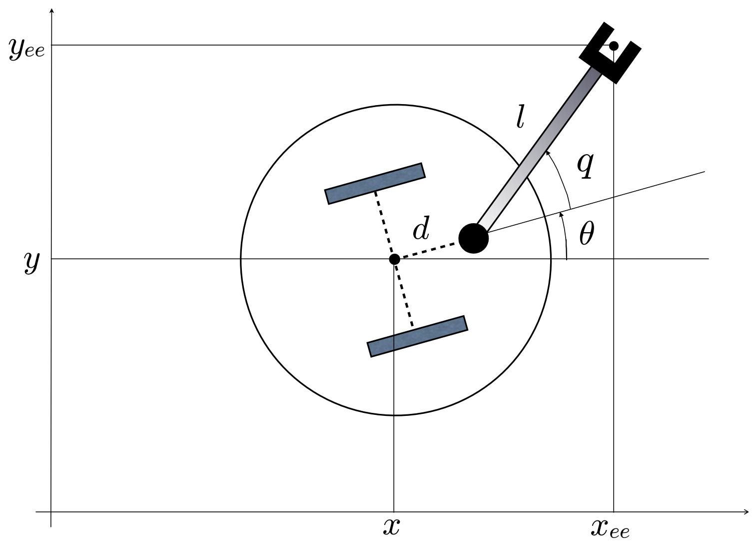
Autonomous and Mobile Robotics
Solution of Final Class Test, 2012/2013

Solution of Problem 1



Refer to the figure for the definition of the relevant variables. The configuration vector of the mobile manipulator is $\boldsymbol{q} = (x \;\; y \;\; \theta \;\; q)^T$. The corresponding kinematic model is

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ u \end{pmatrix}$$

with obvious meaning for v, ω, u. The coordinates of the end-effector are

$$\begin{aligned} x_{ee} &= x + d\cos\theta + l\cos(\theta+q) \\ y_{ee} &= y + d\sin\theta + l\sin(\theta+q) \end{aligned}$$

To design a trajectory tracking controller for this output, differentiate it w.r.t. time and use the kinematic model equation:

$$\begin{aligned} \dot{x}_{ee} &= v\cos\theta - d\,\omega\sin\theta - l\sin(\theta+q)(\omega+u) \\ \dot{y}_{ee} &= v\sin\theta + d\,\omega\cos\theta + l\cos(\theta+q)(\omega+u) \end{aligned}$$

which may be conveniently rewritten as

$$\begin{pmatrix} \dot{x}_{ee} \\ \dot{y}_{ee} \end{pmatrix} = \begin{pmatrix} \cos\theta & -d\,\sin\theta - l\sin(\theta+q) & -l\sin(\theta+q) \\ \sin\theta & d\,\cos\theta + l\cos(\theta+q) & l\cos(\theta+q) \end{pmatrix} \begin{pmatrix} v \\ \omega \\ u \end{pmatrix} = \boldsymbol{T}(\theta,q) \begin{pmatrix} v \\ \omega \\ u \end{pmatrix} \tag{1}$$

Matrix $\boldsymbol{T}(\theta,q)$ has always full row rank. In fact, its rank does not change if we replace the second column with the difference of the second and the third column, obtaining

$$\boldsymbol{T}'(\theta,q) = \boldsymbol{T}(\theta,q) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -d\,\sin\theta & -l\sin(\theta+q) \\ \sin\theta & d\,\cos\theta & l\cos(\theta+q) \end{pmatrix}$$

and this matrix has always rank 2 if $d \neq 0$ (compute the minor corresponding to the first two columns). We may therefore define the following input transformation

$$\begin{pmatrix} v \\ \omega \\ u \end{pmatrix} = \boldsymbol{T}^{\dagger}(\theta, q) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{2}$$

where $\boldsymbol{T}^{\dagger}(\theta, q) = \boldsymbol{T}^T(\boldsymbol{T}\boldsymbol{T}^T)^{-1}$ is the pseudoinverse of $\boldsymbol{T}(\theta, q)$. Using this in (1) we get

$$\begin{pmatrix} \dot{x}_{ee} \\ \dot{y}_{ee} \end{pmatrix} = \boldsymbol{T}(\theta, q)\boldsymbol{T}^{\dagger}(\theta, q) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The input-output channels (from the new inputs w_1, w_2 to the outputs x_{ee}, y_{ee}, respectively) are now simple integrators. We can then guarantee global exponential convergence of the output to the desired trajectory by letting

$$\begin{aligned} w_1 &= \dot{x}^*_{ee} + k_1(x^*_{ee} - x_{ee}) \\ w_2 &= \dot{y}^*_{ee} + k_2(y^*_{ee} - y_{ee}) \end{aligned}$$

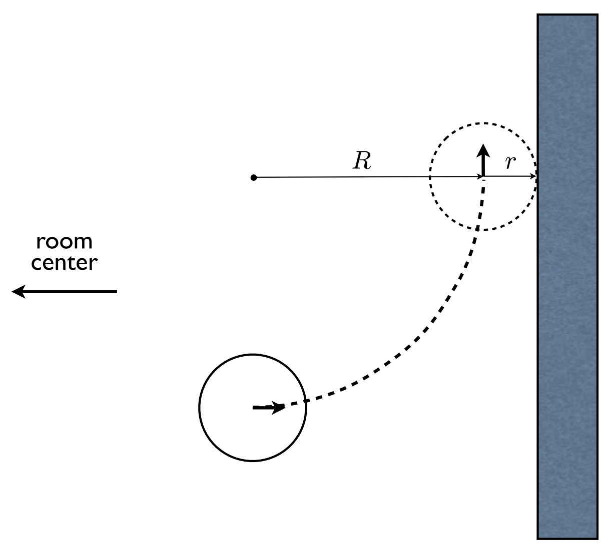
with k_1, k_2 positive gains. The actual control law to be implemented is readily obtained by substituting the latter expressions for w_1, w_2 in (2).

Finally, equation (1) shows that the end-effector position is *not* a flat output. In fact, once $\dot{x}^*_{ee}$, $\dot{y}^*_{ee}$ are specified by the assigned trajectory, there exist an infinity of velocity inputs that realize them:

$$\begin{pmatrix} v^* \\ \omega^* \\ u^* \end{pmatrix} = \boldsymbol{T}^{\dagger}(\theta, q) \begin{pmatrix} \dot{x}^*_{ee} \\ \dot{y}^*_{ee} \end{pmatrix} + (\boldsymbol{I} - \boldsymbol{T}^{\dagger}(\theta, q)\boldsymbol{T}(\theta, q))\boldsymbol{z}$$

where $\boldsymbol{z}$ is an arbitrary 3-vector.

Solution of Problem 2



Consider the initial arrangement shown in figure. Clearly, the only way for the unicycle to avoid collision with the wall is to turn left (or right) as much as possible, by choosing $\omega = \bar{\omega}$ ($\omega = -\bar{\omega}$) for as many consecutive intervals as needed. This obviously corresponds to the robot moving along an arc of circle of radius $R = \bar{v}/\bar{\omega}$. The desired clearance is therefore $r + R = r + \bar{v}/\bar{\omega}$.

To reduce the clearance, one may either decrease $\bar{v}$ or increase $\bar{\omega}$. To eliminate it, extend the primitive set by including $v = 0$ (this will allow rotation on the spot) or $v = -\bar{v}$ (this will allow backward motions).

Solution of Problem 3

Define the extended state vector to be estimated as $\boldsymbol{\chi} = (x_1\ y_1\ x_2\ y_2\ x_l\ y_l)^T$, where (x_i, y_i) are the cartesian coordinates of robot i $(i = 1, 2)$ and (x_l, y_l) are the cartesian coordinates of the landmark. The discrete-time model describing the motion of the extended robots+landmark system is

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{x1,k} \\ u_{y1,k} \\ u_{x2,k} \\ u_{y2,k} \end{pmatrix} + \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ v_{4,k} \\ 0 \\ 0 \end{pmatrix}$$

where $(u_{x1,k}\ \ u_{y1,k}\ \ u_{x2,k}\ \ u_{y2,k})^T$ collects the robots' velocity inputs and $v_{i,k}$ is a white gaussian noise with zero mean and covariance $V_{i,k}$ $(i = 1, \ldots, 4)$. Note how the landmark being fixed reflects on the last two rows of the above equation. This is clearly a linear model of the form

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{A}\boldsymbol{\chi}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{v}_k$$

where $\boldsymbol{A} = \boldsymbol{I}$, the 6×6 identity matrix, $\boldsymbol{u}_k$ is the input vector, and $\boldsymbol{v}_k$ is the process noise vector, whose covariance is

$$\boldsymbol{V}_k = \begin{pmatrix} V_{1,k} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{2,k} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{3,k} & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{4,k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Coming to the measurement model, we have four measurements coming from the two robot sensors:

$$\boldsymbol{h}_k = \begin{pmatrix} h_1(\boldsymbol{\chi}_K) \\ h_2(\boldsymbol{\chi}_K) \\ h_3(\boldsymbol{\chi}_K) \\ h_4(\boldsymbol{\chi}_K) \end{pmatrix} + \begin{pmatrix} w_{1,k} \\ w_{2,k} \\ w_{3,k} \\ w_{4,k} \end{pmatrix} = \begin{pmatrix} \sqrt{(x_{1,k} - x_{2,k})^2 + (y_{1,k} - y_{2,k})^2} \\ \sqrt{(x_{1,k} - x_l)^2 + (y_{1,k} - y_l)^2} \\ \sqrt{(x_{2,k} - x_{1,k})^2 + (y_{2,k} - y_{1,k})^2} \\ \sqrt{(x_{2,k} - x_l)^2 + (y_{2,k} - y_l)^2} \end{pmatrix} + \begin{pmatrix} w_{1,k} \\ w_{2,k} \\ w_{3,k} \\ w_{4,k} \end{pmatrix}$$

where w_i, k is a white gaussian noise with zero mean and covariance $W_{i,k}$ $(i = 1, \ldots, 4)$. Ideally, the first and third components should be identical; however, since they are measured by two different sensors, their measured values will in general be different and therefore the entries must be duplicated in the model.

We are now ready to derive the EKF equations. Note that, since the process dynamics is linear, there is no need to linearize it. As for the linearization of the output equations, we have

$$\boldsymbol{H}_{k+1} = \begin{pmatrix} \left.\dfrac{\partial h_1}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \\ \left.\dfrac{\partial h_2}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \\ \left.\dfrac{\partial h_3}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \\ \left.\dfrac{\partial h_4}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \end{pmatrix}$$

where

$$\begin{aligned}
\frac{\partial \boldsymbol{h}_1}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} \frac{x_1 - x_2}{\delta(\boldsymbol{\chi})} & \frac{y_1 - y_2}{\delta(\boldsymbol{\chi})} & \frac{x_2 - x_1}{\delta(\boldsymbol{\chi})} & \frac{y_2 - y_1}{\delta(\boldsymbol{\chi})} & 0 & 0 \end{pmatrix} \\
\frac{\partial \boldsymbol{h}_2}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} \frac{x_1 - x_l}{\eta_1(\boldsymbol{\chi})} & \frac{y_1 - y_l}{\eta_1(\boldsymbol{\chi})} & 0 & 0 & 0 & 0 \end{pmatrix} \\
\frac{\partial \boldsymbol{h}_3}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} \frac{x_1 - x_2}{\delta(\boldsymbol{\chi})} & \frac{y_1 - y_2}{\delta(\boldsymbol{\chi})} & \frac{x_2 - x_1}{\delta(\boldsymbol{\chi})} & \frac{y_2 - y_1}{\delta(\boldsymbol{\chi})} & 0 & 0 \end{pmatrix} \\
\frac{\partial \boldsymbol{h}_4}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} 0 & 0 & \frac{x_2 - x_l}{\eta_2(\boldsymbol{\chi})} & \frac{y_2 - y_l}{\eta_2(\boldsymbol{\chi})} & 0 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\delta(\boldsymbol{\chi}) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
\eta_1(\boldsymbol{\chi}) &= \sqrt{(x_1 - x_l)^2 + (y_1 - y_l)^2} \\
\eta_2(\boldsymbol{\chi}) &= \sqrt{(x_2 - x_l)^2 + (y_2 - y_l)^2}
\end{aligned}$$

The EKF equations are finally obtained as follows.

1. State and covariance prediction:

$$\begin{aligned}
\hat{\boldsymbol{\chi}}_{k+1|k} &= \hat{\boldsymbol{\chi}}_k + \boldsymbol{B}\boldsymbol{u}_k \\
\boldsymbol{P}_{k+1|k} &= \boldsymbol{P}_k + \boldsymbol{V}_k
\end{aligned}$$

2. Correction:

$$\begin{aligned}
\hat{\boldsymbol{\chi}}_{k+1} &= \hat{\boldsymbol{\chi}}_{k+1|k} + \boldsymbol{R}_{k+1}\boldsymbol{\nu}_{k+1} \\
\boldsymbol{P}_{k+1} &= \boldsymbol{P}_{k+1|k} - \boldsymbol{R}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}
\end{aligned}$$

   where the innovation is

$$\boldsymbol{\nu}_{k+1} = \boldsymbol{h}_{k+1} - \boldsymbol{h}(\hat{\boldsymbol{\chi}}_{k+1|k})$$

   and the Kalman gain matrix is computed as

$$\boldsymbol{R}_{k+1} = \boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + \boldsymbol{W}_{k+1})^{-1}$$

   where $\boldsymbol{W}_{k+1} = \text{diag}\{W_{1,k+1}, \ldots, W_{4,k+1}\}$.

In these equations, $\boldsymbol{P}_k$ obviously denotes the covariance of the estimate, which will be initialized at a certain value reflecting the uncertainty on the initial estimate $\hat{\boldsymbol{\chi}}_0$.