Autonomous and Mobile Robotics
Solution of Final Class Test, 2013/2014

Solution of Problem 1

Substituting the first and second model equations into the third, one immediately obtains the underlying differential constraint

$$x_2\dot{x}_1 - x_1\dot{x}_2 + \dot{x}_3 = 0 \qquad \text{or} \qquad \begin{pmatrix} x_2 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = 0$$

As for system controllability, calling $\boldsymbol{g}_1 = (1 \;\; 0 \;\; -x_2)^T$ and $\boldsymbol{g}_2 = (0 \;\; 1 \;\; x_1)^T$ the two input vector fields, their Lie Bracket is easily obtained as

$$[\boldsymbol{g}_1, \boldsymbol{g}_2] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

Since $[\boldsymbol{g}_1, \boldsymbol{g}_2]$ is always linearly independent on $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, the accessibility rank condition is satisfied and the system is controllable. This also means that the above differential constraint is nonholonomic.

Coming to the last question, a Lie Bracket control maneuver consists in the following control sequence

$$u(t) = \begin{cases} u_1(t) = +1, u_2(t) = 0 & t \in [0, \epsilon) \\ u_1(t) = 0, u_2(t) = +1 & t \in [\epsilon, 2\epsilon) \\ u_1(t) = -1, u_2(t) = 0 & t \in [2\epsilon, 3\epsilon) \\ u_1(t) = 0, u_2(t) = -1 & t \in [3\epsilon, 4\epsilon), \end{cases}$$

where ϵ is a small time interval. Denoting by $(x_{10} \;\; x_{20} \;\; x_{30})^T$ the initial system state, integration of the system equations over the first time interval $[0, \epsilon]$ easily gives

$$\begin{aligned} x_1(\epsilon) &= x_{10} + \epsilon \\ x_2(\epsilon) &= x_{20} \\ x_3(\epsilon) &= x_{30} - \epsilon\, x_{20}. \end{aligned}$$

Similarly, integrating over the second time interval $[\epsilon, 2\epsilon]$ we get

$$\begin{aligned} x_1(2\epsilon) &= x_{10} + \epsilon \\ x_2(2\epsilon) &= x_{20} + \epsilon \\ x_3(2\epsilon) &= x_{30} - \epsilon\, x_{20} + (x_{10} + \epsilon)\epsilon. \end{aligned}$$

After similar computations for the third and fourth time intervals, one obtains

$$\begin{aligned} x_1(4\epsilon) &= x_{10} \\ x_2(4\epsilon) &= x_{20} \\ x_3(4\epsilon) &= x_{30} + 2\,\epsilon^2. \end{aligned}$$

so that the final state displacement is *exactly equal* to $\epsilon^2\,[\boldsymbol{g}_1, \boldsymbol{g}_2]$. Note that this result is stronger than the theoretical result provided by Taylor expansion for general drifless systems: in fact, it shows that for the considered system the $O(\epsilon^3)$ terms are identically zero.

Solution of Problem 2

- $\mathcal{C} = SE(2) \times SE(2)$, $\dim \mathcal{C} = 6$.
- $\mathcal{C}$ is a subset of $SE(2) \times SE(2)$ (for each position of the first robot, the position of the second robot must be within a circle centered at the first robot), $\dim \mathcal{C} = 6$ (the constraint entailed by the rope is an inequality constraint and therefore it does not decrease the dimension of $\mathcal{C}$).
- $\mathcal{C} = SE(2) \times SO(2) \times SO(2)$ (one possible choice of configuration is: position and orientation of the first robot, orientation of the bar, orientation of the second robot), $\dim \mathcal{C} = 5$.
- $\mathcal{C} = SE(3) \times (SO(2))^6$ (need 3D position and orientation for the spacecraft, plus six joint angles for the robot), $\dim \mathcal{C} = 12$.
- $\mathcal{C} = SE(3)$ (the last link of the robot is a rigid body which can be arbitrarily positioned and oriented in space by moving the spacecraft and/or the robot arm), $\dim \mathcal{C} = 6$.

Solution of Problem 3

Since the position of the beacons is unknown, this is a SLAM problem. Define the extended state vector to be estimated as $\boldsymbol{\chi} = (x \;\; y \;\; \psi \;\; \phi \;\; x_1 \;\; y_1 \;\; x_2 \;\; y_2)^T$, where (x_i, y_i), for $i = 1, 2$, are the cartesian coordinates of the two beacons (the z coordinate of the beacon is known to be zero). The nonlinear discrete-time model describing the motion of the extended UAV+beacons system is then

$$
\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \begin{pmatrix} v_k \cos\psi_k \, T_s \\ v_k \sin\psi_k \, T_s \\ -g/v_k \tan\phi_k \, T_s \\ u_{\phi,k} \, T_s \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ v_{4,k} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

where $\boldsymbol{v}_k = (v_{1,k} \;\; v_{2,k} \;\; v_{3,k} \;\; v_{4,k})^T$ is a white gaussian noise with zero mean and covariance V_k. Note how the beacons being fixed reflects on the last four rows of the above equation.

As for the measurement model, we have two distance readings coming from the robot sensor. An elementary geometric construction provides

$$
\boldsymbol{h}_k = \begin{pmatrix} \sqrt{(x_k - x_{1,k})^2 + (y_k - y_{1,k})^2 + \bar{z}^2} \\ \sqrt{(x_k - x_{2,k})^2 + (y_k - y_{2,k})^2 + \bar{z}^2} \end{pmatrix} + \begin{pmatrix} w_{1,k} \\ w_{2,k} \end{pmatrix}
$$

where $\boldsymbol{w}_k = (w_{1,k} \;\; w_{2,k})^T$ is a white gaussian noise with zero mean and covariance W_k. Note that the sensor-beacon distance depends also on $\bar{z}$, since the beacons are on the ground.

The rest of the problem is trivial: linearize the process and measurement models and then derive the EKF equations.