Control Systems - February 12, 2025

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1) For the system characterized by the state space representation

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & a-1 \end{pmatrix}, \quad D = 0$$

which depends on the parameter $a \in \mathbb{R}$

1. Compute, if possible, a stabilizing state feedback $u = Fx$.
2. Discuss, in terms of the real parameter a, whether the system is stabilizable through output feedback.
3. If a value of a exists for which the system is not stabilizable through output feedback, what would you expect to happen if we use the separation principle in this case?

2) Consider the interconnected system $\mathcal{S}$ shown in Fig. 1 with

$$F(s) = \frac{s}{(s+10)^2}, \qquad G(s) = \frac{1}{s+10}$$

Design a control scheme and give the final controller $C(s)$ that will guarantee a steady state error ≤ 0.01 for the reference $r(t) = t \cdot \delta_{-1}(t)$, a crossover frequency $\omega_c \approx 10$ rad/sec and a phase margin $PM \geq 30°$.

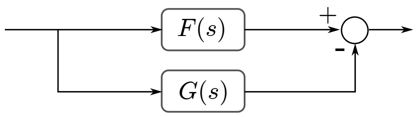


Figure 1: Interconnected system $\mathcal{S}$

3) Consider the plant represented by the following transfer function

$$P(s) = \frac{s+2}{s(s-2)}$$

1. Design a control scheme of *minimal dimension* (i.e., its realization has the smallest state dimension possible) which guarantees that the output will converge at steady state at a given constant reference in spite of the presence of an unknown constant disturbance d acting at the plant's input.
2. Verify the closed loop stability through the Nyquist criterion

1 - Sol.) The system has the following controllability matrix

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \qquad \text{rank}(P) = 2 \qquad \Rightarrow \qquad Im(P) = \text{gen}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = \text{gen}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The controllable subsystem has dimension 2; a change of coordinates T is

$$T^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and the resulting Kalman decomposition is

$$\widetilde{A} = TAT^{-1} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \widetilde{B} = TB = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \widetilde{C} = CT^{-1} = \begin{pmatrix} 1 & 0 & a-1 \end{pmatrix}$$

The system is characterized by the pair of imaginary eigenvalues $\pm j$ (controllable) and -1 (uncontrollable); now the eigenvalues are much easier to compute with respect to the original matrix A (the eigenvalues are invariant to change of coordinates). Other possible changes of coordinates are possible but the controllable and uncontrollable subsystems will always be characterized by the eigenvalues, respectively, $\pm j$ and -1. The uncontrollable subsystem is asymptotically stable so the system is stabilizable through state feedback.

Let us assign, for example, the eigenvalues $\lambda_1^* = -1$ and $\lambda_2^* = -2$ to the dynamic matrix $\widetilde{A}_{11}$ of the controllable subsystem $(\widetilde{A}_{11}, \widetilde{B}_1)$. We have the following desired closed loop polynomial

$$p^*(\lambda) = (\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$$

and therefore, being,

$$P_{11}^{-1} = \begin{pmatrix} \widetilde{B}_1 & \widetilde{A}_{11}\widetilde{B}_1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \qquad p^*(\widetilde{A}_{11}) = \widetilde{A}_{11}^2 + 3\widetilde{A}_{11} + 2I_{2\times 2} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

we finally have

$$\widetilde{F}_1 = -\gamma p^*(\widetilde{A}_{11}) = -\begin{pmatrix} -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \end{pmatrix} \qquad \rightarrow \qquad F = \widetilde{F}T = \begin{pmatrix} \widetilde{F}_1 & 0 \end{pmatrix} T = \begin{pmatrix} -2 & -1 & 0 \end{pmatrix}$$

As a check, we have

$$\text{eig}\,(A+BF) = \text{eig}\begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & -2 \\ -1 & 0 & -1 \end{pmatrix} = \{-1, -1, -2\}$$

For the system to be stabilizable by output feedback, we need to check for which values of a the system is <u>also</u> detectable. The observability matrix is

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & a-1 \\ a-1 & a & 1-a \\ 1 & 0 & -1-a \end{pmatrix} \qquad \rightarrow \qquad \det[\mathcal{O}] = -2a^2$$

so the system for sure will be observable and therefore detectable for all non-zero a. For $a = 0$ the observability matrix becomes

$$\mathcal{O}_{a=0} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \qquad \rightarrow \qquad \text{Ker}(\mathcal{O}_{a=0}) = \text{gen}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Since the dimension of the unobservable subsystem is 2 and it cannot be characterized by the eigenvalues $(-1, j)$ or $(-1, j)$ since for complex eigenvalues the natural modes are associated to the pair $(+j, -j)$, it follows that the unobservable subsystem is characterized by the eigenvalues $\pm j$ and thus is not asymptotically stable. Thus, the system when $a = 0$ is not detectable.

If we try to build an asymptotic observer when $a = 0$, the reconstruction error $e(t)$ will be governed by the equation

$$\dot{e} = (A - KC)e, \qquad e(0) = e_0$$

Since $A - KC$ in the case $a = 0$ will still have the eigenvalues $\pm j$, starting from from a non-zero reconstruction error e_0 the reconstruction error $e(t)$ will have oscillating components which will not converge to zero and therefore the observer variables will never converge to the original state. Moreover, if we try to apply the feedback $u = F\xi$ from the observed/reconstructed variables, the plant state will not converge to zero since it will be forced by a signal which does not go to zero; recall that the closed loop system is governed by the equations

$$\dot{x} = (A + BF)x + BFe \quad (1)$$
$$\dot{e} = (A - KC)e \quad (2)$$

This shows that, although no input is applied to the closed loop system (free evolution), the state (x, e) will not converge to zero asymptotically (not all the closed loop eigenvalues have negative real part).

Typical errors:

- Computing the eigenvalues of the 3×3 dynamic matrix A was not straightforward since A has no particular structure and the characteristic polynomial was third order with no evident roots. Therefore one could find the eigenvalues after the Kalman decomposition (on an upper block triangular matrix) since the sytem was not fully controllable (from the direct rank test).
- Also doing the PBH test was not so evident (and requires the knowledge of the eigenvalues).
- As always, if after the change of coordinates for the Kalman decomposition w.r.t. controllability you do not have the expected structure for $\widetilde{A} = TAT^{-1}$ and $\widetilde{B} = TB$, there was an error.

2 - Sol.) The interconnected system is characterized by the transfer function (parallel between $F(s)$ and $-G(s)$)

$$P(s) = F(s) - G(s) = \frac{s}{(s+10)^2} - \frac{1}{s+10} = \frac{-10}{(s+10)^2} = -\frac{0.1}{(1+0.1s)^2}$$

with gain $K_p = -0.1$. Note that the interconnected system has dimension 3 (sum of the dimensions of each subsystem) while we have only two poles in the resulting transfer function; there is an asymptotically stable hidden dynamics which is characterized by the eigenvalue -10 (the parallel interconnection does not change the eigenvalues). This asymptotically stable hidden dynamics does not influence the solution since the specifications are on the Input/Output behavior. In order to guarantee a closed loop type 1 system with error smaller than 0.01, we need to (necessary condition) add a pole in $s = 0$ in the controller and guarantee that

$$\frac{1}{|K_L|} = \frac{1}{|K_c K_p|} \le 0.01 \qquad \rightarrow \qquad |K_c| \ge 1000 \qquad \rightarrow \qquad K_c = -1000$$

The modified plant is

$$\widehat{P}(s) = \frac{100}{s(1+0.01s)^2}$$

From its Bode plots we have

$$|\widehat{P}(j10)| = 14\text{dB}, \qquad \angle\widehat{P}(j10) = -180^\circ$$

Therefore lead and lag combined actions will solve the problem (see theory); for example $m_a = 8$ and $\omega\tau = 1$ gives 38° and 2.5 dB, the lag $m_i = 7$ tih $\omega\tau = 100$ will attenuate by -16.5 dB and introduce a lag of 4° so the final phase margin will be 34° at the desired crossover frequency.

Typical errors:

- Quite amazing that so many students still make errors doing this simple calculation

$$\frac{s}{(s+10)^2} - \frac{1}{s+10}$$

- Many (**too many**) errors in solving this inequality

$$\frac{1}{|K_c K_p|} \le 0.01$$

  or determining the correct plant's gain K_p.

3 - Sol.) The necessary condition for astatism requires a pole in $s = 0$ in the controller (pole in $s = 0$ before the entry point of the disturbance on the forward path). The temporary loop function is

$$\widehat{L}(s) = \frac{s+2}{s^2(s-2)}$$

which is minimum phase, has $n - m = 2$ and a center of asymptotes $s_0 = 2$. This in general implies that we cannot stabilize the system with high positive gain (large values of the controller K); however, for small values of K the closed loop poles will be close to the open loop ones and since one open loop pole is positive, it is unlikely that for some values of K the closed loop system will be asymptotically stable. More directly, looking at the resulting closed loop polynomial with $C(s) = K/s$, one has

$$p_{\text{temp}}(s, K) = s^2(s - 2) + K(s + 2) = s^3 - 2s^2 + Ks + 2K$$

We notice that the necessary condition for having all the roots with real negative part is not satisfied. Being the system minimum-phase, as an alternative if we add a negative zero $(s+z)$ (with $z > 0$) we obtain $n - m = 1$ and the closed system will for sure be asymptotically stable for high values of the gain K. In this case the closed loop polynomial becomes

$$p(s, K) = s^2(s - 2) + K(s + 2)(s + z) = s^3 + (K - 2)s^2 + K(2 + z)s + 2Kz$$

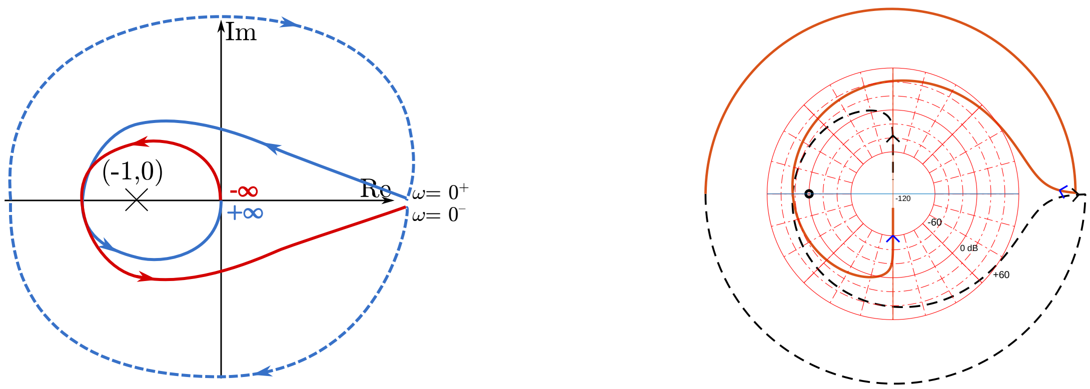
The necessary conditions require $K > 2$ and $z > 0$. From the Routh table

$$\left|\begin{array}{cc} 1 & K(2+z \\ K-2 & sKz \\ K(Kz + 2K - 4z - 4) & \\ 2Kz & \end{array}\right.$$

we have the following condition

$$K(z + 2) \geq 4(z + 1) \qquad \rightarrow \qquad K \geq 4\frac{z + 1}{z + 2}, \qquad \text{with} \quad z > 0$$

The Nyquist plot (asymptotically stable closed loop case) is shown in Fig. 2.



(a) Nyquist plot by hand

(b) Nyquist plot in log scale

Figure 2: Problem 3 - Nyquist plot (here $z = 5$ and $K = 50$).

Typical errors:

- The presence of the pole in $s = 0$ in the controller is only a necessary condition; stability must be guaranteed.
- There is no need to proceed to building the Rout table of

  $$p_{\text{temp}}(s, K) = s^2(s - 2) + K(s + 2) = s^3 - 2s^2 + Ks + 2K$$

  since the necessary condition is not met.
- Some have drawn the Nyquist plot with $C(s) = K/s$; this is possible in order to verify that the system is not stabilizable but it is clearly a loss of time.
- When you realize that the controller $C(s) = K/s$ is not sufficient to stabilize the closed loop system you should find a stabilizing controller otherwise there is no steady state and the temporary controller is useless.