Loop shaping exercise

Excerpt[1] from "Controlli Automatici - Esercizi di Sintesi", L. Lanari, G. Oriolo, EUROMA - La Goliardica, 1997. It's a generic book with some typical problems in control, not a collection of exam problems.

Warning - All the computation have been done using numerical tools in order to show the exact values and reasoning. An exam exercise will be, from the numerical point of view, much simpler.

Exercise

Given the control system of Fig. 1, with the plant having the following transfer function

$$P(s) = \frac{10}{(1+0.1s)(1+0.01s)}.$$

Find the controller $G(s)$ such that the following requirements are met:

a) *the steady state error to a step reference $r(t)$ is zero;*

b) *the steady state error to a unit ramp reference $r(t) = t\delta_{-1}(t)$ is, in absolute value, smaller equal to 0.01;*

c) *the steady state output response to a constant disturbance $d(t)$ is zero;*

d) *the open loop crossover frequency is $\omega_c^* \approx 30$ rad/sec with a corresponding phase margin $PM^* \geq 20°$. [These are almost equivalent requirements, on the closed loop, of a bandwidth close to 50 rad/sec and a resonance peak not greater than 9 dB.]*

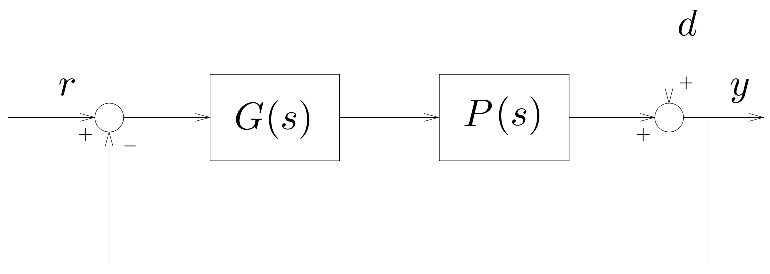

Figure 1: The given control system

How will the design be modified if **d)** *is changed into:*

[1] Free online version available at
`http://www.diag.uniroma1.it/~lanari/FdA9/FdA9MatDid/LibroEs_2pps.pdf`

d') *the open loop crossover frequency is* $\omega_c^* \approx 7$ *rad/sec with a corresponding phase margin* $PM^* \geq 40°$*? [These are almost equivalent requirements, on the closed loop, of a bandwidth close to* 10 *rad/sec and a resonance peak not greater than* 4 *dB.]*

or

d") $\omega_c^* \approx 30$ *rad/sec with a corresponding phase margin* $PM^* \geq 55°$*? (*$B_3 \approx$ 50 *rad/sec and* $M_r \leq 2$ *dB).*

We seek the compensator $G(s)$ in the general form

$$G(s) = \frac{K_G}{s^h} R(s).$$

Parameters K_G and h are chosen such that the steady state requirements are guaranteed (just necessary conditions), while with $R(s)$ we meet the dynamic specifications. $R(s)$ is usually chosen to have unit gain[2] so that the controller gain coincides with K_G. The closed loop stability will be guaranteed, for the considered problem, by a direct application of Bode's theorem on stability.

Notice first that $P(s)$ has no poles at the origin (i.e. in $s = 0$), therefore, in order to obtain a zero steady state error corresponding to a constant reference, we need to introduce a pole in $s = 0$ ($h = 1$) in the controller and thus in the open loop function (being a unit feedback system, the open loop function coincides with the loop function). The control system (closed loop system) then becomes of type 1.

If the reference is a unit ramp, the non-zero steady state error (after the introduction of the pole at the origin) will be

$$|\tilde{e}_1| = \frac{1}{|K_P K_G|} = \frac{1}{10|K_G|} \leq 0.01 \quad \Longrightarrow \quad |K_G| \geq 10,$$

since the plant's gain is $K_p = 10$. Choosing a negative K_G would lead to a negative loop gain and thus not allowing the application of Bode's stability theorem. Moreover, a negative loop gain would make stabilization much harder. We therefore choose $K_G = 10$.

The introduction of the pole at the origin also renders the control system astatic w.r.t. to any constant disturbance d acting at the plant's output; therefore the requirement is already satisfied.

Coming to **d)**, the open loop system $F(j\omega)$ is

$$F(j\omega) = P(j\omega)G(j\omega) = \frac{K_G}{(j\omega)^h} P(j\omega)R(j\omega) = \hat{F}(j\omega)R(j\omega),$$

where $\hat{F}(j\omega)$ denotes the *modified plant* (to satisfy the steady state requirements **a)**, **b)** and **c)**) frequency response corresponding to the transfer function

$$\hat{F}(s) = \frac{10}{s} \frac{10}{(1 + 0.1s)(1 + 0.01s)}.$$

[2] In general steady state specifications may require a loop gain greater, in absolute value, than a minimum value depending on the maximum allowed error. In this case, when choosing K_G so satisfy these inequality constraints with the smallest possible value (in absolute value), we could still consider an increase in the overall loop gain (by choosing $R(0) \geq 1$) but not a decrease and therefore the gain $R(0)$ cannot be chosen smaller than one.

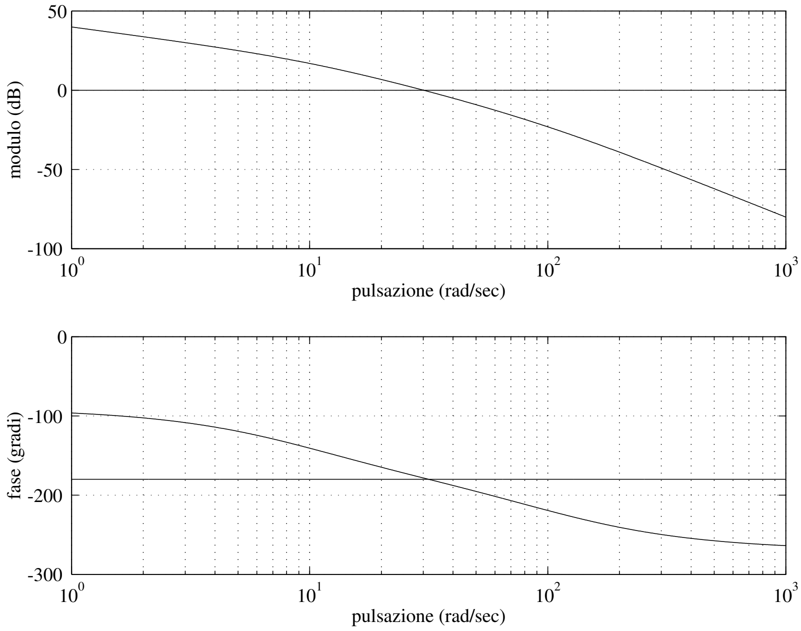

Figure 2: Bode diagrams of $\hat{F}(j\omega)$

From the Bode's diagram of Fig. 2, the current crossover frequency $\hat{\omega}_c$ and phase margin $\widehat{PM}$ of $\hat{F}(j\omega)$ are numerically found to be

$$\hat{\omega}_t \approx 30 \text{ rad/sec} \quad \text{and} \quad \widehat{PM} \approx 1.6°.$$

Note that the closed loop system is very close to the critical point $(-1, 0)$ (phase margin almost zero) and thus, for the considered system, very close to instability.

Comparing the required values of the crossover frequency and corresponding phase margin, we note that the actual crossover frequency coincides with the desired one and thus the only required action is to increase the phase in 30 rad/sec. In particular it is necessary to increase the phase by $20 - 1.6 = 18.4°$ trying to maintain unchanged the crossover frequency (i.e. trying to introduce no change in the magnitude at that frequency).

We can use a *lead function*

$$R_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}, \qquad \tau_a > 0,\ m_a > 1,$$

which has the typical Bode diagrams reported in Fig. 3 for different values of m_a. In these diagrams, defined *universal diagrams*, the abscissa represents a normalized frequency $\omega\tau_a$. Note that the zero binomial factor has the cut off frequency in $1/\tau_a$, while the pole binomial factor has the cut off frequency in m_a/τ_a. Therefore in the diagrams of Fig. 3, the normalized frequency $10^0 = 1$ corresponds the zero cut off frequency $1/\tau_a$.

In this case the choice of the parameters is driven by the following considerations. The lead action should be such that it increases the phase by 18.4° at the frequency $\omega_c^* = \hat{\omega}_c = 30$ rad/sec. From the universal diagrams of Fig. 3, it can be seen that this lead can be achieved by several pairs $(m_a, \omega\tau_a)$. However the resulting amplifications are quite different. Since we want to maintain the crossover frequency as close as possible to the actual value, we need to introduce as little amplification as possible. We therefore need to choose the smallest normalized frequency $\omega\tau_a$ which guarantees, for an appropriate value of m_a, the required lead.

For our case this means choosing $\omega\tau_a = 0.4$ and $m_a = 10$. In $\omega\tau_a = 0.4$, the magnitude plot for $m_a = 10$ shows a very limited amplification and therefore a reduced change in the crossover frequency ω_c. This is extremely important since a large change in the crossover frequency would require a different lead action and at a different frequency location.

We still need to choose τ_a which corresponds to placing the required lead at the required frequency ω_c^*, that is

$$\omega_c^* \tau_a = 30\,\tau_a = 0.4 \qquad \Longrightarrow \qquad \tau_a = \frac{0.4}{30} = \frac{1}{75}.$$

The final lead function is

$$R_a(s) = \frac{1 + \frac{1}{75}s}{1 + \frac{1}{750}s},$$

and therefore the final controller is

$$G(s) = \frac{K_G}{s}\, R_a(s) = \frac{10}{s}\,\frac{1 + \frac{1}{75}s}{1 + \frac{1}{750}s}.$$

The open loop (or loop function in this case) $F(j\omega) = \hat{F}(j\omega)R_a(j\omega)$ Bode diagrams are reported in Fig. 4 and, for a wider frequency range, in Fig. 5. Note that, as expected, the amplification introduced by the lead function has slightly moved the crossover frequency from 30 rad/sec to approximately 31 rad/sec, while the phase margin (computed at the true crossover frequency 31 rad/sec) has increased to more than 20°.

Finally, having a unique crossover frequency and having guaranteed a positive loop gain, since the resulting phase margin is positive and the open loop system (including the controller) has no poles with positive real part, the closed loop is asymptotically stable. Although not explicitly required, asymptotic stability of the control system is an implicit binding requirement since steady state exists only for asymptotic stable systems.

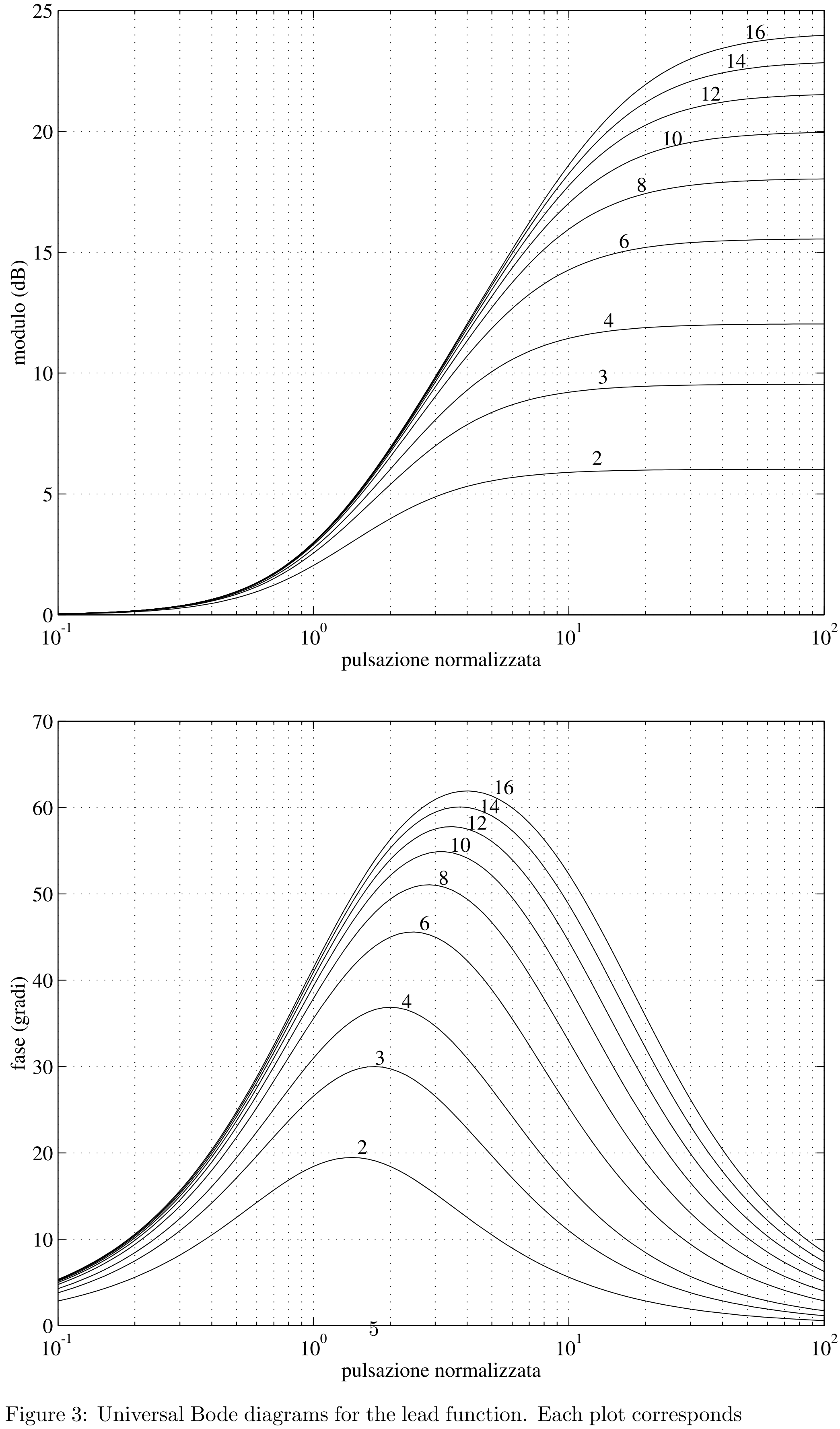

Figure 3: Universal Bode diagrams for the lead function. Each plot corresponds to a value of m_a. The same plots can be used for the lag function by changing the sign on the ordinate values.

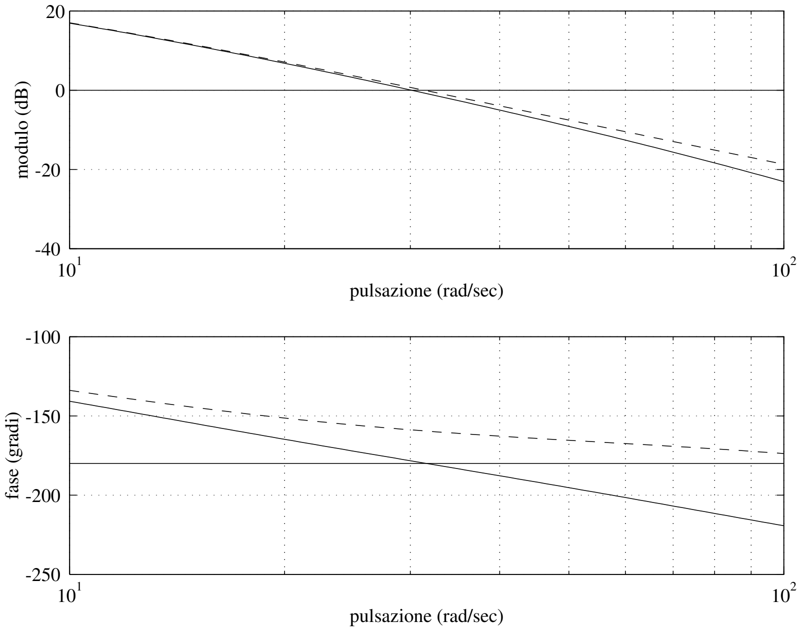

Figure 4: Bode diagrams of $F(j\omega)$ ($--$) and $\hat{F}(j\omega)$ (—)

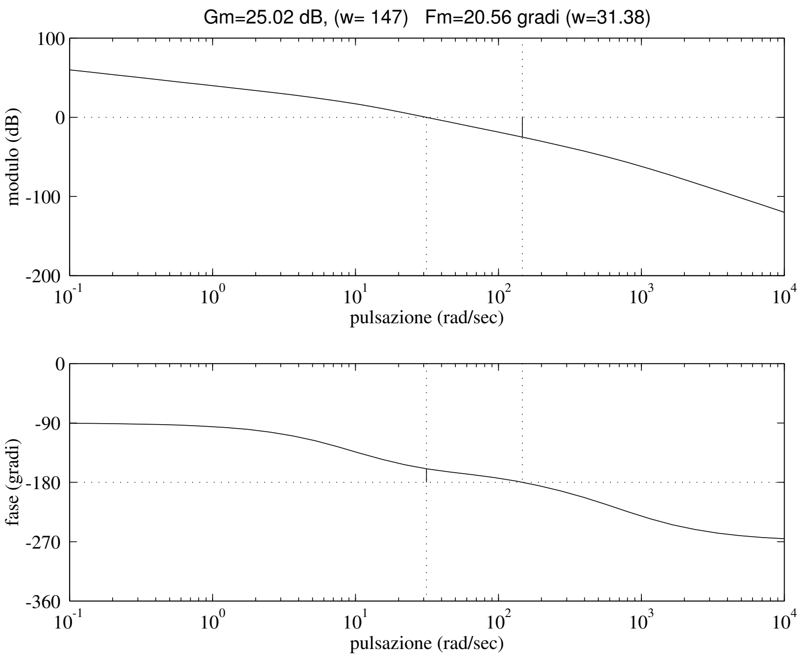

Figure 5: Bode diagrams of $F(j\omega)$. The resulting gain and phase margin are reported in the plot title together with the corresponding frequencies.

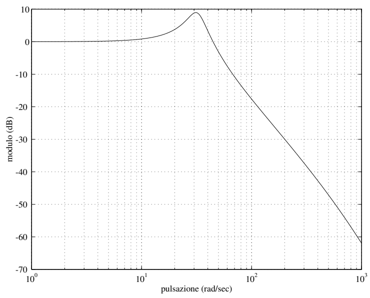

Figure 6: Magnitude in dB of $T(j\omega)$

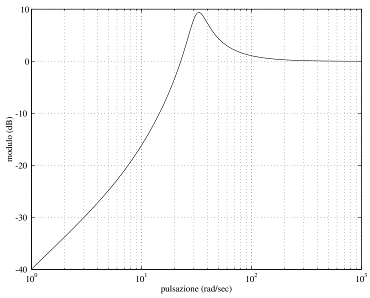

Figure 7: Magnitude in dB of $S(j\omega)$

Since the original specifications (bandwidth close to 50 rad/sec, resonance peak not greater than 9 dB) were on the closed loop system (control system), the complementary sensitivity function frequency response

$$T(j\omega) = \frac{F(j\omega)}{1 + F(j\omega)} = \frac{R_a(j\omega)\hat{F}(j\omega)}{1 + R_a(j\omega)\hat{F}(j\omega)},$$

is shown in Fig. 6 where the resulting resonance peak is smaller than 9 dB and the bandwidth is approximately 50 rad/sec as requested. For completeness the magnitude of the sensitivity function $S(s)$ frequency response is also reported in Fig. 7. As expected, the pole at the origin in the forward path and before the entry point of the disturbance translates into the presence of a zero at the origin in the sensitivity function. The magnitude of $S(j\omega)$ in dB comes from $-\infty$ when ω increases from 0..

If instead of solving **d)** we have to solve the specifications **d')**, we first compare the actual magnitude and phase of the modified plant $\hat{F}(j\omega)$ at the new desired crossover frequency. From the plot in Fig. 2 we have

$$|\hat{F}(j\omega_c^*)| = 21 \text{ dB}, \qquad \angle \hat{F}(j\omega_c^*) = -128°$$

and therefore we need an attenuation of 21 dB while we can accept a maximum delay of $140 - 128 = 12°$.

Attenuation cannot be achieved by introducing an extra smaller than one (negative in dB) gain since this would violate the steady state requirement of **b)**. A lag function

$$R_i(s) = \frac{1 + \frac{\tau_i}{m_i} s}{1 + \tau_i s}, \qquad \tau_i > 0,\ m_i > 1,$$

is therefore required. The corresponding Bode diagrams can be obtained from those of Fig. 3 by changing the sign of the ordinates. The pole binomial term has a cut off frequency $1/\tau_i$ while the one corresponding to the zero is m_i/τ_i. In the universal diagrams of Fig. 3, when dealing with a lag function, the normalized frequency $10^0 = 1$ corresponds to the pole cut off frequency.

In order to determine the values of the lag function parameters m_i and τ_i, we first notice that the required attenuation is of 21 dB at the desired crossover frequency $\omega_c^* = 7$ rad/sec. From the universal diagrams of Fig. 3 we see that this attenuation can be achieved for several pairs $(m_i, \omega\tau_i)$ which give different phase lags. In order to make this lag small it is useful to choose the normalized frequency sufficiently to the right in the diagrams. For example, values of $\omega\tau_i$ close to 10^2 are reasonable. However choosing an excessively large normalized frequency corresponds to introducing the attenuation earlier in the frequency range and thus attenuating at low frequency where it is not strictly necessary. It is always advisable, when possible, to avoid excessive attenuation at low frequency.

We choose $m_i = 12$ and we note that, at the normalized frequency $\omega\tau_i = 60$, we also introduce a phase lag of $12°$ which is the maximum allowed lag. To place this desired magnitude and phase effect at the desired crossover frequency ω_c^*, we choose

$$\omega_c^*\,\tau_i = 7\,\tau_i = 60 \qquad \Longrightarrow \qquad \tau_i = \frac{60}{7}.$$

The lag function is thus

$$R_i(s) = \frac{1 + \frac{5}{7} s}{1 + \frac{60}{7} s},$$

while the final compensator is

$$G'(s) = \frac{K_G}{s} R_i(s) = \frac{10}{s} \frac{1 + \frac{5}{7} s}{1 + \frac{60}{7} s}.$$

Again, stability is guaranteed by Bode's stability theorem.

To see the effect of this design, the open loop frequency response

$$F'(j\omega) = \hat{F}(j\omega) R_i(j\omega),$$

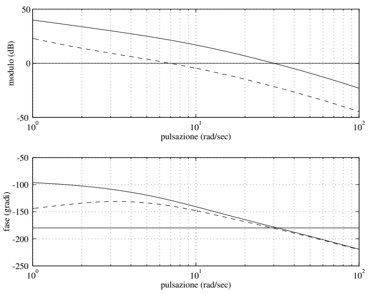

Figure 8: Magnitude of $F'(j\omega)$ ($--$) and $\hat{F}(j\omega)$ (—)

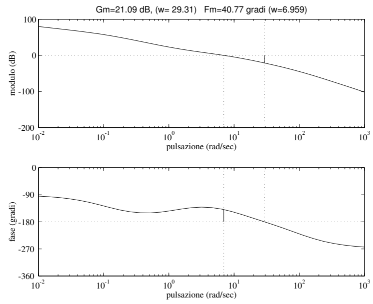

Figure 9: Magnitude of $F'(j\omega)$ in a larger frequency range

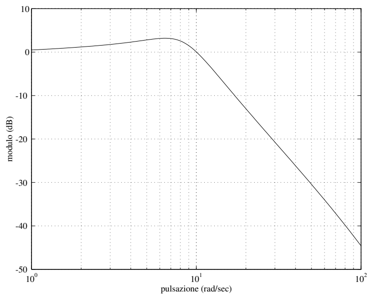

Figure 10: Magnitude of $W'(j\omega)$

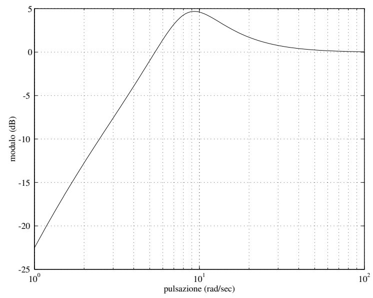

Figure 11: Magnitude of $S'(j\omega)$

Bode diagrams are shown in Fig. 8 and, for a wider frequency range, in Fig. 9. In particular we notice that the lag function has brought the crossover frequency in $\omega_c = 7$ rad/sec with a phase margin greater than $40°$, as requested.

The effect of the chosen controller needs to be verified on the closed loop behavior. The complementary sensitivity function $T'(j\omega)$ magnitude is shown in Fig. 10. The resonance peak M_r is smaller than 4 dB, while the bandwidth B_3 is approximately 11 rad/sec. The resulting sensitivity function $S'(j\omega)$ magnitude is plotted in Fig. 11 and shows how the low frequency behavior has worsen w.r.t. Fig. 7 due to the lag function.

For the case **d")** we again have $\omega_c^* = \hat{\omega}_c = 30$ rad/sec, while the minimum desired phase margin is $PM^* = 55°$. It is therefore necessary to achieve a phase lead of 53.4° at $\hat{\omega}_c$. From the universal diagrams in Fig. 3 it is clear that any lead function allowing such an increase in the phase will also introduce a large amplification at the required frequency. We therefore also need to introduce a lag function to attenuate such an amplification and bring back the crossover frequency in ω_c^*.

From Fig. 3 we observe that a lead function with $m_a = 11$ gives a phase lead greater than 56° at the normalized frequency $\omega\tau_a = 3$. Since we want to achieve this phase increase in $\omega_c^* = 30$ rad/sec, we set $\tau_a = 3/30$ and therefore

$$R_a''(s) = \frac{1 + \frac{1}{10}s}{1 + \frac{1}{110}s}.$$

The introduction of the lead function is shown in Fig. 12. Note that in ω_c^* the function $\hat{F}(j\omega)R_a''(j\omega)$ has an approximate magnitude of 9 dB and phase 58°. It should be reminded that the choice of the lead function has been made in order to introduce a greater lead than necessary so to compensate the future lag introduced by the lag function.

The second step requires an attenuation of 9 dB in ω_c^*. We then choose a lag function with $m_i = 3$ and $\omega\tau_i = 60$, from which $\tau_i = 60/30$. The resulting lag function is

$$R_i''(s) = \frac{1 + \frac{2}{3}s}{1 + 2s}.$$

The overall compensator is given by

$$G''(s) = \frac{10}{s} \frac{1 + \frac{1}{10}s}{1 + \frac{1}{110}s} \frac{1 + \frac{2}{3}s}{1 + 2s}.$$

The Bode diagrams of the compensated plant

$$F''(j\omega) = \hat{F}(j\omega)R_a''(j\omega)R_i''(j\omega)$$

are shown in Fig. 12 and, for a wider frequency range, in Fig. 13. The resulting phase margin is larger than the required 55° at a crossover frequency which is slightly smaller than 31 rad/sec.

For completeness, the Bode diagrams of the closed loop frequency response $W''(j\omega)$ is shown in Fig. 14. As required, the resonance peak is smaller than 2 dB and the bandwidth is around 50 rad/sec.

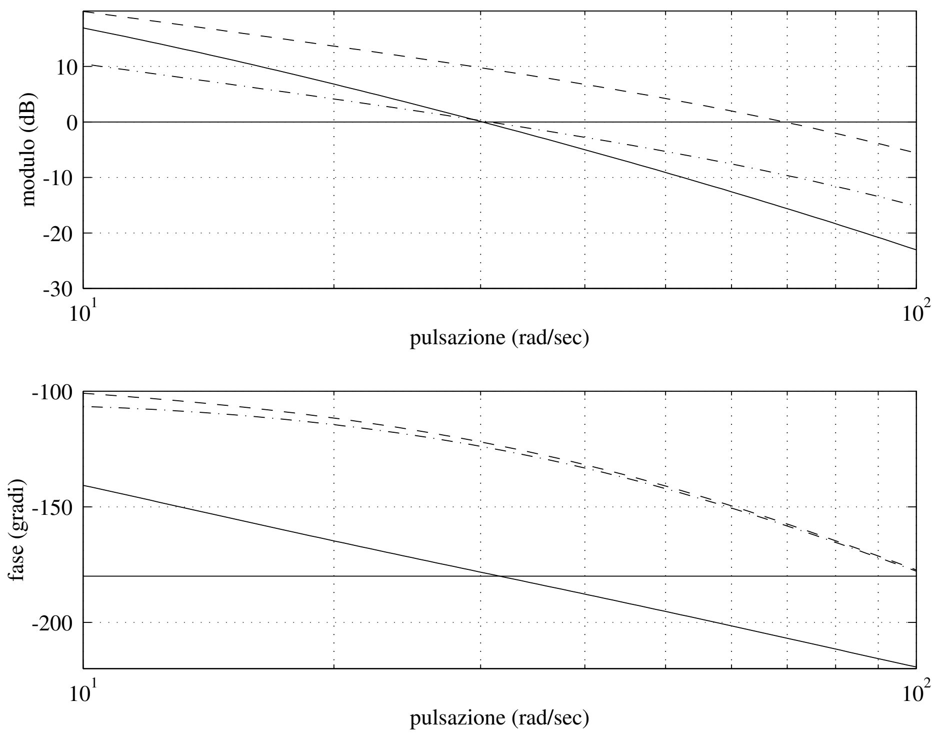

Figure 12: Bode diagrams of $\hat{F}(j\omega)$ (—), of $\hat{F}(j\omega)R''_a(j\omega)$ (−−) and of $F''(j\omega) = \hat{F}(j\omega)R''_a(j\omega)R''_i(j\omega)$ (−·)

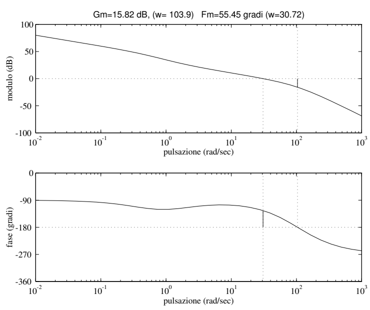

Figure 13: Bode diagrams of $F''(j\omega)$

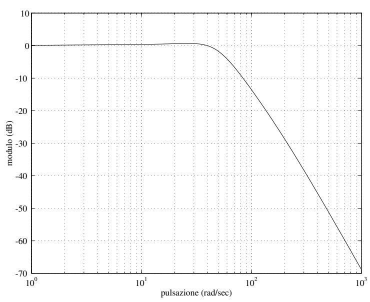

Figure 14: Magnitude of $W''(j\omega)$