Some past exam problems in Control Systems - Part 2

January, 2018 - Updated 30/12/2019

01) (Dominant dynamics) Let the system $\mathcal{S}$ and its low frequency approximation based on dominant dynamics $\mathcal{S}_\mathrm{a}$ be given by

$$\mathcal{S}: \quad F(s) = \frac{100}{(s+1)(0.1s+1)^2} \qquad\qquad \mathcal{S}_\mathrm{a}: \quad F_\mathrm{a}(s) = \frac{100}{(s+1)}$$

We want to compare the two systems in terms of closed-loop stability.

- Compare the Bode diagrams and the related Nyquist plots.
- Verify the previous conclusions referred to stability with any other technique you prefer.

01 - Solution summary) From the Bode diagrams and corresponding Nyquist plots we note that $F_\mathrm{a}(j\omega)$, for $\omega \geq 0$ lies all in the fourth quadrant and has an infinite gain margin while $F(j\omega)$ moves from the fourth to the third and ends in the second quadrant (as ω increases from 0 to ∞) thus crossing the negative real axis. From the Bode plots, we note that the phase margin for $F(j\omega)$ is negative and thus the closed loop system is unstable (the phase of $F(j\omega)$ in $\omega = 10$ rad/sec is $-\pi$ and thus, being the crossover frequency at higher frequency and the phase decreasig, the phase margin is negative). The closed loop system relative to the dominant dynamics approximation is asymptotically stable. The result can be easily verified with the Routh criterion. (See slides "Control Basics II", section on "Unmodeled dynamics")

02) Let the plant and controller be given by

$$P(s) = \frac{1}{s+1} \qquad C(s) = \frac{K_c}{s}$$

Choose the value of K_c such that the following <u>both</u> requirements are met

- a sinusoidal disturbance $\sin(\omega t)$ acting on the output of the plant with $\omega \in [0, 0.1]$ rad/s is attenuated by at least 18 dB
- a sinusoidal measurement noise $\sin(\omega t)$ acting on the feedback loop with $\omega \in [10, 100]$ rad/s is attenuated by at least 38 dB.

Reasoning based on approximate relations is accepted.

02 - Solution summary) The first specification requires the magnitude of the sensitivity function to be smaller than -18 dB in the $[0, 0.1]$ rad/s frequency range. Using the known approximation, this is equivalent to requiring that the magnitude of the loop function should be greater than 18 dB in the same frequency range. The specification on the measurement noise translates into asking that the complementary sensitivity magnitude should be smaller than -38 dB in the frequency range $[10, 100]$ rad/s and thus, through the approximation, that the loop function should have a magnitude smaller than -38 dB in the same frequency range. Choosing $K_c = 1$ solves the problem. Note that, for the obtained loop function the location of the crossover frequency is consistent with the approximations used and that the resulting closed loop is asymptotically stable.

3) Let the plant be $\dot{x} = x + u$ with output $y = 3x$.

- Design a static output feedback stabilizing controller $C_1(s) = k_c$. Let p be the closed-loop resulting pole (choose a numeric value).
- Design an alternative (dynamic) controller based on the separation principle. Choose the state feedback so to assign p, while the estimation error has dynamics given by the eigenvalue p_o.

We now want to compare the two alternative solutions to the output stabilization problem.

- Let a reference r and a measurement noise n be present as illustrated in the scheme, for both schemes compare the behaviour $r \to y$ and $n \to y$.
- Any suggestion on how to choose the desired eigenvalues in the separation-based control scheme?

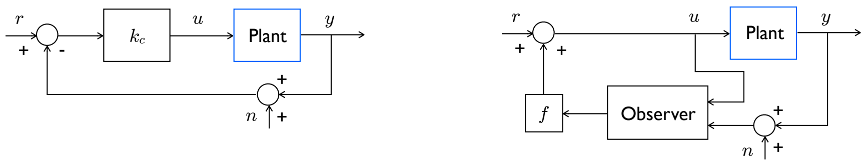

Figure 1: The two alternative choices

03 - Solution summary) The plant, with state space representation $(a, b, c, d) = (1, 1, 3, 0)$, has transfer function $P(s) = c\,b/(s-a) = 3/(s-1)$ then any $k_c > 1/3$ will stabilize the closed loop system (left figure in Fig. 1) which takes on the form (k_c is chosen so that $a - b\,k_c\,c = 1 - 3k_c = p$)

$$\dot{x} = (a - b\,k_c\,c)x + b\,k_c\,r - b\,k_c\,n$$
$$y = c\,x$$

Being the system one-dimensional (and clearly being the system observable and controllable), the state feedback and observer design are trivial. For the state feedback, choose f such that $a + b\,f = 1 + f = p$, while for the observation error dynamics choose k such that $a - k\,c = 1 - 3k = p_o$. The resulting closed loop system is given by (refer to the right figure in Fig. 1)

$$\begin{aligned}\dot{x} &= ax + b\,f\xi + b\,r \\ \dot{\xi} &= k\,c\,x + (a + b\,f - k\,c)\xi + b\,r + k\,n \\ y &= c\,x\end{aligned}$$

From the theory (complementary sensitivity function) it is known that the reference and measurement noise affect the output in a similar way for the scheme on the left of Fig. 1; this is represented by the complementary sensitivity function

$$T(s) = \frac{b\,k_c\,c}{s - a + b\,k_c\,c} = \frac{a-p}{s-p}$$

For the observer-based scheme on the right, we have two different transfer functions

$$\begin{aligned}W_{ry}(s) &= \begin{pmatrix} c & 0 \end{pmatrix} \begin{pmatrix} s-a & -b\,f \\ -k\,c & s-a-b\,f+k\,c \end{pmatrix}^{-1} \begin{pmatrix} b \\ b \end{pmatrix} = \frac{b\,c(s-a+k\,c)}{(s-a-b\,f)(s-a+k\,c)} \\ &= \frac{b\,c}{(s-a-b\,f)} = \frac{b\,c}{(s-p)}\end{aligned}$$

while

$$\begin{aligned}W_{ny}(s) &= \begin{pmatrix} c & 0 \end{pmatrix} \begin{pmatrix} s-a & -b\,f \\ -k\,c & s-a-b\,f+k\,c \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ k \end{pmatrix} = \frac{b\,c\,k\,f}{(s-a-b\,f)(s-a+k\,c)} \\ &= \frac{b\,c\,k\,f}{(s-p)(s-p_o)}\end{aligned}$$

Comparing the two transfer functions we see that the observer has a filtering role w.r.t. the measuring noise (clear from the figure) and thus we have an indication on how to choose the parameter k (and thus p_o) which affects the reconstruction error dynamics.

The given solution is clearly longer than usual since the full symbolic case has been treated.

4) Let the plant be

$$P(s) = \frac{1}{s + 0.01}$$

Find a control scheme which ensures that

- a reference $r(t) = \delta_{-1}(t)$ is tracked with an absolute error smaller than 10^{-4}
- a constant disturbance acting on the input of the plant does not alter the output steady-state behaviour
- a crossover frequency of 1 rad/s and a phase margin of at least 30°.

04 - Solution summary) One of the simplest loop shaping exercise. The specification on the disturbance requires a pole in $s = 0$ in the controller which makes the closed loop system of type 1 and therefore the steady state error to

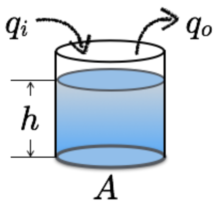

Figure 2: Liquid tank

a constant reference, if the control system is asymptotically stable, will be zero. Then the first requirement is also automatically satisfied. The extended (or modified) plant has a crossover frequency which already meets the specifications ($\omega_c = 1$ rad/sec) while the phase margin is almost zero. We need to increase the phase in $\omega = 1$ rad/sec and compensate for any amplification introduced by the chosen lead function. This can be achieved with a controller gain smaller than one in magnitude since there are no particular requirements on the positive loop gain.

05) Let the cylinder tank shown in Fig. 2 have a constant section of area $A = 1$ m^2, with a liquid of volume $V(t)$ and level $h(t)$. The incoming and outgoing flow volume flow rates are respectively denoted by $q_i(t)$ and $q_o(t)$. The system dynamics is represented by the following differential equation

$$\dot{V}(t) = q_i(t) - q_o(t)$$

Let the liquid level $h(t)$ inside the tank be measurable and $q_i(t)$ be the control input.

- Write both the state-space representation and the transfer function of the considered system
- Design a control scheme which generates the control input $q_i(t)$ such that, at steady-state, the liquid is guaranteed to stay at a desired constant reference value h_d in spite of a constant unknown disturbance $q_o(t) = q_o$. Explain in details your control scheme and the variables involved.
- Show with a root locus the behaviour of your control scheme when the controller gain varies (N.B. the overall controller is not necessarily static).
- Draw the three approximate sensitivity functions and comment the obtained plots.
- Show closed loop stability through the Nyquist criterion.

04 - Solution summary) The state space model is obtained by noting that $V(t) = A\,h(t)$ (see slides "Control Basics I") and being the output $y = h$, the scalar system is represented by $(a, b, c, d) = (0, 1/A, 1, 0)$. The transfer function relating the plant's input $q_i(t) - q_o(t)$ to the output h is therefore $P(s) = 1/(A\,s)$. Since the constant unknown disturbance q_o enters at the plant's input, the pole in $s = 0$ in the plant is useless w.r.t. the disturbance and we need to introduce a pole in $s = 0$ in the controller. This, provided the closed loop system is asymptotically stable, also guarantees that at steady state the output will tend to the constant desired reference value h_d (type 1 system). The extended (or modified) plant has two poles at the origin and thus the closed loop system is clearly not asymptotically stable (the Nyquist plot goes through the point $-1, 0)$). A negative zero (since the controller already has a pole) or a lead function will guarantee a positive margin. The value of the (positive) controller gain will determine the crossover frequency and the corresponding phase margin. Depending upon the chosen stabilizing solution the root locus as the controller gain varies will be different. One should explain what can be inferred by the root locus in terms of transient behavior. The three approximate sensitivity functions and the Nyquist plot are standard questions.

5) (Double integrator) Let the plant be

$$P(s) = \frac{1}{s^2}$$

- Find a state space realisation of the plant.
- Assuming the state is measurable, find a stabilising controller.
- Assuming only the output is measurable, find, if it exists, a state observer and, if possible, use it for stabilising the plant. Give the closed-loop state-space representation.

05 - Solution summary) A state space realisation is given by the controller canonical form with $D = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Being the system controllable (by definition) the state feedback is directly obtained by the Ackermann's formula. The system is also (by definition since we started from the transfer function) observable and therefore the determination of the observer is just a direct application of the course slides formulas.

6) Let a system be characterised by the following dynamic matrix

$$A = \begin{pmatrix} -6 & 8 \\ -3 & 4 \end{pmatrix}$$

- Can the given system have, for some initial condition, a constant state evolution? If yes, motivate your answer and show your solution.
- Does the result change if we just want a constant output trajectory (discuss in terms of the output choice)?

06 - Solution summary) The A matrix has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$. Provided the initial condition does not excite the converging mode $e^{\lambda_2 t}$, the state free evolution will just remain constant. This can be achieved by any initial condition belonging to the eigenspace associated to λ_1, that is parallel to $u_1 = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ (see slides "Internal Stability - LTI systems"). The output matrix C can be chosen so to make the converging mode unobservable (use, for example, the PBH test to determine such a C). A possible choice is $C = \begin{pmatrix} -1 & 2 \end{pmatrix}$. In this case any initial condition will give a constant output free response.