3

$$\begin{aligned} z_1(s) &= a_1 s + b_1 \\ z_3(s) &= a_2 s^3 + b_2 s^2 + c_2 s + d_2 \end{aligned}$$

whose derivatives w.r.t. to s are

$$\begin{aligned} z_1'(s) &= a_1 \\ z_3'(s) &= 3\,a_2 s^2 + 2\,b_2 s + c_2 \end{aligned}$$

By imposing the previous conditions, one finds

$$\begin{aligned} z_1(s) &= s \\ z_3(s) &= -s^3 + 2s^2 \end{aligned}$$

and correspondingly

$$z_2(s) = \frac{z_3'(s)}{z_1'(s)} = -3s^2 + 4s$$

Solution of Problem 3

1. The output variable is x. According to the unicycle equations, we have directly

$$\dot{x} = v \cos\theta$$

This simple, scalar input-output map can be linearized by using the input transformation $v = u/\cos\theta$, where u is the new input. This leads to

$$\dot{x} = u$$

and therefore it is sufficient to set $u = k_x(x_d - x)$ (i.e., $v = k_x(x_d - x)/\cos\theta$) to drive exponentially the output to x_d, provided that $k_x > 0$.

So far, the steering velocity ω is free. We can use it to keep the robot parallel to the corridor, so that it will not collide with the lateral walls. Since we have

$$\dot{\theta} = \omega$$

the control law $\omega = -k_\theta\, \theta$, with $k_\theta > 0$, will drive the robot orientation θ exponentially to zero. Assuming[1] that $|\theta(0)| < \pi/2$, this will also guarantee that $|\theta| < \pi/2$ at all times, so that the input transformation for v is never affected by the potential singularity.

2. To implement the previous controller, we need real-time estimates of x and θ, but not of y. Also, the two sensor measurements (θ and d) do not depend on y. Therefore, we can restrict our attention to first and third equations of the unicycle, and build a two-dimensional filter.

Using, e.g., Euler integration, the noise-free discrete-time model (process dynamics) is easily obtained as

$$\begin{aligned} x_{k+1} &= x_k + v_k\, T_s \cos\theta_k \\ \theta_{k+1} &= \theta_k + \omega_k\, T_s \end{aligned}$$

where T_s is the sampling interval. As usual, the discrete-time velocity inputs v_k and ω_k can be reconstructed from wheel encoder readings (see, e.g., the formulas in the AMR slides on 'Odometric Localization').

The noise-free output equation (measurement model) is

$$h_k = \begin{pmatrix} \theta_k \\ \ell - x_k \end{pmatrix}$$

where we have used the fact that $d = \ell - x$ thanks to the placement of the world frame.

The rest of the solution is straightforward: linearize the process dynamics (note that the measurement model is already linear) and then derive the EKF equations.

[1] This condition can always be achieved by a preliminary rotation on the spot.