Control Systems - January 30, 2023

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1) Consider the system

$$P(s) = \frac{(s-10)^2}{(s^2+100)(s+10)}$$

and a static controller $C(s) = K$.

1. Establish if it is possible to stabilize the given plant with the static controller $C(s) = K$ (in a unit feedback interconnection).
2. Draw the corresponding root locus.
3. Determine the values of the gain K for which some closed loop poles (if any) cross the imaginary axis.
4. Confirm the stability analysis (as a function of K) through the Nyquist criterion. All possible situations should be discussed as a function of K. The associated Bode plots are required.

2) Consider the dynamic system characterized by the matrices

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \qquad D = 0.$$

1. Find the state feedback law $u = Fx$ which assigns coincident eigenvalues to the closed loop.
2. Is it possible to obtain the same result with a static controller $C(s) = K_c$ via output feedback? If yes, determine the value of K_c.

3) Consider the dynamic system characterized by the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \qquad D = 0.$$

1. Compute the (output) impulse response.
2. Study the system stability.

4) Consider the plant $P(s) = \dfrac{10}{s}$. Determine, a controller $C_1(s)$ in a feedback control scheme such, that at steady state,

1. a constant unknown disturbance affecting the plant's input is not affecting the output;
2. the reference $r(t) = -t\delta_{-1}(t)$ is perfectly followed by the controlled output.

Determine an alternative controller $C_2(s)$ which guarantees a quicker convergence to zero of the tracking error with respect to $C_1(s)$.

Possible useful numbers for some problem: $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$, $\sqrt{5} \approx 2.24$, $\sqrt{7} \approx 2.64$, $\sqrt{11} \approx 3.31$.

1 - Sol.) The system $P(s)$ is non-minimum phase and therefore, although it has $n - m = 1$, it cannot be stabilized by high gain (however this does not mean it cannot be stabilized with a different controller). The closed loop pole polynomial is given by

$$p(s, K) = (s^2 + 100)(s + 10) + K(s - 10)^2 = s^3 + (K + 10)s^2 + (100 - 20K)s + 100(K + 10)$$

and the corresponding Routh table is

$$\begin{array}{|cc} 1 & 100 - 20K \\ K + 10 & 100(K + 10) \\ \alpha & \\ K + 10 & \end{array}$$

with

$$\alpha = -\frac{1}{K + 10}(100(K + 10) - (K + 10)(100 - 20K)) = -20K$$

and so the closed loop system is asymptotically for $-10 < K < 0$.

The number of sign changes in the first column of the Routh table as a function of K is reported in Table 1.

		-10		0	
1	$+$		$+$		$+$
$K + 10$	-		$+$		$+$
$-20K$	$+$		$+$		-
$K + 10$	-		$+$		$+$
# sign changes	3		0		2

Table 1: Sign changes in first column of the Routh table as a function of K

We notice that when K increases and crosses -10 all three poles move from the right to the left half-plane and therefore, quite surprisingly, simultaneously cross the imaginary axis. Clearly, being two open loop poles on the imaginary axis, when K further increases and crosses $K = 0$ two closed loop poles go to the right half-plane.

The true root locus is shown in Fig. 1 (left) where we notice that there is a small part of the negative root locus which enters the left half-plane corresponding to small (since we are approaching the open loop poles) values of $|K|$. This is consistent with the interval of values for K which guarantee closed loop asymptotic stability.

An alternative and compatible root locus is also shown (right) if the singular points are not computed explicitly as in this case since this would mean solving the third order polynomial

$$\frac{1}{s + 10j} + \frac{1}{s - 10j} + \frac{1}{s + 10} - \frac{2}{s - 10j} = \frac{s^3 - 30s^2 - 300s - 3000}{\cdots} = 0$$

The solutions $(39.51, -4.76 \pm 7.3j)$ are still candidate singular points (except for the real solution) and not compatible with the right plot.

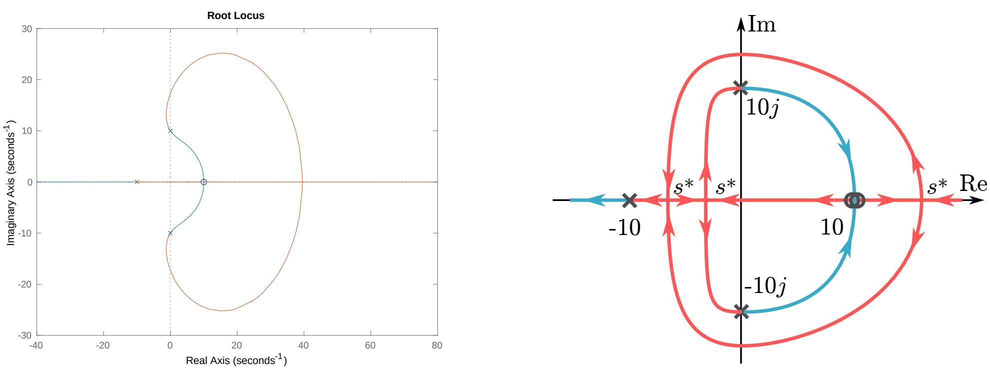

Figure 1: Exercise 1: Root locus: true (left) - alternative (right)

The Bode canonical form of $KP(s)$ is

$$KP(s) = \frac{K}{10}\frac{(1 - s/10)^2}{(1 + s^2/100)(1 + s/10)}.$$

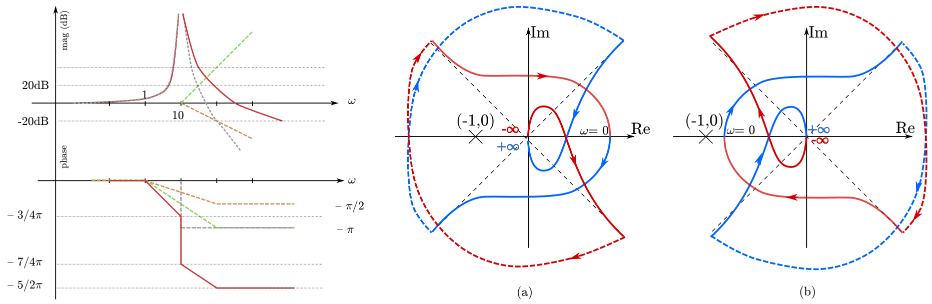

Figure 2: Exercise 1: Bode plots for $K = 10$ and Nyquist plot (a) for $K = 10$ – (b) for $-10 < K < 0$.

The Bode and Nyquist plots are shown in Fig. 2; the closed loop stability is confirmed when $-10 < K < 0$ since $N_{cc} = 0 = n_L^+$.

Typical errors:

- A root locus that is not giving the same result as the analysis through the Routh criterion should trigger an alert.
- When putting the transfer function or the frequency response in the Bode canonical form the term $(s-10)^2$ is erroneously written as $-100(1-s/10)^2$ instead of $100(1-s/10)^2$.
- In the necessary condition, the inequality $100 - 20K > 0$ has been frequently wrongly rewritten (it's an inequality, these errors should not happen).
- Plotting the Nyquist plot seems to be still a problem, in particular the closure at infinity.

2 - Sol.) Due to the triangular structure of the matrix A, the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 1$ and $\lambda_3 = -1$. Let's first verify if the problem can be solved by checking the controllability of the system. Computing the controllability matrix P it turns out that it is singular ($\det(P) = 0$) and its rank is 2. Therefore there exists a one-dimensional uncontrollable subsystem. We need to check if this uncontrollale subsystem is asymptotically stable (the system is therefore stabilizable with state feedback) or not. We could check controllability of the unstable eigenvalue $\lambda_2 = 1$ using the PBH test, however it will be useful to do the decomposition and therefore we compute the image of P.

$$P = \begin{pmatrix} -1 & 3 & -9 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \mathrm{Im}[P] = \mathrm{gen}\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\} = \mathrm{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Note that T^{-1} can be chosen as the identity matrix (and therefore $T^{-1} = I = T$). In other words the system is already in Kalman's controllability canonical form and the uncontrollable subsystem is characterized by the eigenvalue $\lambda_3 = -1$. We could have already noticed from the structure of the matrices A and B that for sure the eigenvalue $\lambda_3 = -1$ is uncontrollable and since the uncontrollable subsystem has dimension 1, for sure the other two eigenvalues are controllable. The controllable subsystem is given by

$$\mathcal{S}_c: \qquad A_{11} = \begin{pmatrix} -3 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Since we want to assign coincident eigenvalues and the eigenvalue $\lambda_3 = -1$ is not affected by any state feedback, we should assign the remaining eigenvalues also as $\lambda_1^* = \lambda_2^* = -1$, i.e.,

$$p^*(\lambda) = (\lambda+1)^2 = \lambda^2 + 2\lambda + 1 \qquad p^*(A_{11}) = A_{11}^2 + 2A_{11} + I_{2\times 2} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

The controllable subsystem controllability matrix is

$$P_c = [B_1, A_{11}B_1] = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad P_c^{-1} = \begin{pmatrix} -1 & -3 \\ 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \gamma_c = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

so that the state feedback F_c is

$$F_c = -\gamma_c\, p^*(A_{11}) = \begin{pmatrix} 0 & 4 \end{pmatrix}$$

and in the original coordinates, being $T = I_{3\times 3}$

$$F = \begin{pmatrix} F_c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \end{pmatrix}$$

When, instead, we do an output feedback with the static controller $C(s) = K_c$, we should first recall that the system $\mathcal{S}$ has one hidden (uncontrollable) dynamics characterized by the eigenvalue $\lambda_3 = -1$ and this eigenvalue is not affected by the output feedback. Since the system is observable being

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \qquad \det[\mathcal{O}] = 4 \neq 0$$

the poles of $\mathcal{S}$ are given by the eigenvalues of A_{11} and the transfer matrix is given by

$$P(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} (sI_{2\times 2} - A_{11})^{-1} B_1 = \frac{-1}{(s-1)(s+3)}$$

where we have taken the first two components of the C matrix since the state of the controllable subsystem is given by the first two components of the state. We now have that the two poles (and thus eigenvalues) of the closed loop system, as a function of K_c, will vary as indicated by the root locus. In particular, setting $K = -K_c$ we can trace the root locus of $KP(s)$ with pole polynomial

$$p(s, K) = (s-1)(s+3) + K = s^2 + 2s + K - 3 \qquad \text{or} \qquad p(s, K_c) = s^2 + 2s - K_c - 3$$

which clearly has a singular point between the two open loop poles -3 and 1 corresponding to a positive value of K (or negative of K_c). Since it is required to obtain the same result as with the state feedback, the question is if there exists a value of K_c (or K) which gives two closed loop poles coincident in -1, or, in other words, if there is a singular point in -1. The candidate singular points equation is

$$\frac{1}{s-1} + \frac{1}{s+3} = 0 \qquad \Leftrightarrow \qquad 2s + 2 = 0 \qquad \Leftrightarrow \qquad s^* = -1$$

so $s^* = -1$ is a singular point and the corresponding value of $K = 4$ (or $K_c = -4$) is obtained solving

$$p(s^*, K) = 0$$

It is therefore possible with $K_c = -4$ to obtain all the closed loop eigenvalues coincident in -1 via output feedback. We could have directly checked if there was a K_c such that

$$p(s, K_c) = s^2 + 2s - K_c - 3 = (s+1)^2.$$

Alternatively, setting $u = -K_c y = -K_c Cx$ in the state equation $\dot{x} = Ax + Bu$ we have the closed loop system under static output feedback

$$\dot{x} = (A - BK_cC)x$$

For our system

$$A - BK_cC = \begin{pmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} - K_c \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & K_c & K_c \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which, due to the block triangular structure, has the obvious eigenvalue in -1 (the uncontrollable one) and the eigenvalues of

$$\bar{A} = \begin{pmatrix} -3 & K_c \\ 1 & 1 \end{pmatrix} \qquad \Leftrightarrow \qquad p_{\bar{A}}(\lambda) = \det\left[\begin{pmatrix} \lambda+3 & -K_c \\ -1 & \lambda-1 \end{pmatrix}\right] = \lambda^2 + 2\lambda - 3 - K_c$$

so we have to choose $K_c = -4$ to have the remaining two eigenvalues coincident in -1.

Although it is not, a priori, in the allowed class of controllers (static controller) and therefore this solution is not suggested, it is interesting to see what happens if we look for a dynamic controller which tries to solve the same problem, i.e., assigns the poles in -1. Since the plant's transfer function is

$$P(s) = \frac{-1}{(s-1)(s+3)}$$

the controller should be of dimension 1 ($r = n - 1$) and of the form

$$C(s) = \frac{as + b}{s + c}$$

The resulting closed loop has the denominator

$$d_{\mathrm{CL}}(s) = (s-1)(s+3)(s+c) - (as+b) = s^3 + s^2(c+2) + s(2c-a-3) - 3c - b$$

and if we assign three $(r+n)$ poles in -1, the desired closed loop pole polynomial is

$$d^*_{\mathrm{CL}}(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

Equating the coefficients and solving, we find $a = -4$, $b = -4$ and $c = 1$, that is the following controller

$$C(s) = \frac{-4s-4}{s+1} = -4$$

which is the required static controller.

Typical errors:

- Choosing a complicated base for the $\mathrm{Im}[P]$ clearly makes the computation longer and more prone to errors.
- Building an observer would lead to a dynamic controller, i.e., a controller which has a state.
- In order to find the transfer function, we should take advantage of the knowledge that there is an uncontrollable subsystem; this simplifies the computation greatly with respect to computing the full $C(sI-A)^{-1}B$.
- Recall that in order, for example, to compute A^2B for the controllability matrix, it is easier and faster to compute it using the previous result AB; in other words compute it as $A^2B = A.AB$ instead of computing A^2 and then multiply by B.
- When you compute the inverse M^{-1} of a matrix M, always check that it satisfies $M.M^{-1} = I$.
- Read the text, the goal was to obtain a closed loop system with coincident eigenvalues, both with a state feedback and a static output feedback.

3 - Sol.) To compute the impulse response $w(t) = Ce^{At}B$ we need to compute the matrix exponential which takes on the form (being $A^2 = 0$)

$$e^{At} = I_{3\times3} + At = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore we have

$$w(t) = Ce^{At}B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -1 \quad \text{or better} \quad = -\delta_{-1}(t)$$

We could also have computed the impulse response as

$$w(t) = \mathcal{L}^{-1}\left[C(sI-A)^{-1}B + D\right]$$

that is, having

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{pmatrix}^{-1} = \frac{1}{s^3}\begin{pmatrix} s^2 & 0 & 0 \\ 0 & s^2 & s \\ 0 & 0 & s^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & 0 & 0 \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{pmatrix} \qquad \text{or directly} \quad = \mathcal{L}\left[e^{At}\right]$$

we finally obtain

$$w(t) = \mathcal{L}^{-1}\left[-\frac{1}{s}\right] = -\delta_{-1}(t)$$

From the presence of the natural mode t we clearly understand that the system is unstable.
Alternatively, having three coincident eigenvalues $\lambda_1 = 0$ and therefore algebraic multiplicity $m_{\mathrm{a}} = 3$, we can compute the geometric multiplicity by searching for the kernel of $A - \lambda_1 I_{3\times3}$

$$\mathrm{Ker}\,(A-\lambda_1 I_{3\times3}) = \mathrm{Ker}\,(A) = \mathrm{gen}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} \quad \Rightarrow \quad \dim\left(\mathrm{Ker}\,(A-\lambda_1 I_{3\times3})\right) = 2 \quad \Rightarrow \quad m_{\mathrm{g}} = 2.$$

Recall that, since the rank of $A - \lambda_1 I_{3\times 3} = A$ is 1, then we could have directly concluded through the Rank-Nullity theorem that the nullspace has dimension $3 - 1 = 2$. Since $m_{\rm g} < m_{\rm a}$ and the eigenvalue has zero real part, the system is unstable.

Of course we could have also directly recognized the presence of 2 Jordan blocks (and therefore the geometric multiplicity is 2) corresponding to the eigenvalue $\lambda_1 = 0$, of dimension 1 and 2

$$A = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

.

Typical errors:

- Stability is an internal property of the state evolution and therefore looking at the transfer function or the impulse response – an Input/Output characterization of the system behavior – is not sufficient to establish the stability of the system.
- In this case we cannot use the spectral form to compute the matrix exponential (coincident eigenvalues).
- Some have studied the stability of the closed loop system (the system in a unit feedback) but this was not the question.

4 - Sol.) The first requirement is astatism w.r.t. a constant disturbance acting at the plant's input and this requires the presence of a pole in $s = 0$ before the entry point of the disturbance, that is in the controller. Since the plant has already a pole in $s = 0$, now the closed loop system (with the additional pole of the controller) is of type 2 and therefore, if asymptotically stable, will also guarantee a zero steady state error w.r.t. to a reference input $r(t) = t\delta_{-1}(t)$ (order 1) which solves the second point provided we have closed loop asymptotic stability. From the simple Bode plots of the modified plant

$$\hat{P}(s) = \frac{10}{s^2},$$

since the phase is constantly equal to $-\pi$, the phase margin is equal to zero for any crossover frequency. We are in a situation where the Bode stability theorem can be applied. We can proceed, for stabilization, either with the loop shaping or with the root locus.

- **Loop shaping -** We can choose a lead function (any would do) to increase the phase in a given interval. For example let's choose $m_a = 10$ and $\tau_a = 1$ leading to

  $$R_a(s) = \frac{1+s}{1+0.1s}$$

  There is not even the necessity to change the controller gain since the obtained crossover frequency guarantees a positive phase margin so that the overall controller is

  $$C_1(s) = \frac{1+s}{s(1+0.1s)}$$

  and stability is guaranteed by Bode's stability theorem.
- **Root locus -** We can equivalently obtain a simple controller reasoning on the root locus of

  $$\hat{P}(s) = K\frac{10}{s^2} = \frac{K'}{s^2}$$

  where we have set $K' = 10K$. Since the modified plant $\hat{P}(s)$ is characterized by $n - m = 2$ and a center of asymptotes $s_o = 0$, we can simply move the center of asymptotes choosing the zero/pole pair for example as $(-1, -10)$ so that the new center of asymptotes is in $s'_o = -4.5$. In this case we have a probable root locus (with no singular points) but the computation of the singular points

  $$\frac{2}{s} + \frac{1}{s+10} - \frac{1}{s+1} = 0 \qquad \Leftrightarrow \qquad 2(s+1)(s+10) + s(s+1) - s(s+10) = 2s^2 + 13s + 20 = 0$$

  which has the two real – and thus singular points – solutions $s_1^* = -4$ and $s_2^* = -2.5$, shows that the real root locus is different. A different choice of the zero/pole pair (for example $(-1, -4)$) gives the simpler root locus. In any case, we see that after the addition of the zero/pole pair the closed loop system is asymptotically stable for any positive K' and thus also for any positive K. So the controller, choosing $K = 1$, is

  $$C_1(s) = \frac{s+1}{s(s+10)}$$

- **Third alternative -** Having already introduced a pole in our controller, we can also think of making the loop function having $n - m = 1$ by adding a negative zero so that the loop function remains minimum phase. Then we know that for sure there exists a sufficiently high positive gain K_c that will stabilize the closed loop system. Choosing for example a zero in $s = -1$, we have the following proper controller

$$C(s) = \frac{K_c(s+1)}{s}$$

and the corresponding closed loop polynomial

$$p(s, K_c) = s^2 + K_c s + K_c$$

The closed loop system is asymptotically stable for $K_c > 0$.

In order to speed up the closed loop system response, we can increase the closed loop bandwidth (from the reference to the output, i.e., of the complementary sensitivity function) by increasing the crossover frequency of the loop function. This can be obtained by increasing the controller gain but in the first case we need to be more careful (see Fig. 3). In the third case we can also choose a zero more distant from the origin and a value of the gain which guarantees closed loop poles with more negative real part.

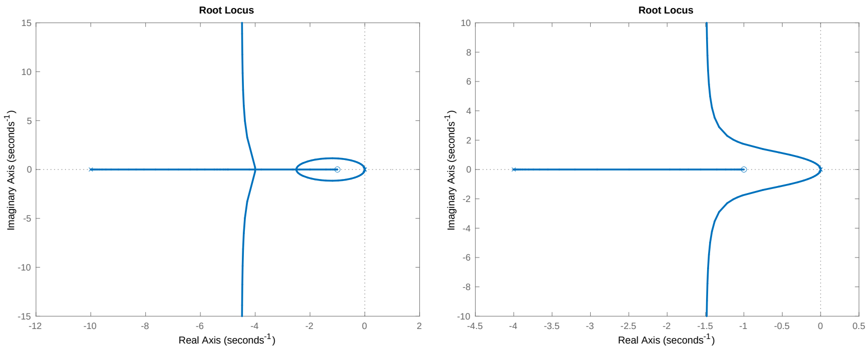

Figure 3: Exercise 4: Two possible root locus.

In the left plot of Fig. 3, an increase in K_c one closed loop pole moves towards the zero and therefore cannot be made arbitrarily fast thus limiting the closed loop bandwidth. However that pole will get closer and closer to the open loop zero and thus there will be an almost cancellation or, equivalently, its contribution will not be significant. We can increase the closed loop bandwidth as much as we want.

Typical errors:

- The most common and serious error is not considering the stability of the closed loop system after adding the pole in $s = 0$.
- Some state that a gain is sufficient to stabilize the modified plant $\hat{P}(s)$ since the closed loop system is characterized by the pole polynomial

$$s^2 + 10K_c$$

since, if K_c is positive then all coefficients of the second order polynomial have the same sign (necessary & sufficient condition for asymptotic stability) but forget the 0 coefficient of the first order term s since it is missing.