Control Systems - February 4, 2020 - (**Solution**)

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1) In Fig. 1 a force f pushes a mass m_1 which slides with no friction on a flat horizontal surface. The mass m_2 lies on top of the mass m_1 and viscous friction is present between the two masses (the friction force is proportional with coefficient μ to the relative velocity of the two masses).

1. Write the dynamic equations governing the motion of the system. We are not interested in the position of the masses so, since there is no term depending on the positions, choose as state vector the masses absolute velocities.
2. Determine the system eigenvalues and corresponding natural modes.
3. Give a physical interpretation of the obtained eigenvalues.

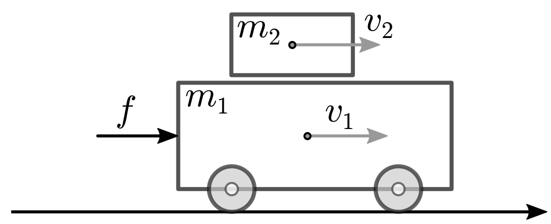


Figure 1: Ex. 1 – The two-mass system.

Sol. 1 We have Newton's equations

$$\begin{aligned} m_1\dot{v}_1 &= -\mu(v_1 - v_2) + f \\ m_2\dot{v}_2 &= \mu(v_1 - v_2) \end{aligned}$$

and therefore, choosing as suggested the state vector $[v_1,\, v_2]^T$, the dynamic matrix A is

$$A = \begin{pmatrix} -\mu/m_1 & \mu/m_1 \\ \mu/m_2 & -\mu/m_2 \end{pmatrix} \quad \Rightarrow \quad p_A(\lambda) = \lambda^2 + \lambda\mu(\frac{1}{m_1} + \frac{1}{m_2}) = \lambda\left(\lambda + \mu(\frac{1}{m_1} + \frac{1}{m_2})\right)$$

which shows the presence of the eigenvalues $\lambda_1 = 0$ and the negative eigenvalue $\lambda_2 = -\mu(1/m_1 + 1/m_2)$. If we choose the system center of mass velocity

$$v_c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

it's dynamics is

$$\dot{v}_c = \frac{m_1\dot{v}_1 + m_2\dot{v}_2}{m_1 + m_2} = \frac{-\mu(v_1 - v_2) + f + \mu(v_1 - v_2)}{m_1 + m_2} = \frac{f}{m_1 + m_2}$$

and reflects the fact that the center of mass, if no force is applied (then $\dot{v}_c = 0$), will have a constant velocity equal to its initial value. Similarly, if we take the relative velocity between the two masses $v_r = v_1 - v_2$, its dynamics is

$$\dot{v}_r = \dot{v}_1 - \dot{v}_2 = -\frac{\mu}{m_1}(v_1 - v_2) + \frac{1}{m_1}f - \frac{\mu}{m_2}(v_1 - v_2) = -\mu\left(\frac{1}{m_1} + \frac{1}{m_2}\right)v_r + \frac{1}{m_1}f$$

and therefore, if no force is applied, the two velocities v_1 and v_2 tend to be equal asymptotically since their difference goes to 0 asymptotically.

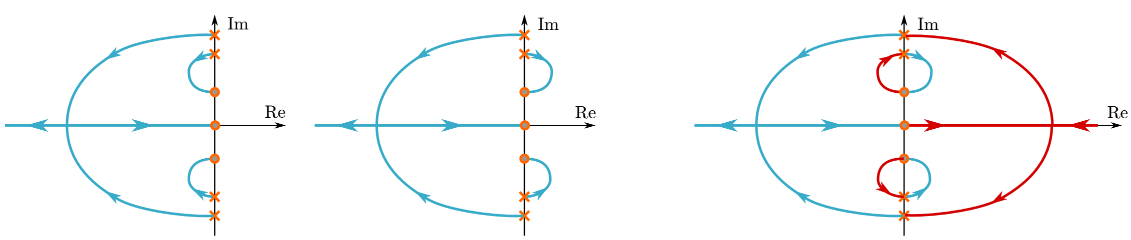
Typical errors:

- wrong dynamic equations (this is similar to the 2-mass system done during the course);
- wrong choice of the state: it was explicitly stated to choose the absolute velocities of each mass as state so the state vector derivative, in the model, is the vector of the absolute accelerations;
- one of the most important error has been justifying the values of the eigenvalues (correct or not) by how the system would behave in the presence of an applied force; the eigenvalues fully characterize the zero-input state response (or state free response);
- similarly, obtaining an unstable eigenvalue should have risen some questions.

2) Draw two possible positive root loci for $F(s)$ and use the Routh criterion to determine the correct one.

$$F(s) = K\frac{s(s^2+1)}{(s^2+2)(s^2+3)}$$

Sol. 2 Two possible positive root locus, shown in Fig. 2, differ from being, for any $K > 0$, asymptotically stable (Left) or unstable (Right).



(a) Two possible positive root locus

(b) Correct positive and negative root locus

Figure 2: Ex. 2 – Root locus.

The closed loop system has the following pole polynomial

$$p(s,K) = (s^2+2)(s^2+3) + Ks(s^2+1) = s^4 + Ks^3 + 5s^2 + Ks + 6$$

with the Routh table

$$
\begin{array}{|ccc}
1 & 5 & 6 \\
K & K & \\
4 & 6 & \\
-K & & \\
6 & &
\end{array}
$$

Recall that we can multiply an entire row by a positive number without altering the Routh criterion (here the penultimate row). The closed loop system will never be asymptotically stable for $K \geq 0$ (two changes of sign for $K > 0$ and therefore two poles with positive real part) and therefore the correct positive root locus is the one on the right in Fig. 2. Also recall that in a regular point (for example the open loop poles and zeros on the imaginary axis) the tangent of the positive and negative root locus is the same (at a given point) so the negative root locus cannot go along the imaginary axis. There were other similar possible positive root locus plots.

Typical errors:

- wrong poles and zeros (as incredible this could be, some got the solutions for example of $s^2 + 1 = 0$ wrong, ... students of a Master degree in engineering);
- drawing a positive root locus which would necessarily create intersections, outside the multiple open loop poles and zeros (which here were not present), with the negative root locus;
- creating intersections between branches of the positive root locus (complex singular points) brings conflicting directions of the branches;
- creating a branch along the imaginary axis (for example from the pole $j\sqrt{2}$ to the zero j) means that a complex pole (and its conjugate) remains always with real part equal to zero and this is not compatible with the Routh table;
- drawing branches which are not symmetric w.r.t. the real axis (if a complex pole (or zero) exists, then also its conjugate is a pole (or zero));
- some have tried to stabilize the plant by introducing a pole/zero pair. This was not required and out of scope. Stick to the questions, no need to show you know how to solve a different problem.

3) Let an interconnected system be composed as the series of $\mathcal{S}_1$ with the parallel of $\mathcal{S}_2$ and $\mathcal{S}_3$. In the series, the output of $\mathcal{S}_1$ is the input of the parallel. Each system $\mathcal{S}_i$, with $i = 1, ..., 3$, has u_i and y_i as input and output and x_i as state,

$$\dot{x}_1 = -2x_1 - u_1, \quad y_1 = x_1 + u_1, \quad \dot{x}_2 = -x_2 + 2u_2, \quad y_2 = x_2, \quad \dot{x}_3 = -x_3 - u_3, \quad y_3 = x_3 + u_3,$$

- Find the state space representation of the interconnected system and study controllability and observability (decide if it is really necessary to do the corresponding decompositions).
- Compute the overall transfer function (from $u = u_1$ to $y = y_2 + y_3$) and discuss if the result is a direct consequence of the previous analysis.

Sol. 3 The interconnection equations are

$$u = u_1, \quad y = y_2 + y_3, \quad u_2 = u_3 = y_1$$

and therefore the interconnected system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -2x_1 - u_1 \\ -x_2 + 2u_2 \\ -x_3 - u_3 \end{pmatrix} = \begin{pmatrix} -2x_1 - u \\ -x_2 + 2x_1 + 2u \\ -x_3 - x_1 - u \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} u$$

with output

$$y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x + u$$

Note the presence od the feedthrough term $D = 1$. So we have

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad D = 1$$

From the triangular structure of the dynamic matrix, the eigenvalues are clearly $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = -2$.

There are several different possible approaches, the main difficulty lies in the presence of a repeated eigenvalue. Note that if the PBH test, in the presence of a repeated eigenvalue λ_i, gives full rank then one can say that the eigenvalue is controllable (or observable) while if the rank condition fails it just indicates that not all the λ_i are controllable (or observable) but not how many are or are not.

We first compute the controllability and observability matrices which both have rank 1

$$R = \begin{pmatrix} -1 & 2 & -4 \\ 2 & -4 & 8 \\ -1 & 2 & -4 \end{pmatrix}, \qquad O = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

which means the system has a 2-dimensional uncontrollable subsystem and also a 2-dimensional unobservable subsystem. Doing the decomposition will show which eigenvalues are unobservable and/or uncontrollable.

For completeness we do both decompositions. First w.r.t. controllability, we choose[1]

$$T_R^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_R = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, \quad \Rightarrow \quad A_R = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

while w.r.t. observability

$$T_O^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_O = T_O^{-1}, \quad \Rightarrow \quad A_O = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -3 & -2 \\ 1 & 1 & 0 \end{pmatrix}, \quad C_O = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

[1]Another typical choice

$$T_R^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad A_R = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We note that the eigenvalue $\lambda_3 = -2$ is the only one controllable but is not observable and therefore no eigenvalue will become a pole.

In this specific example, if we do the PBH test for $\lambda_3 = -2$ it turns out that this eigenvalue is controllable and therefore the other two (rank of R is 1, dimension of the controllable subspace is 1) are uncontrollable. Checking observability through the PBH test, it turns out that $\lambda_3 = -2$ is unobservable so at the end no eigenvalue will be both controllable and observable.

To compute the overall transfer function, one could proceed following the definition (being a 3-dimensional system, this means long computations prone to errors)

$$F(s) = C(sI - A)^{-1}B + D$$

or proceed by computing the interconnection of the single systems represented by their transfer function $F_i(s)$. Transfer function of parallel of $\mathcal{S}_2$ and $\mathcal{S}_3$ is

$$F_2(s) = \frac{2}{s+1}, \quad F_3(s) = \frac{s}{s+1}, \qquad \Rightarrow \qquad F_{\|}(s) = F_2(s) + F_3(s) = \frac{s+2}{s+1}$$

while in state space (this is not necessary, it's just if one follows the interconnection by step procedure also in the state space)

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_2 + 2u_2 \\ -x_3 - u_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + u$$

and therefore

$$R_{\|} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}, \qquad O_{\|} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

that is, noticing that one eigenvalue in -1 has become a pole, the other eigenvalue in -1 is both uncontrollable and unobservable (this was known from the theory). Finally, being

$$F_1(s) = \frac{-1}{s+2} + 1 = \frac{s+1}{s+2}$$

the series interconnection gives

$$F(s) = F_{\|}(s)F_1(s) = \frac{s+2}{s+1}\frac{s+1}{s+2} = 1$$

This is confirmed by noticing that taking the series of $\mathcal{S}_1$ with $\mathcal{S}_{\|}$ a cancellation of the zero in -1 of $\mathcal{S}_1$ with the pole of $\mathcal{S}_{\|}$ occurs thus leading to a loss of controllability of -1 (or the creation of extra uncontrollable dynamics characterized by the eigenvalue -1) while the cancellation of the pole in -2 of $\mathcal{S}_1$ with the zero of $\mathcal{S}_{\|}$ leads to the creation of extra unobservable hidden dynamics characterized by the eigenvalue -2. The final interconnected system does not have controllable and observable dynamics and therefore $C(sI - A)^{-1}B = 0$ but since $D = 1$ the overall transfer function is

$$F(s) = C(sI - A)^{-1}B + D = 1$$

One could have jumped directly to this conclusion using these last motivations.

Typical errors:

- not being able to deal with systems interconnection (for example some have kept the three inputs as inputs of the overall system);
- forgetting the D term in $F_1(s)$ and $F_3(s)$;
- not being able to use the available information derived from PBH or Kalman decomposition, too many computations without trying to figure out if there is a better way;
- computing the transfer function of the interconnection of two systems, for example $\mathcal{S}_2$ and $\mathcal{S}_3$, and deriving a state space realization from it. First if the interconnected system has less poles than eigenvalues then there is clearly the creation of hidden dynamics and therefore the realization is not representative of the original interconnection; even if one does not cancel the common term in the numerator and denominator (corresponding to the common term) one cannot do a realization since the numerator and denominator are not coprime.

4) For the plant $P(s)$, design a controller and a control scheme (draw it) such that

$$P(s) = \frac{s-1}{s+1}$$

- the tracking error at steady state is not greater than 0.1 (in absolute value) w.r.t. $r(t) = t\delta_{-1}(t)$
- a constant disturbance at the plant's input has no effect at steady state on the controlled output
- the crossover frequency is as close as possible to 1 rad/s with a phase margin of at least 30°.

Sol. 4 Standard loop shaping exercise. The first and second specification require the presence of a pole in $s = 0$ in the controller; moreover, to obtain a steady state error smaller that the required amount the loop gain has to be

$$\frac{1}{|K_L|} = \frac{1}{|K_c K_p|} \leq 0.1, \quad K_p = -1, \quad \Rightarrow \quad K_c \leq -10$$

From the Bode plots of the modified plant $-10P(s)/s$, we note that in $\omega_c^* = 1$ rad/s the magnitude has to be attenuated by 20 dB while the phase needs to be increased by at least 30°. Classic lead/lag function choice.

Typical errors:

- most common error has been not seeing the negative gain of the plant or not considering it, leading to choosing $K_c \geq 10$. When drawing the modified plant Bode plots, the phase would have a $-\pi$ extra lag;
- some have solved the inequality arriving to $0 < K_c \leq 10$ (wrong) and thus having the possibility to attenuate using the controller gain;
- looking at the phase at the actual crossover frequency rather than at the desired one;
- wrong Bode plots (pole and zero have equal $1/|\tau|$ but opposite sign of τ).

Figure 3: Ex. 5 – The automatic steering control scheme (Left) and the chosen signals (Right)

5) The automatic steering control scheme for ships shown in Fig. 3 has been introduced first by N. Minorsky around 1930. Find the transfer function from the desired ship's course r to the actual course y. You can omit the dependence from s in the transfer functions and the Laplace transform of the signals.

Sol. 5 We can proceed, using the Laplace transforms of the signals in Fig. 3 (Right), by writing the relations

$$a = K(r - y), \quad b = a - (H_1 + H_2)e, \quad d = \frac{G_1}{1 + G_1 H_2} b, \quad e = G_2 d, \quad y = \frac{1}{s} e$$

and obtaining the relation between r and y as

$$\frac{y(s)}{r(s)} = \frac{G_1(s)G_2(s)K(s)}{G_1(s)G_2(s)K(s) + s(1 + G_1(s)H_2(s) + (H_1(s) + H_3(s))G_1(s)G_2(s))}$$

Otherwise one could first close the inner loop and use the series

$$\frac{e(s)}{b(s)} = \frac{G_1(s)}{1 + G_1(s)H_2(s)} G_2(s),$$

note that the double feedback is equivalent to a feedback with the parallel of $H_1(s)$ and $H_3(s)$ so that

$$\frac{e(s)}{a(s)} = \frac{\frac{G_1(s)}{1+G_1(s)H_2(s)} G_2(s)}{1 + (H_1(s) + H_3(s)) \frac{G_1(s)}{1+G_1(s)H_2(s)} G_2(s)} = \frac{G_1 G_2}{1 + G_1 H_2 + (H_1 + H_3) G_1 G_2}$$

then the series

$$\frac{G_1 G_2}{(1 + G_1 H_2 + (H_1 + H_3) G_1 G_2)} \frac{K}{s}$$

and the final outer unit feedback confirms the previous result

$$\frac{y(s)}{r(s)} = \frac{\frac{G_1 G_2}{(1+G_1 H_2+(H_1+H_3)G_1 G_2)} \frac{K}{s}}{1 + \frac{G_1 G_2}{(1+G_1 H_2+(H_1+H_3)G_1 G_2)} \frac{K}{s}} = \frac{G_1 G_2 K}{G_1 G_2 K + s(1 + G_1 H_2 + (H_1 + H_3) G_1 G_2)}$$

Leaving the implicit expression using the intermediate transfer functions does not give the full clear solution.

Typical errors:

- in most cases it was a question of computation errors (or forgetting terms);
- however some saw a parallel between H_1 and H_3 (this is correct) and the forward path (this is wrong) formed by the transfer function from b to e in Fig. 3.