

# Self assessment - 01

30/10/14 - Updated 31/01/20 - 03/01/25

## 1 Exercise

Let the input  $u(t)$  and output  $y(t)$  of a system satisfy the following linear differential equation

$$y^{(5)}(t) + 4y^{(4)}(t) + 3y^{(3)}(t) - 2y^{(2)}(t) + y^{(1)}(t) + y(t) - u(t) = 0$$

where  $y^{(i)}(t)$  denotes the  $i$ -th time derivative of  $y(t)$ . For this system:

1. find a state space representation
2. compute the transfer function and say if there exists any uncontrollable or unobservable mode
3. say if the system is asymptotically stable or not.

## 2 Exercise

Let the system  $S$  respond, from zero initial conditions, with

$$y(t) = \left(1 - t + \frac{t^2}{2} - e^{-t}\right) \delta_{-1}(t)$$

to the input

$$u(t) = \delta(t) - 2e^{-3t} \delta_{-1}(t)$$

Find the impulse response  $w(t)$  of  $S$ .

## 3 Exercise

Find the output forced response (output zero-state response)  $y(t)$  of the system represented by

$$F(s) = \frac{50}{s^2 + 15s + 50}$$

to the input  $u(t)$  shown in Fig. 1

## 4 Exercise

For each system having the dynamics matrix  $A_i$  discuss the stability property

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 4 & -2 \\ 0 & -3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -3 & 0 \\ -3 & 1 & 3 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -1 & -12 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

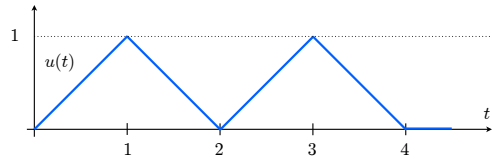


Figure 1: Ex. 3, input  $u(t)$

## 5 Exercise

Assuming the coincidence of poles and eigenvalues, study the stability property of the following systems.

$$P_1(s) = \frac{s-1}{s^2}, \quad P_2(s) = \frac{s-1}{s(s+1)}, \quad P_3(s) = \frac{s+1}{s^3+12s^2+3s}, \quad P_4(s) = \frac{s+1}{s^3+12s^2+s+10}$$

$$P_5(s) = \frac{s^2-18}{s^3+12s^2+s-12}, \quad P_6(s) = \frac{-1}{s^3+2s^2+s+1}, \quad P_7(s) = \frac{s-10}{s^5+s^4+2s^3+s^2+3s+4}$$

## 6 Exercise

For the system having dynamics matrix

$$A = \begin{pmatrix} k & 1 \\ 0 & 0 \end{pmatrix}$$

determine, depending upon the values of  $k \in \mathbb{R}$ , the natural modes and study stability.

## 7 Exercise

Find the forced response of the system

$$P(s) = \frac{s-1}{s+1}$$

to the input  $u(t) = e^t \delta_{-1}(t) - 2t \delta_{-1}(t)$ .

## 8 Exercise

For the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= (1 \quad -1) x \end{aligned}$$

find the forced zero-state response to the input  $u(t)$  shown in Fig. 2 using

$$\mathcal{L}[\sin(\omega t) \delta_{-1}(t)] = \frac{\omega}{s^2 + \omega^2}$$

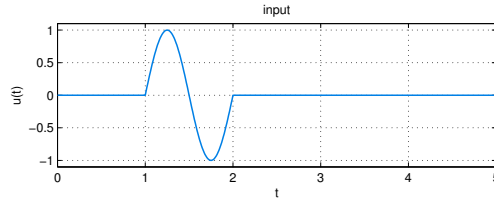


Figure 2: Ex. 8, input  $u(t)$

## 9 Exercise

Find the natural modes of the system having dynamics matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

## 10 Exercise

Compute the free state and output response of the system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 2 & 1 \end{pmatrix} x(t) \end{aligned}$$

from the initial condition

$$x(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

## 11 Exercise

Determine the initial conditions of the system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t) \\ y &= \begin{pmatrix} 1 & -1 \end{pmatrix} x(t) \end{aligned}$$

for which we obtain a non-diverging free output.

## 12 Exercise

For the system given by

$$\dot{x}(t) = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} x(t)$$

determine the initial conditions, if any, such that the zero-input output response remains constant.

## A Exercise 1

State will have dimension 5. One possible choice is given by  $y$  and its derivatives up to  $y^{(4)}$

$$x^T(t) = [y(t) \quad y^{(1)}(t) \quad y^{(2)}(t) \quad y^{(3)}(t) \quad y^{(4)}(t)]^T$$

With this choice we obtain

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2 & -3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0 \quad 0 \quad 0 \quad 0] \quad D = 0$$

To find the transfer function we could use the formula involving  $(A, B, C, D)$  but this would require the inversion of the  $5 \times 5$  matrix  $(sI - A)$ . More directly we can recognize in the structure of the obtained  $A$ ,  $B$  and  $C$  the controller canonical form and therefore we can directly state that

$$F(s) = \frac{1}{s^5 + 4s^4 + 3s^3 - 2s^2 + s + 1}$$

Otherwise, since the transfer function relates the input to the output zero-state response (i.e. with  $x(0) = 0$ ), applying the derivative theorem to the differential equation leads to

$$s^5 Y(s) + 4s^4 Y(s) + 3s^3 Y(s) - 2s^2 Y(s) + sY(s) + Y(s) = U(s)$$

and to the transfer function  $F(s) = Y(s)/U(s)$  previously found. System stability can be inferred from the eigenvalues, but since the denominator of the transfer function has degree equal to  $n = 5$ , the poles coincide with the eigenvalues. The Routh necessary condition is not satisfied and therefore the system is not asymptotically stable.

## B Exercise 2

We can find the impulse response from the inverse Laplace transform of the transfer function which can be found as the ratio of the zero-state response transform with the input transform that is, being

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1} \\ U(s) &= 1 - \frac{2}{s+3} \end{aligned}$$

we have

$$F(s) = \frac{1/(s^3(s+1))}{(s+1)/(s+3)} = \frac{(s+3)}{s^3(s+1)^2}$$

We then just have to do an expansion in partial fractions.

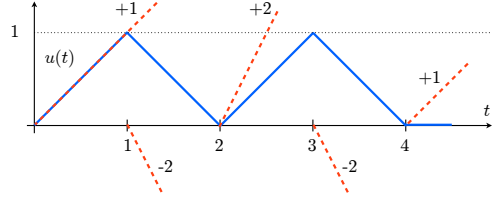


Figure 3: Ex. 3, input  $u(t)$

### C Exercise 3

This exercise requires the correct use of the Laplace transform time shifting property and rewriting  $u(t)$  as a linear combination of functions which have simple Laplace transform. The input  $u(t)$ , as shown in Fig. 3, can also be written as

$$\begin{aligned} u(t) &= t\delta_{-1}(t) - 2(t-1)\delta_{-1}(t-1) + 2(t-2)\delta_{-1}(t-2) - 2(t-3)\delta_{-1}(t-3) + (t-4)\delta_{-1}(t-4) \\ &= \sum_{k=0}^4 a_k(t-k)\delta_{-1}(t-k) \end{aligned}$$

and therefore

$$U(s) = (1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + e^{-4s}) \frac{1}{s^2}$$

and  $Y(s) = F(s)U(s)$ .

Recall that if, in general,  $Y(s) = Y_0(s)e^{-sT}$  then

$$y(t) = y_0(t-T)\delta_{-1}(t-T)$$

and therefore defining

$$Y_0(s) = F(s) \frac{1}{s^2} = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+5} + \frac{R_3}{s+10}$$

we have, once the residues have been computed,

$$y(t) = \sum_{k=0}^4 a_k y_0(t-k)\delta_{-1}(t-k)$$

with  $R_{11} = -3/10$ ,  $R_{12} = F(0) = 1$ ,  $R_2 = 2/5$  and  $R_3 = -1/10$ . The input and final output are plotted in Fig. 4.

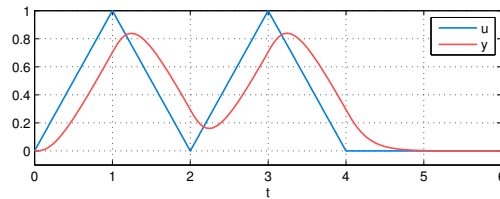


Figure 4: Ex. 3, input  $u(t)$  and corresponding response  $y(t)$

## D Exercise 4

When possible we try to exploit some particular matrix structure to simplify calculation as much as possible.

- Note that matrix  $A_1$  is *block diagonal*

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} A_\alpha & 0 \\ 0 & A_\beta \end{pmatrix} \quad \text{with} \quad A_\alpha = \begin{pmatrix} -1 \end{pmatrix} \quad A_\beta = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}$$

and therefore

$$\text{eig} \left\{ \begin{pmatrix} A_\alpha & 0 \\ 0 & A_\beta \end{pmatrix} \right\} = \text{eig} \{A_\alpha\} \cup \text{eig} \{A_\beta\}$$

We have  $\text{eig} \{A_\alpha\} = -1$  and

$$p_{A_\beta}(\lambda) = \det(\lambda I - A_\beta) = \lambda^2 - 10 \quad \Rightarrow \quad \text{eig} \{A_\beta\} = \{+\sqrt{10}, -\sqrt{10}\}$$

Since one eigenvalue  $\sqrt{10}$  is positive (and therefore has positive real part) the system is unstable. As an alternative we could have applied the Routh criterion to the characteristic polynomial

$$p_{A_1}(\lambda) = \lambda^3 + \lambda^2 - 10\lambda - 10$$

The necessary condition is not satisfied and therefore we can only assess that not all the eigenvalues have negative real part. Note that the Routh table has a row (with only one element) equal to zero since the first two rows are linearly dependent.

$$\begin{vmatrix} 1 & -10 \\ 1 & -10 \\ 0 \end{vmatrix}$$

(N.B. There are rules to overcome this situation).

- Being the matrix  $A_2$  *upper triangular*, the eigenvalues coincide with the elements on the diagonal  $\lambda_1 = -1$ ,  $\lambda_2 = 3$  and  $\lambda_3 = 3$ . The eigenvalue  $\lambda_3 = 3$  makes the system unstable.
- Similarly, being  $A_3$  *lower triangular* again the eigenvalues are the elements on the main diagonal and therefore, having  $\lambda_3 = 3$ , the system is unstable.
- Matrix  $A_4$  is *block triangular*

$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} A_\alpha & \star \\ 0 & A_\beta \end{pmatrix}$$

(with  $\star$  matrix having the right dimensions) and therefore we have

$$\text{eig} \left\{ \begin{pmatrix} A_\alpha & \star \\ 0 & A_\beta \end{pmatrix} \right\} = \text{eig} \{A_\alpha\} \cup \text{eig} \{A_\beta\}$$

Being

$$p_{A_\beta}(\lambda) = \lambda^2 - 3\lambda - 1$$

one root (eigenvalue of  $A_\beta$ ) of  $p_{A_\beta}(\lambda) = 0$  has positive real part since the elements of the first column of the Routh table (which coincide for a second order equation with the polynomial coefficients) change sign once. The corresponding system is unstable. One could also have noted that the matrix is in the *controller canonical form* and therefore its characteristic polynomial is

$$p_A(\lambda) = \lambda^3 - 3\lambda^2 - \lambda = \lambda(\lambda^2 - 3\lambda - 1)$$

- Note that  $A_5$  is in the *controller canonical form* and therefore its characteristic polynomial is

$$p_{A_5}(\lambda) = \lambda^3 + 12\lambda^2 + \lambda + 10$$

with Routh table

$$\begin{array}{c|cc} & 1 & 1 \\ & 12 & 10 \\ & 1/6 & \\ & 10 & \end{array}$$

The corresponding system is asymptotically stable.

- First note that  $A_6$  is lower triangular and therefore the system has the double eigenvalue  $\lambda_1 = 0$  – which prevents the system from being asymptotically stable – and  $\lambda_2 = -3$ . Moreover  $A_6$  is also block diagonal with

$$A_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In order to understand if this double eigenvalue in 0 leads to instability or not, recall that the eigenvalues of a matrix coincide with those of its transpose

$$\text{eig}\{A_\alpha\} = \text{eig}\{A_\alpha^T\}$$

This can be shown with the similarity transformation  $T$  such that

$$A_\alpha^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = T A_\alpha T^{-1} = T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} T^{-1} \quad \text{with} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = T^{-1}$$

and therefore  $A_\alpha$  and  $A_\alpha^T$  are similar and share the same eigenvalues. We can then note that  $A_\alpha^T$  is a Jordan block of dimension 2 (index = 2) for the eigenvalue  $\lambda_1 = 0$  which makes the system unstable. Alternatively, one can compute the geometric multiplicity of the eigenvalue  $\lambda_1 = 0$  which is

$$m_g(\lambda_1) = \dim \{\text{Ker}[A_\alpha - \lambda_1 I]\} = \dim \{\text{Ker}[A_\alpha]\} = 1 < m_a(\lambda_1) = 2$$

## E Exercise 5

For the considered systems we have the following results.

- The system has a double pole in  $s = 0$  and therefore it is not asymptotically stable. To see it is unstable we can either recall that an eigenvalue will appear as a pole with at most multiplicity equal to its index (dimension of the largest Jordan block). Therefore here we would have an index equal to 2 and this leads to instability. This is evident from the realization

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}, \quad D_1 = 0$$

which shows the presence of the Jordan block.

Alternatively, since the transfer function is the Laplace transform of the impulse response  $p_1(t)$ , we have

$$P_1(s) = \frac{s-1}{s^2} = \frac{R_1}{s} + \frac{R_2}{s^2}, \quad \text{with} \quad R_1 = 1, \quad R_2 = -1$$

and the impulse response

$$p_1(t) = \mathcal{L}^{-1}[P_1(s)] = (R_1 + R_2 t) \delta_{-1}(t)$$

shows the presence of the diverging natural mode  $t\delta_{-1}(t)$ .

- Being the poles of  $P_2(s)$  equal to  $p_1 = 0$  and  $p_2 = -1$ , the system is marginally stable (or more properly Lyapunov stable).

- The denominator of  $P_3(s)$  can be factored as

$$s^3 + 12s^2 + 3s = s(s^2 + 12s + 3)$$

thus we have a pole in  $s = 0$  and two poles with negative real part. The system is therefore marginally stable (or more properly Lyapunov stable).

- The Routh criterion for  $P_4(s)$  is satisfied therefore the system is asymptotically stable. The roots are (found numerically)  $p_1 = -11.9862$ ,  $p_{2/3} = -0.0069 \pm 0.9134j$ .

$$\begin{array}{c|cc} & 1 & 1 \\ & 12 & 10 \\ & 1/6 & \\ & 10 & \end{array}$$

- The system is not asymptotically stable since the necessary condition is not satisfied. Building the Routh table

$$\begin{array}{c|cc} & 1 & 1 \\ & 12 & -12 \\ & 2 & \\ & -12 & \end{array}$$

shows that there is only one change of sign so one root with positive real part. As a check, numerically the roots are  $p_1 = -11.8297$ ,  $p_2 = -1.0959$  and  $p_3 = 0.9256$ .

- Routh criterion shows asymptotic stability. The roots are  $p_1 = -1.7549$ ,  $p_{2/3} = -0.1226 \pm 0.7449j$ .

$$\begin{array}{c|cc} & 1 & 1 \\ & 2 & 1 \\ & 1/2 & \\ & 1 & \end{array}$$

- Being the Routh table

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ & 1 & 1 & 4 \\ & 1 & -1 & \\ & 2 & 4 & \\ & -3 & & \\ & 4 & & \end{array}$$



the system with transfer function  $P_7(s)$  is unstable due to the presence of two poles with positive real part (2 sign variations in the first column). The poles are  $p_1 = -1$ ,  $p_{2/3} = -0.7177 \pm 1.3651j$  and  $p_{4/5} = 0.7177 \pm 1.0801j$ .

## F Exercise 6

Being the matrix triangular

$$p_A(\lambda) = (\lambda - k)\lambda \Rightarrow \text{eigenvalues: } \begin{cases} \lambda_1 = 0, \lambda_2 = k, & \text{if } k \neq 0; \\ \lambda_1 = \lambda_2 = 0, & \text{if } k = 0. \end{cases}$$

If  $k \neq 0$  the natural modes are  $e^{\lambda_1 t} = e^{kt}$  and  $e^{\lambda_2 t} = 1$ ; if  $k > 0$  the system is unstable while for  $k < 0$  we have marginal stability (or preferably Lyapunov stability).

For the case  $k = 0$  we have several equivalent options.

With  $k = 0$  matrix  $A$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has an obvious Jordan block of dimension 2 (index 2) and therefore the zero eigenvalue leads to instability. The natural modes are  $e^{0t} = 1$  and  $te^{0t} = t$ .

Equivalently, with  $k = 0$ , the dynamics equations are

$$\dot{x} = Ax, \quad x \in \mathbf{R}^2, \quad \rightarrow \quad \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{cases}$$

The second equation has the constant solution  $x_2(t) = x_2(0)$  and therefore  $x_1(t) = x_2(0)t + x_1(0)$ . These are the components of the free (unforced, zero-input) state evolution

$$x_{zi}(t) = e^{At}x(0) = \begin{cases} x_2(0)t + x_1(0) \\ x_2(0) \end{cases}$$

which clearly shows the diverging behavior for generic initial conditions.

In the Laplace domain, note that

$$\mathcal{L}[e^{At}] = (sI - A)^{-1} = \frac{1}{s^2} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}$$

which can be expanded as (Heaviside)

$$(sI - A)^{-1} = \frac{1}{s} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{s^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Being the matrices (residues) independent from  $s$ , the inverse Laplace transform is

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_{-1}(t) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \delta_{-1}(t)$$

Post-multiplication (right multiplication) by the initial condition  $x(0)$  leads to the same unforced response previously found.

## G Exercise 7

The forced response transform is given by

$$Y(s) = P(s)U(s)$$

and therefore, being

$$U(s) = \frac{1}{s-1} - 2\frac{1}{s^2} = \frac{s^2 - 2s + 2}{s^2(s-1)}$$

we get

$$Y(s) = \frac{s-1}{s+1} \frac{s^2 - 2s + 2}{s^2(s-1)} = \frac{s^2 - 2s + 2}{s^2(s+1)} = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+1}$$

with  $R_{11} = -4$ ,  $R_{12} = 2$  and  $R_2 = 5$ . An interesting aspect of this example is that the diverging exponential component of the input  $e^t$  is not present at the output (while the polynomial component is) since the system has “filtered” this diverging forcing term through the presence of the zero in  $s = 1$  in the transfer function  $P(s)$ .

## H Exercise 8

The forced response transform is given by

$$Y(s) = P(s)U(s)$$

and therefore the only difficulty lies in finding  $U(s)$  from known Laplace transform and properties. The input  $u(t)$  is a truncated sinusoidal function with frequency 1 Hz or  $2\pi$  rad/s as shown in Fig. 5-A.

As shown in Fig. 5-D, the input  $u(t)$  can be seen as the result of a time shift of 1 sec (Fig. 5-B) to which an opposite and time shifted of 2 sec sinusoid (Fig. 5-C) has been added and therefore

$$u(t) = [\sin(2\pi(t-1))] \delta_{-1}(t-1) - [\sin(2\pi(t-2))] \delta_{-1}(t-2)$$

Defining  $y_0(t)$  as the output corresponding to the input  $\sin 2\pi t$ , the output  $y(t)$  is given by

$$y(t) = y_0(t-1)\delta_{-1}(t-1) - y_0(t-2)\delta_{-1}(t-2)$$

We therefore just need to compute  $y_0(t)$  as the inverse Laplace transform of

$$Y_0(s) = P(s) \frac{2\pi}{s^2 + 4\pi^2}$$

with  $P(s)$  the transfer function of the given system (state space representation is in the controller canonical form)

$$P(s) = \frac{-s+1}{s^2+2s+1} = -\frac{s-1}{(s+1)^2}$$

We have

$$Y_0(s) = -\frac{s-1}{(s+1)^2} \frac{2\pi}{s^2+4\pi^2} = \frac{R_1}{s+2\pi j} + \frac{R_1^*}{s-2\pi j} + \frac{R_{21}}{s+1} + \frac{R_{22}}{(s+1)^2}$$

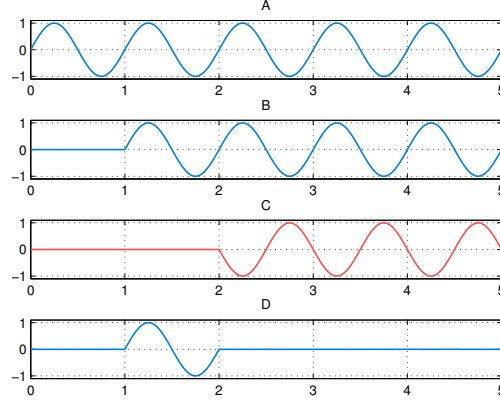


Figure 5: Ex. 8, input  $u(t)$

with

$$\begin{aligned}
 R_1 &= [(s + 2\pi j)Y(s)]_{s=-2\pi j} = \frac{\pi(4\pi^2 - 3)}{(1 + 4\pi^2)^2} + \frac{1 - 12\pi^2}{2(1 + 4\pi^2)^2}j \\
 R_{21} &= \left[ \frac{d}{ds} ((s + 1)^2 Y(s)) \right]_{s=-1} = \frac{-2\pi((2\pi)^2 - 3)}{(1 + (2\pi)^2)^2} \\
 R_{22} &= [(s + 1)^2 Y(s)]_{s=-1} = \frac{4\pi}{1 + (2\pi)^2}
 \end{aligned}$$

and therefore, defining the residue  $R_1$  as  $R_1 = a + jb$ , we have

$$\frac{R_1}{s + 2\pi j} + \frac{R_1^*}{s - 2\pi j} = 2a \frac{s}{s^2 + (2\pi)^2} + 2b \frac{2\pi}{s^2 + (2\pi)^2}$$

which admits the inverse Laplace transform

$$2a \cos 2\pi t + 2b \sin 2\pi t$$

The overall  $y_0(t)$  is then given by

$$y_0(t) = 2a \cos 2\pi t + 2b \sin 2\pi t + R_{21}e^{-t} + R_{22}te^{-t}$$

## I Exercise 9

Being the matrix block diagonal, the eigenvalues are the union of the eigenvalues of

$$A_1 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \end{pmatrix}$$

that is  $\lambda_1 = 1$ ,  $\lambda_2 = i$  and  $\lambda_3 = \lambda_2^* = -i$  (the characteristic polynomial of  $A$  is  $p_A(\lambda) = (\lambda - 1)(\lambda^2 + 1)$ ). The natural modes are therefore

$$e^{\lambda_1 t} = e^t, \quad \sin t \quad (\text{or equivalently } \cos t)$$

## J Exercise 10

The characteristic polynomial is  $p_A(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3)$  and therefore  $\lambda_1 = -1$ ,  $\lambda_2 = -3$ . The eigenvector associated to  $\lambda_1 = -1$  is

$$u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or any parallel. the eigenvector associated to  $\lambda_2 = -3$  is

$$u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Choose  $U = \begin{pmatrix} u_1 & u_2 \end{pmatrix}$  so that

$$U^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

From the spectral form we obtain

$$\begin{aligned} x_{zi}(t) = e^{At}x(0) &= \left( e^{\lambda_1 t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T \right) x(0) \\ &= \left\{ e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \right\} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \left\{ e^{-t} \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \right\} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} \end{aligned}$$

while

$$y_{zi}(t) = C e^{At} x(0) = C x_{zi}(t) = e^{-t} + 3e^{-3t}$$

As an alternative, we can find the coefficients  $c_1 = v_1^T x_0$  and  $c_2 = v_2^T x_0$  which give the initial condition  $x_0$  in the  $(u_1, u_2)$  basis

$$x(0) = c_1 u_1 + c_2 u_2 = u_1 + u_2$$

and therefore, being  $v_i^T u_j = \delta_{ij}$ ,  $c_1$  and  $c_2$  scalars,

$$\begin{aligned} e^{At}x(0) &= \left( e^{\lambda_1 t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T \right) x(0) \\ &= \left( e^{\lambda_1 t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T \right) (c_1 u_1 + c_2 u_2) \\ &= c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 \end{aligned}$$

## K Exercise 11

The eigenvalues and associated eigenvectors are

$$\lambda_1 = -1 \rightarrow u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \lambda_2 = 1 \rightarrow u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The output zero-input response is therefore

$$\begin{aligned}
 y_{zi}(t) &= Ce^{At}x(0) = C \left\{ e^{\lambda_1 t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T \right\} x(0) \\
 &= \left\{ e^{\lambda_1 t} C u_1 v_1^T + e^{\lambda_2 t} C u_2 v_2^T \right\} x(0) \\
 &= \left\{ e^{\lambda_1 t} C u_1 v_1^T \right\} x(0)
 \end{aligned}$$

being  $Cu_2 = 0$  and therefore any initial condition solves the problem. This is understandable since  $Cu_2 = 0$  implies that the diverging natural mode  $e^{\lambda_2 t} = e^t$  is unobservable.

## L Exercise 12

The eigenvalues and associated eigenvectors are

$$\lambda_1 = 0 \rightarrow u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 5 \rightarrow u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

and therefore we have a constant natural mode (for  $\lambda_1 = 0$ ) and a diverging one (for  $\lambda_2 = 5$ ). In order for the diverging natural mode not to compare in the output free response, the initial state needs to belong to the eigenspace relative to  $\lambda_1 = 0$ , that is

$$x(0) = \alpha u_1 = \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix}$$

Since  $\lambda_1 = 0$  the output response from  $x(0)$  not only will be non diverging but also constant.