Self assessment - 01

30/10/2014 - Updated 31/01/2020

1 Exercise

Let the input $u(t)$ and output $y(t)$ of a system satisfy the following linear differential equation

$$y^{(5)}(t) + 4y^{(4)}(t) + 3y^{(3)}(t) - 2y^{(2)}(t) + y^{(1)}(t) + y(t) - u(t) = 0$$

where $y^{(i)}(t)$ denotes the i-th time derivative of $y(t)$. For this system:

1. find a state space representation
2. compute the transfer function and say if there exists any uncontrollable or unobservable mode
3. say if the system is asymptotically stable or not.

2 Exercise

Let the system S respond, from zero initial conditions, with

$$y(t) = \left(1 - t + \frac{t^2}{2} - e^{-t}\right)\delta_{-1}(t)$$

to the input

$$u(t) = \delta(t) - 2e^{-3t}\delta_{-1}(t)$$

Find the impulse response $w(t)$ of S.

3 Exercise

Find the output forced response (output zero-state response) $y(t)$ of the system represented by

$$F(s) = \frac{50}{s^2 + 15s + 50}$$

to the input $u(t)$ shown in Fig. 1

4 Exercise

For each system having the dynamics matrix A_i discuss the stability property

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 4 & -2 \\ 0 & -3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -3 & 0 \\ -3 & 1 & 3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -1 & -12 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

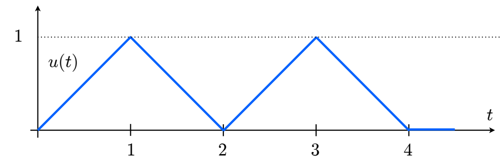

Figure 1: Ex. 3, input $u(t)$

5 Exercise

Assuming the coincidence of poles and eigenvalues, study the stability property of the following systems.

$$P_1(s) = \frac{s-1}{s^2}, \quad P_2(s) = \frac{s-1}{s(s+1)}, \quad P_3(s) = \frac{s+1}{s^3+12s^2+3s}, \quad P_4(s) = \frac{s+1}{s^3+12s^2+s+10}$$

$$P_5(s) = \frac{s^2-18}{s^3+12s^2+s-12}, \quad P_6(s) = \frac{-1}{s^3+2s^2+s+1}, \quad P_7(s) = \frac{s-10}{s^5+s^4+2s^3+s^2+3s+4}$$

6 Exercise

For the system having dynamics matrix

$$A = \begin{pmatrix} k & 1 \\ 0 & 0 \end{pmatrix}$$

determine, depending upon the values of $k \in R$, the natural modes and study stability.

7 Exercise

Find the forced response of the system

$$P(s) = \frac{s-1}{s+1}$$

to the input $u(t) = e^t \delta_{-1}(t) - 2t\delta_{-1}(t)$.

8 Exercise

For the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & -1 \end{pmatrix} x \end{aligned}$$

find the forced zero-state response to the input $u(t)$ shown in Fig. 2 using

$$\mathcal{L}\left[\sin(\omega t)\,\delta_{-1}(t)\right] = \frac{\omega}{s^2+\omega^2}$$

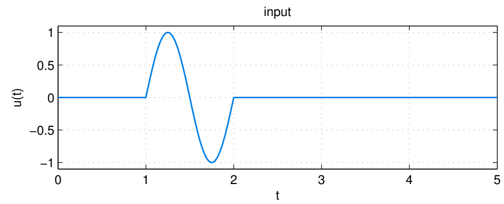

Figure 2: Ex. 8, input $u(t)$

9 Exercise

Find the natural modes of the system having dynamics matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

10 Exercise

Compute the free state and output response of the system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 2 & 1 \end{pmatrix} x(t) \end{aligned}$$

from the initial condition

$$x(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

11 Exercise

Determine the initial conditions of the system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t) \\ y &= \begin{pmatrix} 1 & -1 \end{pmatrix} x(t) \end{aligned}$$

for which we obtain a non-diverging free output.

12 Exercise

For the system given by

$$\dot{x}(t) = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} x(t)$$

determine the initial conditions, if any, such that the zero-input output response remains constant.

A Exercise 1

State will have dimension 5. One possible choice is given by y and its derivatives up to $y^{(4)}$

$$x^T(t) = \begin{bmatrix} y(t) & y^{(1)}(t) & y^{(2)}(t) & y^{(3)}(t) & y^{(4)}(t) \end{bmatrix}^T$$

With this choice we obtain

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2 & -3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad D = 0$$

To find the transfer function we could use the formula involving (A, B, C, D) but this would require the inversion of the 5×5 matrix $(sI - A)$. More directly we can recognize in the structure of the obtained A, B and C the controller canonical form and therefore we can directly state that

$$F(s) = \frac{1}{s^5 + 4s^4 + 3s^3 - 2s^2 + s + 1}$$

Otherwise, since the transfer function relates the input to the output zero-state response (i.e. with $x(0) = 0$), applying the derivative theorem to the differential equation leads to

$$s^5Y(s) + 4s^4Y(s) + 3s^3Y(s) - 2s^2Y(s) + sY(s) + Y(s) = U(s)$$

and to the transfer function $F(s) = Y(s)/U(s)$ previously found. System stability can be inferred from the eigenvalues, but since the denominator of the transfer function has degree equal to $n = 5$, the poles coincide with the eigenvalues. The Routh necessary condition is not satisfied and therefore the system is not asymptotically stable.

B Exercise 2

We can find the impulse response from the inverse Laplace transform of the transfer function which can be found as the ratio of the zero-state response transform with the input transform that is, being

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1} \\ U(s) &= 1 - \frac{2}{s+3} \end{aligned}$$

we have

$$F(s) = \frac{1/(s^3(s+1))}{(s+1)/(s+3)} = \frac{(s+3)}{s^3(s+1)^2}$$

We then just have to do an expansion in partial fractions.

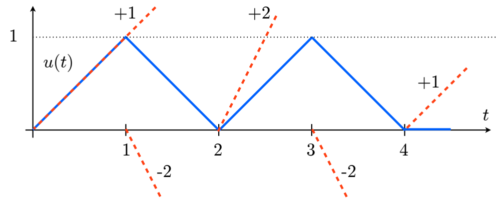

Figure 3: Ex. 3, input $u(t)$

C Exercise 3

This exercise requires the correct use of the Laplace transform time shifting property and rewriting $u(t)$ as a linear combination of functions which have simple Laplace transform. The input $u(t)$, as shown in Fig. 3, can also be written as

$$\begin{aligned} u(t) &= t\delta_{-1}(t) - 2(t-1)\delta_{-1}(t-1) + 2(t-2)\delta_{-1}(t-2) - 2(t-3)\delta_{-1}(t-3) + (t-4)\delta_{-1}(t-4) \\ &= \sum_{k=0}^{4} a_k (t-k)\delta_{-1}(t-k) \end{aligned}$$

and therefore

$$U(s) = \left(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + e^{-4s}\right)\frac{1}{s^2}$$

and $Y(s) = F(s)U(s)$.
Recall that if, in general, $Y(s) = Y_0(s)e^{-sT}$ then

$$y(t) = y_0(t-T)\delta_{-1}(t-T)$$

and therefore defining

$$Y_0(s) = F(s)\frac{1}{s^2} = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+5} + \frac{R_3}{s+10}$$

we have, once the residues have been computed,

$$y(t) = \sum_{k=0}^{4} a_k y_0(t-k)\delta_{-1}(t-k)$$

with $R_{11} = -3/10$, $R_{12} = F(0) = 1$, $R_2 = 2/5$ and $R_3 = -1/10$. The input and final output are plotted in Fig. 4.

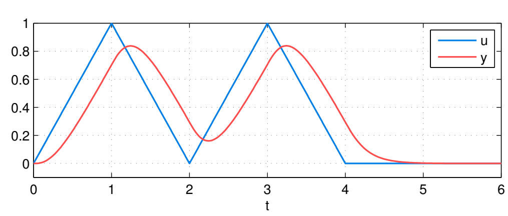

Figure 4: Ex. 3, input $u(t)$ and corresponding response $y(t)$

D Exercise 4

When possible we try to exploit some particular matrix structure to simplify calculation as much as possible.

• Note that matrix A_1 is *block diagonal*

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} A_\alpha & 0 \\ 0 & A_\beta \end{pmatrix} \quad \text{with} \quad A_\alpha = \begin{pmatrix} -1 \end{pmatrix} \;\; A_\beta = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}$$

and therefore

$$\text{eig}\left\{\begin{pmatrix} A_\alpha & 0 \\ 0 & A_\beta \end{pmatrix}\right\} = \text{eig}\{A_\alpha\} \cup \text{eig}\{A_\beta\}$$

We have aut $\{A_\alpha\} = -1$ and

$$p_{A_\beta}(\lambda) = \det(\lambda I - A_\beta) = \lambda^2 - 10 \quad \Rightarrow \quad \text{eig}\{A_\beta\} = \left\{+\sqrt{10}, -\sqrt{10}\right\}$$

Since one eigenvalue $\sqrt{10}$ is positive (and theefore has positive real part) the system is unstable. As an alternative we could have applied the Routh criterion to the characteristic polynomial

$$p_{A_1}(\lambda) = \lambda^3 + \lambda^2 - 10\lambda - 10$$

The necessary condition is not satisfied and therefore we can only assess that not all the eigenvalues have negative real part. Note that the Routh table has a row (with only one element) equal to zero since the first two rows are linearly dependent.

1	-10
1	-10
0	

(N.B. There are rules to overcome this situation).

• Being the matrix A_2 *upper triangular*, the eigenvalues coincide with the elements on the diagonal $\lambda_1 = -1$, $\lambda_2 = 3$ and $\lambda_3 = 3$. The eigenvalue $\lambda_3 = 3$ makes the system unstable.

• Similarly, being A_3 *lower triangular* again the eigenvalues are the elements on the main diagonal and therefore, having $\lambda_3 = 3$, the system is unstable.

• Matrix A_4 is *block triangular*

$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} A_\alpha & \star \\ 0 & A_\beta \end{pmatrix}$$

(with $\star$ matrix having the right dimensions) and therefore we have

$$\text{eig}\left\{\begin{pmatrix} A_\alpha & \star \\ 0 & A_\beta \end{pmatrix}\right\} = \text{eig}\{A_\alpha\} \cup \text{eig}\{A_\beta\}$$

Being

$$p_{A_\beta}(\lambda) = \lambda^2 - 3\lambda + 1$$

both roots (eigenvalues) of $p_{A_\beta}(\lambda) = 0$ have positive real part since the elements of the first column of the Routh table (which coincide for a second order equation with the polynomial coefficients) change sign twice. The corresponding system is unstable.

• Note that A_5 is in the *controller canonical form* and therefore its characteristic polynomial is

$$p_{A5}(\lambda) = \lambda^3 + 12\lambda^2 + \lambda + 10$$

with Routh table

$$\begin{array}{|cc} 1 & 1 \\ 12 & 10 \\ 1/6 & \\ 10 & \end{array}$$

The corresponding system is asymptotically stable.

• First note that A_6 is lower triangular and therefore the system has the double eigenvalue $\lambda_1 = 0$ – which prevents the system from being asymptotically stable – and $\lambda_2 = -3$. Moreover A_6 is also block diagonal with

$$A_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In order to understand if this double eigenvalue in 0 leads to instability or not, recall that the eigenvalues of a matrix coincide with those of its transpose

$$\text{eig}\{A_\alpha\} = \text{eig}\{A_\alpha^T\}$$

This can be shown with the similarity transformation T such that

$$A_\alpha^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = T A_\alpha T^{-1} = T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} T^{-1} \quad \text{with} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = T^{-1}$$

and therefore A_α and A_α^T are similar and share the same eigenvalues. We can then note that A_α^T is a Jordan block of dimension 2 (index = 2) for the eigenvalue $\lambda_1 = 0$ which makes the system unstable.

E Exercise 5

For the considered systems we have the following results.

• The system has a double pole is $s = 0$ and therefore it is not asymptotically stable. To see it is unstable we can either recall that an eigenvalue will appear as a pole with at most multiplicity equal to its index (dimension of the largest Jordan block). Therefore here we would have an index equal to 2 and this leads to instability. This is evident from the realization

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}, \quad D_1 = 0$$

which shows the presence of the Jordan block.

Alternatively, since the transfer function is the Laplace transform of the impulse response $p_1(t)$, we have

$$P_1(s) = \frac{s-1}{s^2} = \frac{R_1}{s} + \frac{R_2}{s^2}, \quad \text{with} \quad R_1 = 1, \quad R_2 = -1$$

and the impulse response

$$p_1(t) = \mathcal{L}^{-1}\left[P_1(s)\right] = (R_1 + R_2\, t)\, \delta_{-1}(t)$$

shows the presence of the diverging natural mode $t\delta_{-1}(t)$.

• Being the poles of $P_2(s)$ equal to $p_1 = 0$ and $p_2 = -1$, the system is marginally stable (or more properly Lyapunov stable).

• The denominator of $P_3(s)$ can be factored as

$$s^3 + 12s^2 + 3s = s(s^2 + 12s + 3)$$

thus we have a pole in $s = 0$ and two poles with negative real part. The system is therefore marginally stable (or more properly Lyapunov stable).

• The Routh criterion for $P_4(s)$ is satisfied therefore the system is asymptotically stable. The roots are (found numerically) $p_1 = -11.9862$, $p_{2/3} = -0.0069 \pm 0.9134j$.

$$\left|\begin{array}{cc} 1 & 1 \\ 12 & 10 \\ 1/6 & \\ 10 & \end{array}\right.$$

• The system is not asymptotically stable since the necessary condition is not satisfied. Building the Routh table

$$\left|\begin{array}{cc} 1 & 1 \\ 12 & \text{-12} \\ 2 & \\ \text{-12} & \end{array}\right.$$

shows that there is only one change of sign so one root with positive real part. As a check, numerically the roots are $p_1 = -11.8297$, $p_2 = -1.0959$ and $p_3 = 0.9256$.

• Routh criterion shows asymptotic stability. The roots are $p_1 = -1.7549$, $p_{2/3} = -0.1226 \pm 0.7449j$.

$$\left|\begin{array}{cc} 1 & 1 \\ 2 & 1 \\ 1/2 & \\ 1 & \end{array}\right.$$

• Being the Routh table

$$\left|\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & 4 \\ 1 & \text{-1} & \\ 2 & 4 & \\ \text{-3} & & \\ 4 & & \end{array}\right.$$

the system with transfer function $P_7(s)$ is unstable due to the presence of two poles with positive real part (2 sign variations in the first column). The poles are $p_1 = -1$, $p_{2/3} = -0.7177 \pm 1.3651j$ and $p_{4/5} = 0.7177 \pm 1.0801j$.

F Exercise 6

Being the matrix triangular

$$p_A(\lambda) = (\lambda - k)\lambda \quad \Rightarrow \text{ eigenvalues: } \quad \begin{cases} \lambda_1 = 0, \lambda_2 = k, & \text{if } k \neq 0; \\ \lambda_1 = \lambda_2 = 0, & \text{if } k = 0. \end{cases}$$

If $k \neq 0$ the natural modes are $e^{\lambda_1 t} = e^{kt}$ and $e^{\lambda_2 t} = 1$; if $k > 0$ the system is unstable while for $k < 0$ we have marginal stability (or preferably Lyapunov stability).

For the case $k = 0$ we have several equivalent options.

With $k = 0$ matrix A

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has an obvious Jordan block of dimension 2 (index 2) and therefore the zero eigenvalue leads to instability. The natural modes are $e^{0t} = 1$ and $te^{0t} = t$.

Equivalently, with $k = 0$, the dynamics equations are

$$\dot{x} = Ax, \quad x \in \mathbf{R}^2, \qquad \rightarrow \quad \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{cases}$$

The second equation has the constant solution $x_2(t) = x_2(0)$ and therefore $x_1(t) = x_2(0)t + x_1(0)$. These are the components of the free (unforced, zero-input) state evolution

$$x_{zi}(t) = e^{At}x(0) = \begin{cases} x_2(0)t + x_1(0) \\ x_2(0) \end{cases}$$

which clearly shows the diverging behavior for generic initial conditions.

In the Laplace domain, note that

$$\mathcal{L}\left[e^{At}\right] = (sI - A)^{-1} = \frac{1}{s^2}\left\{ \begin{matrix} s & 1 \\ 0 & s \end{matrix} \right)$$

which can be expanded as (Heaviside)

$$(sI - A)^{-1} = \frac{1}{s}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{s^2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Being the matrices (residues) independent from s, the inverse Laplace transform is

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta_{-1}(t) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}t\delta_{-1}(t)$$

Post-multiplication (right multiplication) by the initial condition $x(0)$ leads to the same unforced response previously found.

G Exercise 7

The forced response transform is given by

$$Y(s) = P(s)U(s)$$

and therefore, being

$$U(s) = \frac{1}{s-1} - 2\frac{1}{s^2} = \frac{s^2-2s+2}{s^2(s-1)}$$

we get

$$Y(s) = \frac{s-1}{s+1}\frac{s^2-2s+2}{s^2(s-1)} = \frac{s^2-2s+2}{s^2(s+1)} = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+1}$$

with $R_{11} = -4$, $R_{12} = 2$ and $R_2 = 5$. An interesting aspect of this example is that the diverging exponential component of the input e^t is not present at the output (while the polynomial component is) since the system has "filtered" this diverging forcing term through the presence of the zero in $s = 1$ in the transfer function $P(s)$.

H Exercise 8

The forced response transform is given by

$$Y(s) = P(s)U(s)$$

and therefore the only difficulty lies in finding $U(s)$ from known Laplace transform and properties. The input $u(t)$ is a truncated sinusoidal function with frequency 1 Hz or 2π rad/s as shown in Fig. 5-A.

As shown in Fig. 5-D, the input $u(t)$ can be seen as the result of a time shift of 1 sec (Fig. 5-B) to which an opposite and time shifted of 2 sec sinusoid (Fig. 5-C) has been added and therefore

$$u(t) = [\sin(2\pi(t-1))]\,\delta_{-1}(t-1) - [\sin(2\pi(t-2))]\,\delta_{-1}(t-2)$$

Defining $y_0(t)$ as the output corresponding to the input $\sin 2\pi t$, the output $y(t)$ is given by

$$y(t) = y_0(t-1)\delta_{-1}(t-1) - y_0(t-2)\delta_{-1}(t-2)$$

We therefore just need to compute $y_0(t)$ as the inverse Laplace transform of

$$Y_0(s) = P(s)\frac{2\pi}{s^2+4\pi^2}$$

with $P(s)$ the transfer function of the given system (state space representation is in the controller canonical form)

$$P(s) = \frac{-s+1}{s^2+2s+1} = -\frac{s-1}{(s+1)^2}$$

We have

$$Y_0(s) = -\frac{s-1}{(s+1)^2}\frac{2\pi}{s^2+4\pi^2} = \frac{R_1}{s+2\pi j} + \frac{R_1^*}{s-2\pi j} + \frac{R_{21}}{s+1} + \frac{R_{22}}{(s+1)^2}$$

with

$$\begin{aligned}
R_1 &= [(s+2\pi j)Y(s)]_{s=-2\pi j} = \frac{\pi(4\pi^2-3)}{(1+4\pi^2)^2} + \frac{1-12\pi^2}{2(1+4\pi^2)^2}j \\
R_{21} &= \left[\frac{d}{ds}\left((s+1)^2 Y(s)\right)\right]_{s=-1} = \frac{-2\pi((2\pi)^2-3)}{(1+(2\pi)^2)^2} \\
R_{22} &= \left[(s+1)^2 Y(s)\right]_{s=-1} = \frac{4\pi}{1+(2\pi)^2}
\end{aligned}$$

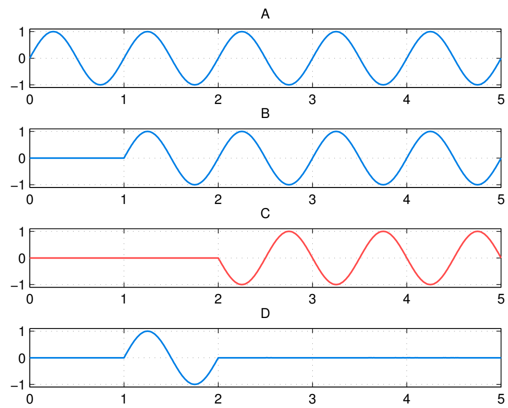

Figure 5: Ex. 8, input $u(t)$

and therefore, defining the residue R_1 as $R_1 = a + jb$, we have

$$\frac{R_1}{s+2\pi j} + \frac{R_1^*}{s-2\pi j} = 2a\frac{s}{s^2+(2\pi)^2} + 2b\frac{2\pi}{s^2+(2\pi)^2}$$

which admits the inverse Laplace transform

$$2a\cos 2\pi t + 2b\sin 2\pi t$$

The overall $y_0(t)$ is then given by

$$y_0(t) = 2a\cos 2\pi t + 2b\sin 2\pi t + R_{21}e^{-t} + R_{22}te^{-t}$$

I Exercise 9

Being the matrix block diagonal, the eigenvalues are the union of the eigenvalues of

$$A_1 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \end{pmatrix}$$

that is $\lambda_1 = 1$, $\lambda_2 = i$ and $\lambda_3 = \lambda_2^* = -i$ (the characteristic polynomial of A is $p_A(\lambda) = (\lambda-1)(\lambda^2+1)$). The natural modes are therefore

$$e^{\lambda_1 t} = e^t, \quad \sin t \quad \text{(or equivalently} \cos t)$$

J Exercise 10

The characteristic polynomial is $p_A(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3)$ and therefore $\lambda_1 = -1$, $\lambda_2 = -3$. The eigenvector associated to $\lambda_1 = -1$ is

$$u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or any parallel. the eigenvector associated to $\lambda_2 = -3$ is

$$u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Choose $U = \begin{pmatrix} u_1 & u_2 \end{pmatrix}$ so that

$$U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

From the spectral form we obtain

$$\begin{aligned}
x_{zi}(t) = e^{At}x(0) &= \left(e^{\lambda_1 t}u_1 v_1^T + e^{\lambda_2 t}u_2 v_2^T\right) x(0) \\
&= \left\{ e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}\right\}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
&= \left\{ e^{-t}\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\right\}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}
\end{aligned}$$

while

$$y_{zi}(t) = Ce^{At}x(0) = Cx_{zi}(t) = e^{-t} + 3e^{-3t}$$

As an alternative, we can find the coefficients $c_1 = v_1^T x_0$ and $c_2 = v_2^T x_0$ which give the initial condition x_0 in the (u_1, u_2) basis

$$x(0) = c_1 u_1 + c_2 u_2 = u_1 + u_2$$

and therefore, being $v_i^T u_j = \delta_{ij}$, c_1 and c_2 scalars,

$$\begin{aligned}
e^{At}x(0) &= \left(e^{\lambda_1 t}u_1 v_1^T + e^{\lambda_2 t}u_2 v_2^T\right) x(0) \\
&= \left(e^{\lambda_1 t}u_1 v_1^T + e^{\lambda_2 t}u_2 v_2^T\right) (c_1 u_1 + c_2 u_2) \\
&= c_1 e^{\lambda_1 t}u_1 + c_2 e^{\lambda_2 t}u_2
\end{aligned}$$

K Exercise 11

The eigenvalues and associated eigenvectors are

$$\lambda_1 = -1 \rightarrow u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \lambda_2 = 1 \rightarrow u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The output zero-input response is therefore

$$\begin{aligned}
y_{zi}(t) &= Ce^{At}x(0) = C\left\{e^{\lambda_1 t}u_1 v_1^T + e^{\lambda_1 t}u_2 v_2^T\right\} x(0) \\
&= \left\{e^{\lambda_1 t}Cu_1 v_1^T + e^{\lambda_1 t}Cu_2 v_2^T\right\} x(0) \\
&= \left\{e^{\lambda_1 t}Cu_1 v_1^T\right\} x(0)
\end{aligned}$$

being $Cu_2 = 0$ and therefore any initial condition solves the problem. This is understandable since $Cu_2 = 0$ implies that the diverging natural mode $e^{\lambda_2 t} = e^t$ is unobservable.

L Exercise 12

The eigenvalues and associated eigenvectors are

$$\lambda_1 = 0 \rightarrow u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 5 \rightarrow u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

and therefore we have a constant natural mode (for $\lambda_1 = 0$) and a diverging one (for $\lambda_2 = 5$). In order for the diverging natural mode not to compare in the output free response, the initial state needs to belong to the eigenspace relative to $\lambda_1 = 0$, that is

$$x(0) = \alpha u_1 = \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix}$$

Since $\lambda_1 = 0$ the output response from $x(0)$ not only will be non diverging but also constant.