Some past exam problems in Control Systems - Part 3

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1) Consider the open-loop system

$$L(s) = \frac{K}{s(s+1)(s/10+1)}$$

1. Determine if and for which values of K the closed-loop system is asymptotically stable.
2. Determine the non-zero critical value of K_{crit} for which the closed-loop system moves from being asymptotically stable to unstable.
3. Draw the Bode plots with $K = K_{\text{crit}}$ (consistent with the knowledge that this is a critical value of the gain).
4. Draw the corresponding Nyquist plot and recall what this critical value implies in terms of the Nyquist plot.
5. Based upon the previous observations and plots, find the closed-loop poles for $K = K_{\text{crit}}$.

1 - Solution
Closed-loop pole polynomial

$$p(s,K) = \frac{1}{10}s^3 + \frac{11}{10}s^2 + s + K$$

K needs to be positive (necessary condition). From the Routh table (multiplying some rows by a positive number),

$$\begin{array}{|cc} 1/10 & 1 \\ 11/10 & K \\ (11\text{-}K) & \\ K & \end{array}$$

the closed-loop system is asymptotically stable for $0 < K < 11$. For $K > 11$ there are two changes of sign in the first column i.e. two roots with positive real part. This can also be seen through the root locus. Therefore the critical value is $K_{\text{crit}} = 11$.

Since for the closed-loop system, as K positive crosses the critical value, the number of poles with positive real part passes from zero to two, this corresponds

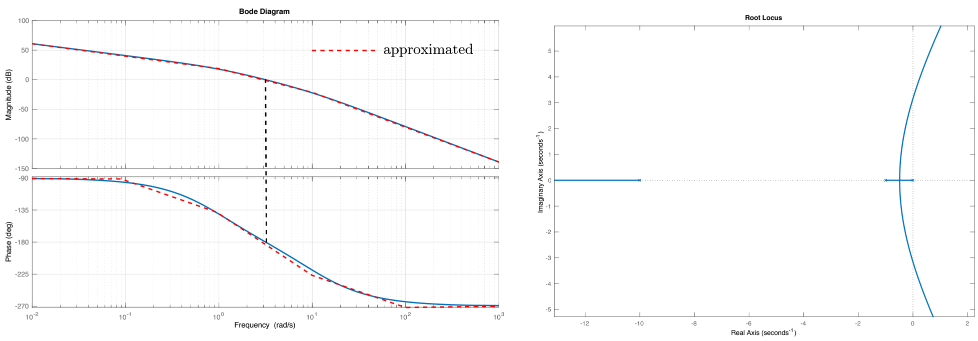

Figure 1: Bode diagram for $K = K_{\text{crit}}$ and root locus

to the Nyquist plot passing through the point $(-1, 0)$ and therefore the Bode plot will have zero gain and phase margin.

02) Let the plant dynamics be represented by

$$\begin{aligned}\dot{x}_1 &= 5x_1 + 8x_2 - 4u \\ \dot{x}_2 &= -3x_1 - 5x_2 + 2u \\ y &= x_2\end{aligned}$$

The state is not measurable.

1. Find a controller such that the output, at steady state, follows exactly the reference $r(t) = t\delta_{-1}(t)$ without being affected by a constant unknown disturbance d acting at the plant's output.

2. How does the closed-loop system react at steady state to the disturbance $d(t) = a\, t\delta_{-1}(t)$, with a unknown real constant?

2 - Solution summary

Matrices are

$$A_1 = \begin{pmatrix} 5 & 8 \\ -3 & -5 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

with characteristic polynomial

$$p_1(\lambda) = (\lambda + 1)(\lambda - 1)$$

and transfer function

$$P_1(s) = \frac{2}{s - 1}$$

and therefore we have stable hidden dynamics characterized by the eigenvalue $\lambda = -1$. For a reference of order 1, zero steady-state error requires 2 poles in $s = 0$ in the open loop which makes also the system astatic w.r.t. the given disturbance. Stability is guaranteed by adding to

$$\hat{P}(s) = \frac{2}{s^2(s - 1)}$$

- two negative zeros so to make $n - m = 1$, and then choose a sufficiently high gain
- or one negative zero to make $n-m = 2$ and a zero/pole pair with the pole sufficiently negative to move the center of asymptotes in the left half-plane, plus a sufficiently high gain.

Once the controller is found, the sensitivity function (characterizing the disturbance–output behavior) will have gain equal to zero since the two poles in $s = 0$ in the open loop will lead to two zeros in $s = 0$ in $S(s)$ which will be of the form

$$S(s) = \frac{s^2 S'(s)}{\#}$$

with $S'(0) \neq 0$ since $S'(s)$ does not have zeros in $s = 0$. Therefore the steady-state response to a ramp disturbance $d(t) = a\, t\delta_{-1}(t)$ is

$$y_{ss} = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sS(s)a\frac{1}{s^2} = \lim_{s\to 0} s\frac{s^2 \dots}{\#}a\frac{1}{s^2} = 0$$

Note that this is a special case in which the final value theorem is applicable since the rational function $sy(s)$ has all its roots at the denominator with real part strictly less than 0.

3) Give the state space representation (A, B, C, D) of a three dimensional system characterized by the only eigenvalue $\lambda = 0$ which, independently from the B and C matrices will give a transfer function with at most two poles. Is the system marginally stable? Motivate clearly your answer.

3 - Solution summary From the theory (slides) the dynamic matrix with eigenvalue $\lambda = 0$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with any B and C will provide at most 2 poles in $s = 0$ since the index (dimension of the largest Jordan block) of the eigenvalue $\lambda = 0$ is 2. Being the geometric multiplicity (equal to 2 since there are two Jordan blocks) different from the algebraic multiplicity (which is equal to 3) the system is unstable.

4) For the represented control scheme in Fig 2, find the transfer function from the reference $r(t)$ to the signal $e(t)$.

Assuming that the closed-loop system is asymptotically stable, does the presence of a pole in $s = 0$ in $C(s)$ guarantee that the output will exactly follow a constant reference $r(t) = r$?

04 - Solution summary) The error e will respond to a reference as

$$e(s) = S(s)r(s) - C_f(s)P(s)S(s)r(s) = (1 - C_f(s)P(s))S(s)r(s)$$

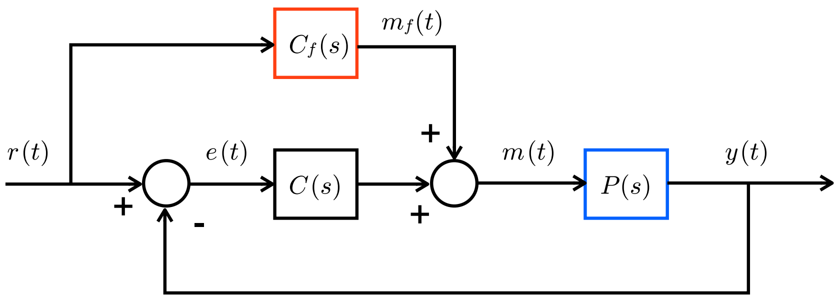

Figure 2: Control system.

Note that

$$\begin{aligned}\frac{e(s)}{r(s)} &= (1 - C_f(s)P(s))S(s) = \left(1 - C_f(s)\frac{N_P(s)}{D_P(s)}\right)\frac{1}{1+L(s)}\\ &= \frac{D_P(s) - C_f(s)N_P(s)}{D_P(s)}\frac{D_P(s)D_C(s)}{D_P(s)D_C(s) + N_P(s)N_C(s)}\\ &= \frac{(D_P(s) - C_f(s)N_P(s))D_C(s)}{D_P(s)D_C(s) + N_P(s)N_C(s)}\end{aligned}$$

From the final value theorem, the presence of a pole in $s = 0$ (i.e., $D_C(0) = 0$) in the controller guarantees that, if the steady state exists, for a constant reference the error will tend to 0 as $t \to \infty$ since the transfer function gain is 0.

05) Consider the plant shown in Fig. 3 with

$$P_1: \quad \dot{x}_1 = -x_1 + u \qquad\qquad P_2(s) = \frac{(s-1)^2}{(s+1)^2}$$
$$y = x_1$$

1. Determine the structure of a controller (and the corresponding control scheme) that guarantees asymptotic rejection, at the regulated output y_2, of any constant disturbance d.
2. Compute, for the control system, the transfer function from d to y_2.
3. Draw the root locus of the resulting loop function (both positive and negative)

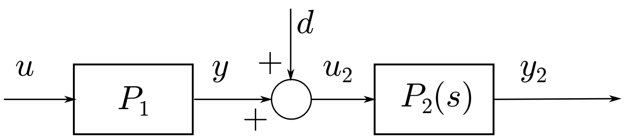

Figure 3: Plant

05 - Solution summary) The first system has transfer function

$$P_1(s) = c(s-a)^{-1}b + d = \frac{1}{s+1}$$

We need to add a pole in $s = 0$ in the controller to guarantee astatism in an output feedback control scheme. The temporary loop function becomes

$$L_a(s) = \frac{1}{s}P_2(s)P_1(s) = \frac{(s-1)^2}{s(s+1)^3}$$

The closed-loop pole polynomial is

$$p(s,K) = s^4+3s^3+3s^2+s+K(s^2-2s+1) = s^4+3s^3+(3+K)s^2+(1-2K)s+K$$

and the corresponding root locus is shown in Fig. 4. We see that there is an interval of positive values for the controller gain K which give closed-loop stability. The singular point candidates, different from the multiple open-loop zeros and poles, are given by the solutions

$$\frac{3}{s+1}+\frac{1}{s}-\frac{2}{s-1} = \frac{3s(s-1)+(s+1)(s-1)-2s(s+1)}{\dots} = \frac{2s^2-5s-1}{\dots} = 0$$

which are both real, one positive and one negative. The resulting root locus is shown in Fig. 4

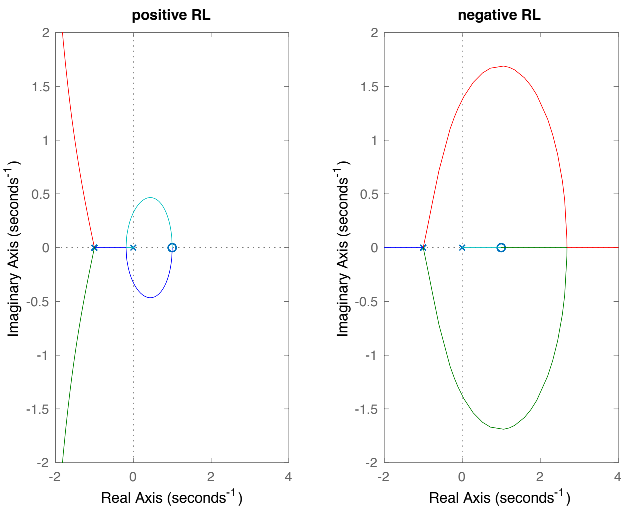

Figure 4: Root locus of problem 05

6) Consider the system represented by

$$\begin{aligned}\dot{x}_1 &= 0.8x_1 - 0.4x_2 - 4u\\ \dot{x}_2 &= -1.4x_1 - 1.8x_2 + 2u\\ y &= -0.6x_1 - 0.2x_2\end{aligned}$$

The state is not measurable. Stabilize the system.

06 - Solution summary) The system is characterized by eigenvalues 1 and -2 and is not fully controllable (while it is fully observable). The uncontrollable dynamics is given by the eigenvalue -2 (either you check with the PBH test or you do the Kalman decomposition w.r.t. controllability)

$$\text{rk}[O] = \text{rk}\left[\begin{pmatrix} -0.6 & -0.2 \\ -0.2 & 0.6 \end{pmatrix}\right] = 2, \qquad R = \begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix} \quad \Rightarrow \quad T^{-1} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

and the transfer function is

$$F(s) = \frac{2}{s-1}$$

which is stabilizable with a simple gain $K > 0.5$.

7) Let the plant be

$$P(s) = \frac{1}{s+10}$$

Find a controller which, simultaneously,

- is able to exactly track (or follow) at steady state a reference $r(t) = 10\,\delta_{-1}(t)$
- is able to reduce at steady state, by at least a factor of 10, the effect on the output of any sinusoidal disturbance $d(t) = d \sin \omega t$ acting on the output when ω belongs to the interval $[1, 10]$ rad/s.

07 - Solution summary) For the first requirement, we necessarily need a pole in $s = 0$. The resulting Bode plots of the actual loop and sensitivity function are shown in Fig. 5 (Top).

Recalling that an output disturbance affects the output, in closed loop, through the sensitivity function $S(s)$, guaranteeing a reduction at steady state by at least a factor of 10 means that the contribution of the disturbance to the output – at steady state the response to a sinusoidal input is a sinusoidal with the same frequency and with magnitude and phase depending on the frequency response at that frequency – should have a magnitude less than 1/10 (in decibels less than -20 dB) in the considered frequency range of the disturbance.

$$|S(j\omega)| \leq 1/10 \quad \Rightarrow \quad |S(j\omega)|_{dB} \leq -20 \text{ dB} \quad \text{for} \quad \omega \in [1, 10]$$

Moreover, considering the approximation of the sensitivity function in terms of the loop function, the previous specification can be met by requiring that the loop function magnitude should be greater than 20 dB in the same frequency range

$$|S(j\omega)|_{dB} \leq -20 \text{ dB} \quad \Rightarrow \quad |L(j\omega)|_{dB} \geq 20 \text{ dB} \quad \text{for} \quad \omega \in [1, 10]$$

To meet this requirement, we see that it is sufficient to add a gain of at least $60 + 3$ dB, that is of approximately $1000\sqrt{2}$, and therefore the final controller (pole in $s = 0$ and gain) is given by

$$C(s) = \frac{1000\sqrt{2}}{s}$$

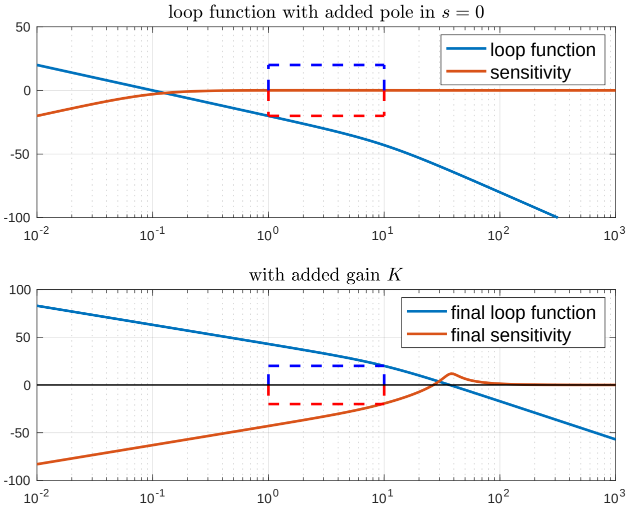

Figure 5: (Top) Bode magnitude plots of the loop and sensitivity functions after the introduction of the pole in $s = 0$, (Bottom) and after the additional gain.

The final magnitude plots are reported in Fig. 5 (Bottom) proving that the requirement has been successfully met.

8) Consider the control scheme illustrated in Fig. 6. Design the two controllers $K_1(s)$ and $K_2(s)$ in order to guarantee that at steady state the closed-loop system follows a constant reference with no error.

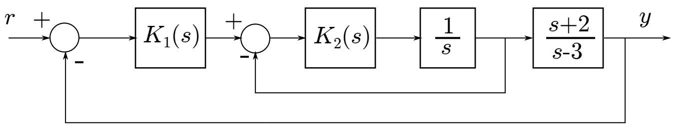

Figure 6: Control scheme exercise 8

08 - Solution summary) From the control scheme we understand that the inner loop will move the pole in $s = 0$ so it will not be useful for the Type 1 requirement. To fulfil this specification we need a pole in $s = 0$ in the first controller which now becomes

$$K_1(s) = \frac{1}{s} K_1'(s)$$

A possible solution could be to choose $K_2(s) = K_2$ pure gain in such a way that closing the inner loop will result in a pole in $s = -2$, that is

$$\text{inner loop} = \frac{K_2/s}{1 + K_2/s} = \frac{K_2}{s + K_2} \quad \rightarrow \quad K_2 = 2$$

This creates a cancellation of a stable pole in the series interconnection of the loop function which simplifies in

$$L(s) = K_1(s) \frac{2}{s+2} \frac{s+2}{s-3} = K_1'(s) \frac{2}{s(s-3)}$$

We need to choose $K_1'(s)$ in order to guarantee asymptotic stability of the closed loop. A simple zero/pole pair properly chosen and a gain (see root locus technique) solves the problem.

Otherwise, choose K_2 to stabilize inner loop, e.g., $K_2 = 1$ so that the open loop becomes

$$L(s) = K_1(s)\frac{1}{s+1}\frac{s+2}{s-3} = K_1'(s)\frac{s+2}{s(s+1)(s-3)}$$

again the same type of solution is possible.

9) Let $(\mathcal{S})$ be the parallel of the following two systems

$$(\mathcal{S}_1): \quad \dot{x}_1 = 2x_1 + u_1, \quad y_1 = 2x_1 \qquad (\mathcal{S}_2): \quad P_2(s) = \frac{s-1}{s(s-2)}$$

Can we stabilize $(\mathcal{S})$ with a state feedback? The answer should be clear and complete.

09 - Solution summary) Computing the transfer function of $\mathcal{S}_1$ we have

$$P_1(s) = \frac{2}{s-2}$$

and either we already notice that the two systems (we could have noticed before, of course) have the same eigenvalue and therefore theory says it will lead to a loss of controllability and observability or we compute the parallel as

$$P_1(s) + P_2(s) = \frac{3s-1}{s(s-2)}$$

and notice that the number of poles is 2 rather than the expected value of 3. We have a hidden dynamics which we can recognize to be uncontrollable and unobservable by interconnecting the two systems in state space with

$$P_2(s) = \frac{s-1}{s(s-2)} \quad \rightarrow \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

The interconnected system is

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}$$

Controllability (and observability) matrix has rank $2 < 3$.

10) Consider the interconnected system shown in Fig. 7 where

$$F(s) = \frac{-2}{s+1}, \qquad G(s) = \frac{1}{s+1}$$

When $r(t) = 0$ the system has just one input $d(t)$ and its state-space representation is

$$(\mathcal{S}): \ \dot{x}(t) = Ax(t) + Bd(t), \qquad y(t) = Cx(t)$$

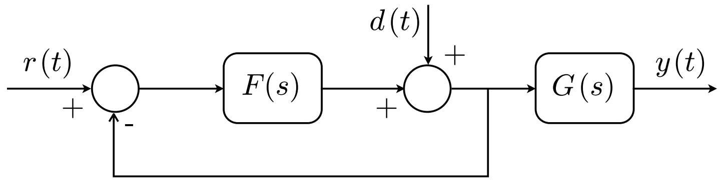

Figure 7: The interconnected system (which gives ($\mathcal{S}$) if $r(t) = 0$)

1. Find the state-space representation of the interconnected system ($\mathcal{S}$).
2. Study the observability and controllability property of ($\mathcal{S}$).
3. Compute the zero-state output response $y(t)$ to a unit step $d(t)$.

10 - Solution summary) When $r(t) = 0$, the loop transfer function from d to the input of $G(s)$ is given by

$$W_1(s) = \frac{1}{1+F(s)} = \frac{1}{1-\frac{2}{s+1}} = \frac{s+1}{s-1} = \frac{2}{s-1} + 1$$

System ($\mathcal{S}$) is therefore the series interconnection between $W_1(s)$ and $G(s)$; however note that there is a zero/pole cancellation when writing the final transfer function and therefore it is not possible, in order to obtain the overall state-space representation, to write the realization of the product of the transfer functions. We have to write the single realizations before interconnecting in series that is, defining as x_1 the state corresponding to $W_1(s)$ and x_2 to $G(s)$,

$$\begin{aligned} \dot{x}_1 &= x_1 + d, \qquad & y_1 &= 2x_1 + d \\ \dot{x}_2 &= -x_2 + y_1, \qquad & y &= x_2 \end{aligned}$$

where y_1 denotes the output of $W_1(s)$ and therefore also the input of $G(s)$. We thus obtain

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} d, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

We know there is the creation of hidden dynamics characterized by the eigenvalue $\lambda = -1$ so there is for sure loss of controllability and/or observability (from the theory we know that a zero–pole cancellation in a series interconnection leads to uncontrollable hidden dynamics). The controllability and observability matrices have rank

$$\text{rk}[P] = \text{rk}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1 < 2 = n, \qquad \text{rk}[O] = \text{rk}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = 2 = n$$

which confirms what we know. The final transfer function from d to y is

$$W(s) = W_1(s)G(s) = \frac{s+1}{s-1}\frac{1}{s+1} = \frac{1}{s-1}$$

and therefore the step response from d is

$$y(s) = \frac{1}{s-1}\frac{1}{s} = \frac{-1}{s} + \frac{1}{s-1} \quad \rightarrow \quad y(t) = -\delta_{-1}(t) + e^t\delta_{-1}(t)$$