Control Systems

Structural properties & State space design

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Outline

- introduce the controllability and observability matrices
- define controllable and observable system
- solve an eigenvalue placement problem for a controllable system (Ackermann formula)
- study the non fully controllable case and define stabilizable systems

What we know

given the system (S) $\left[\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx\end{array}\right.$

- λ_i **controllable** $\Leftrightarrow$ $\mathrm{rank}\left[A - \lambda_i I \quad \middle| \quad B\right] = n$

  implies the corresponding mode will appear in the state impulsive response $e^{At}B = H(t)$

  an uncontrollable mode gives rise to hidden dynamics with eigenvalue λ_i

- if all the eigenvalues are controllable, the **system** is **controllable**

- λ_i **observable** $\Leftrightarrow$ $\mathrm{rank}\begin{bmatrix} A - \lambda_i I \\ \hline C \end{bmatrix} = n$

implies the corresponding mode will appear in the output transition matrix $Ce^{At} = \Psi(t)$

an unobservable mode gives rise to hidden dynamics with eigenvalue λ_i

- if all the eigenvalues are observable, the **system** is **observable** equivalent controllability condition (Kalman) but different perspective

system is **controllable** $\Leftrightarrow \quad \text{rank}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$

$\Leftrightarrow \quad (A,B)$ controllable

$\Leftrightarrow$ all eigenvalues are controllable

equivalent to **non-singular** controllability matrix for SISO systems

- controllability characterises a property of the **input-state** interaction and does not depend on C
- if a system is **controllable** there always exists an input that will transfer any state x_a in any other state x_b in finite time (new characterization, no proof)

$P = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$ is defined as the **controllability matrix**

system is **controllable** $\Leftrightarrow$ **controllability matrix** is **nonsingular** equivalent observability condition (Kalman) but different perspective

system is **observable** $\Leftrightarrow$ $\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

equivalent to **non-singular** observability matrix for SISO systems

$\Leftrightarrow$ (A,C) observable

$\Leftrightarrow$ all eigenvalues are observable

- observability characterises a property of the **state-ouput** interaction and does not depend on B
- if a system is **observable** it is always possible to deduce the initial state from the output ZIR, or equivalently starting from two different initial conditions and applying the same input for the same finite time, the ZIR will end in two different states (new characterization, no proof)

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is defined as the **observability matrix**

$$\boxed{\text{system is } \textbf{observable} \quad \Leftrightarrow \quad \textbf{observability matrix} \text{ is } \textbf{non-singular}}$$

Controllability

we defined the matrix $P = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$ **controllability matrix**

we now study the two cases:

- **controllable** system: $\text{rank}\,(P) = n$ and
- **uncontrollable** (or not fully controllable) system: $\text{rank}\,(P) = m < n$

■ $\text{rank}(P) = n$ if 1 input case ⟶ P nonsingular square matrix ⟶ P^{-1} exists

let γ be the last row of P^{-1} and define the matrix T as

$$T = \begin{bmatrix} \gamma \\ \gamma A \\ \vdots \\ \gamma A^{n-1} \end{bmatrix}$$

- T is nonsingular (no proof)

that is we define a change of coordinates $w = Tx$

- T is such that $TAT^{-1} = A_c \qquad TB = B_c$

$$A_c = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

controller canonical form

with $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$

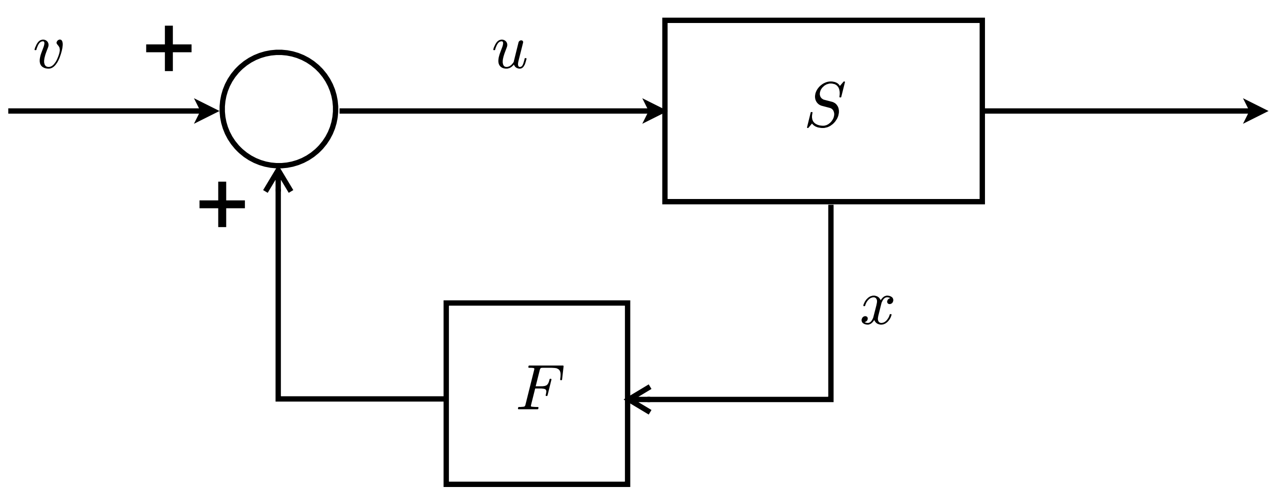
effect of a **state feedback**

- state feedback in w in x

$$u = F_c w + v = F_c T x + v = F x + v$$

in the new coordinates $w = Tx$, the effect of the state feedback is more evident

$$w = Tx \longrightarrow \dot{w} = A_c w + B_c u \longrightarrow \dot{w} = (A_c + B_c F_c) w + B_c v$$



dynamic matrix of the closed-loop system

$$A_{cl} = A_c + B_c F_c$$

assuming

$$F_c = \begin{bmatrix} f_0 & f_1 & \dots & f_{n-1} \end{bmatrix}$$

we have
$$A_c + B_c F_c = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_0 & f_1 & \dots & f_{n-1} \end{bmatrix}$$

- closed-loop dynamic matrix

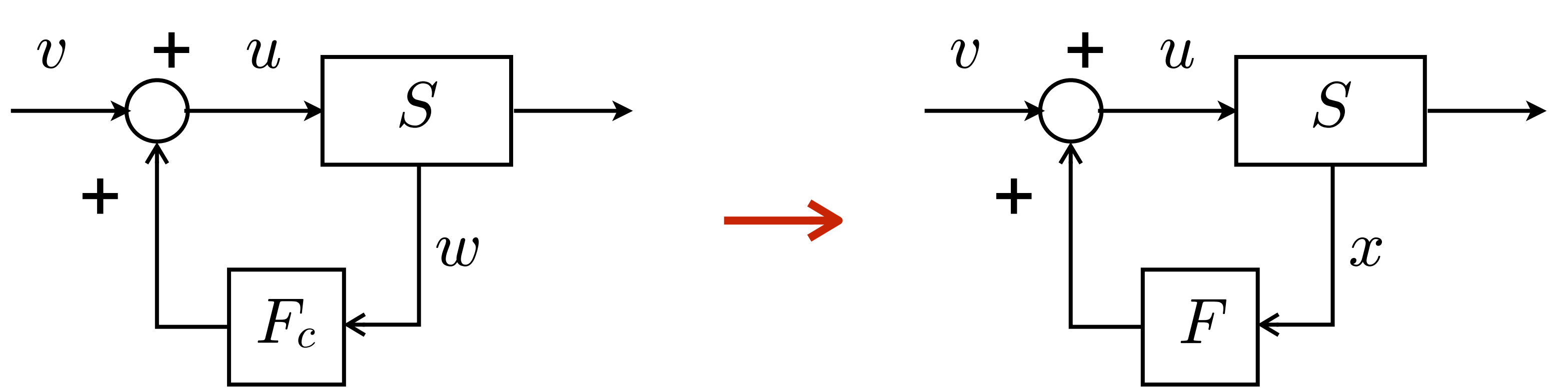
$$A_c + B_c F_c = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ f_0 - a_0 & f_1 - a_1 & \dots & f_{n-1} - a_{n-1} \end{bmatrix}$$

- if $\{\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*\}$ desired eigenvalues which can be seen as solutions of the polynomial

desired closed loop polynomial $$p_A^*(\lambda) = (\lambda - \lambda_1^*) \dots (\lambda - \lambda_n^*) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1 \lambda + \alpha_0$$

choosing $f_i = a_i - \alpha_i \qquad i = 0, \dots, n-1$ we assign the n eigenvalues

- back in the original coordinates $u = Fx$ with $F = F_c T$



but things get even simpler: we can get a simple formula directly in the x original coordinates without going through w

Th. Caley-Hamilton

Let A have the characteristic polynomial

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then A **satisfies its own characteristic polynomial** that is

$$\boxed{p_A(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0}$$

applied to our problem

$$A^n = -a_{n-1}A^{n-1} - \cdots - a_1 A - a_0 I \quad \longleftarrow$$

from Caley-Hamilton

$$p^*(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \quad \longleftarrow$$

$$\Longrightarrow \quad p^*(\lambda) - A^n = \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 A + \alpha_0 I$$

$p^*(\lambda)$ is not the characteristic polynomial of A $\longleftarrow$

this is not equal to 0 since the Caley-Hamilton theorem does not hold let's compute F in the original coordinates

$$F = F_c T$$

$$\begin{aligned} F &= \begin{bmatrix} a_0 - \alpha_0 & a_1 - \alpha_1 & \cdots & a_{n-1} - \alpha_{n-1} \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma A \\ \vdots \\ \gamma A^{n-1} \end{bmatrix} \\ &= \gamma \left[a_0 I + a_1 A + \cdots + a_{n-1} A^{n-1} \right. \\ &\quad \left. -(\alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}) \right] \\ &= \gamma \left[-A^n - (p^*(A) - A^n) \right] \\ &= -\gamma p^*(A) \end{aligned}$$

$$\boxed{F = -\gamma p^*(A)}$$

Ackermann formula

to assign a set of desired eigenvalues $\{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$ solutions of the polynomial $p_A^*(\lambda)$ to the matrix $A+BF$, we can compute (for the controllable system case) the last row γ of the inverse of the controllability matrix and use the Ackermann formula what happens when the system is **not controllable**?

- $\text{rank}(P) = m < n$ $\longrightarrow$ $\dim \text{Range}\,(P) = \dim \text{Image}\,(P) = \dim \text{Im}\,(P) = m$

  $\longrightarrow$ $\text{Im}\,(P) =$ subspace generated by m independent vectors

  we can choose as base of $\text{Im}\,(P)$ the m linearly independent columns of P which we call $\{v_1, v_1, ..., v_m\}$

$$\text{Im}\,(P) = \text{gen}\,\{v_1, v_1, ..., v_m\}$$

define a change of coordinates T (nonsingular) such that

$$T^{-1} = [\underbrace{v_1 \quad v_2 \quad \cdots \quad v_m}_{\text{base of } P} \quad \underbrace{v_{m+1} \quad \cdots \quad v_n}_{\text{completion}}] \quad n \text{ components}$$

completion:
choose the remaining n-m n-dimensional vectors such that all the columns of T^{-1} are linearly independent

note that, by construction,
no vector v_k, for $k = m+1, \ldots, n$ (vector of the completion) belongs to $\text{Im}\,(P)$

example

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

controllability matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

therefore $\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ the same subspace is generated by $\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can be generated as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -0.5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ we can choose different changes of coordinates

$$T^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

different base of $\mathrm{Im}(P)$

different completion but always such that the matrix T is nonsingular

Kalman decomposition w.r.t. controllability

under this change of coordinates $z = Tx$

$$\tilde{A} = TAT^{-1} = \begin{array}{cc} \begin{array}{cc} m & n-m \end{array} & \\ \left[\begin{array}{c:c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hdashline 0 & \tilde{A}_{22} \end{array}\right] & \begin{array}{c} m \\ n-m \end{array} \end{array} \qquad \tilde{B} = TB = \left[\begin{array}{c} \tilde{B}_1 \\ \hdashline 0 \end{array}\right] \begin{array}{c} m \\ n-m \end{array}$$

$$\tilde{C} = CT^{-1} = \left[\begin{array}{c:c} \tilde{C}_1 & \tilde{C}_2 \end{array}\right]$$

no special structure
(controllability depends only upon A and B)

such that

- $\text{rank}\left[\tilde{B}_1 \quad \tilde{A}_{11}\tilde{B}_1 \quad \tilde{A}_{11}^2\tilde{B}_1 \quad \ldots \quad \tilde{A}_{11}^{m-1}\tilde{B}_1\right] = m$
- that is $(\tilde{A}_{11}, \tilde{B}_1)$ controllable
- $\text{eig}\left\{\tilde{A}\right\} = \text{eig}\left\{\tilde{A}_{11}\right\} \bigcup \text{eig}\left\{\tilde{A}_{22}\right\}$

Kalman decomposition w.r.t. controllability (partial proof)

- property of $\mathrm{Im}\,(P)$: **invariance** w.r..t A

if v belongs to $\mathrm{Im}\,(P)$ then Av also belongs to $\mathrm{Im}\,(P)$

$$v \in \mathrm{Im}(P) \quad \Rightarrow \quad A\,v \in \mathrm{Im}(P)$$

if v belongs to $\mathrm{Im}\,(P)$ then it can be expressed as a linear combination of the base of $\mathrm{Im}\,(P)$

$$v \in \mathrm{Im}(P) \quad \Rightarrow \quad v = \sum_{i=1}^{m} \beta_i v_i$$

in the z coordinates, $w = T\,v$ belongs to $\mathrm{Im}\,(P)$ and therefore, being

$$v = T^{-1} w \in \mathrm{Im}(P) \quad \text{and} \quad v = \sum_{i=1}^{m} \beta_i v_i \quad \Rightarrow \quad w = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. if w belongs to $\mathrm{Im}\,(P)$ then its last n-m components are equal to 0 in the new coordinates $w = T\,x$

Kalman decomposition w.r.t. controllability (partial proof)

if $\mathrm{Im}\,(P)$ is **invariant** w.r..t A it is also invariant w.r.t. $A_c = TA\,T^{-1}$ (coordinate independent)

- in the new coordinates $w = T\,x$ if w belongs to $\mathrm{Im}\,(P)$ it has the last n-m components equal to 0

$$w \in \mathrm{Im}(P) \quad \Rightarrow \quad w = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

- if w belongs to $\mathrm{Im}\,(P)$ then also $A_c w$ belongs to $\mathrm{Im}\,(P)$ and therefore also $A_c w$ has the last n-m components equal to 0

$$A_c\, w \in \mathrm{Im}(P) \quad \Rightarrow \quad A_c\, w = \begin{bmatrix} \star \\ 0 \end{bmatrix}$$

- then, for a generic A_c the invariance of w belonging to $\mathrm{Im}\,(P)$ w.r.t. A_c to means that the fowling relationship must hold for **any** vector $[\,]$

$$\begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix} \begin{bmatrix} [\,] \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ 0 \end{bmatrix} \quad \Rightarrow \quad A_{c21}.[\,] = 0 \quad \Rightarrow \quad A_{c21} = 0$$

- note that B belongs to $\mathrm{Im}\,(P)$ then B_c also belongs to $\mathrm{Im}\,(P)$ and therefore has its last n-m components equal to 0

Kalman decomposition w.r.t. controllability

partition z accordingly

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{matrix} \left.\right] m \\ \left.\right] n - m \end{matrix}$$

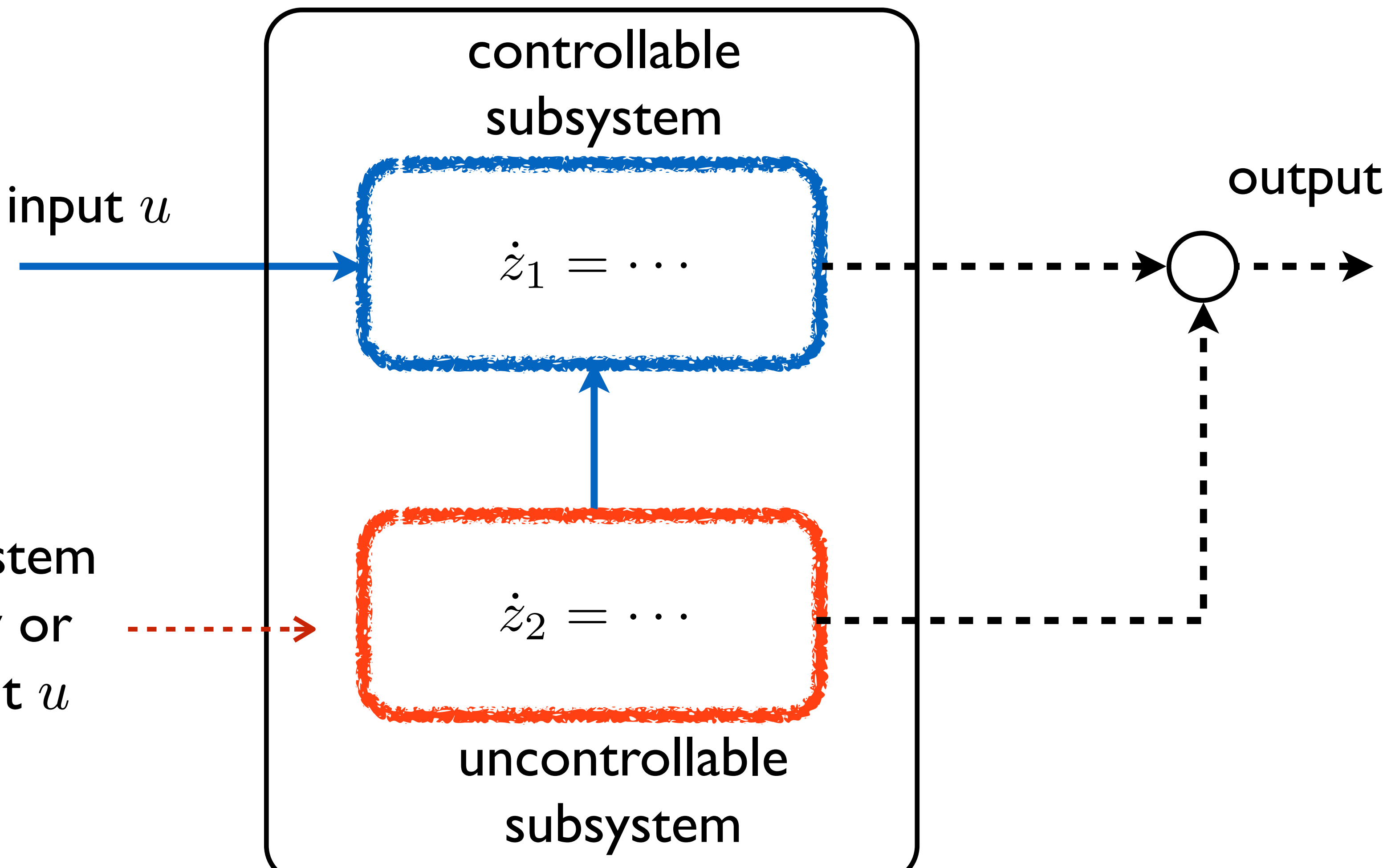
dimensions

two sets of equations can be viewed as **two subsystems**, (S_1) with state z_1 and (S_2) with state z_2

$$\begin{cases} \dot{z}_1 = \tilde{A}_{11} z_1 + \boxed{\tilde{A}_{12} z_2 + \tilde{B}_1 u} \\ \dot{z}_2 = \tilde{A}_{22} z_2 \\ y = \tilde{C}_1 z_1 + \tilde{C}_2 z_2 \end{cases}$$

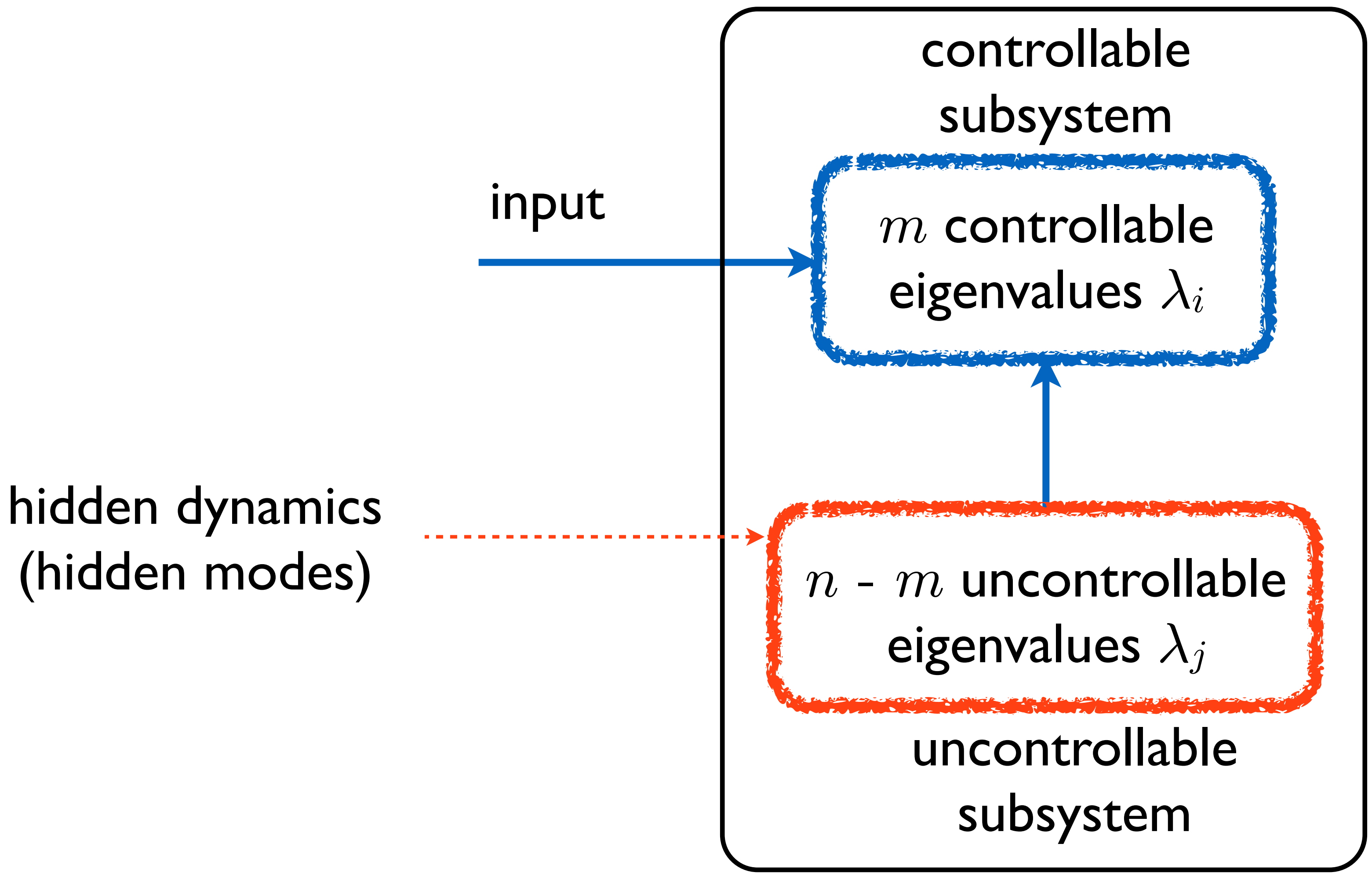
inputs of (S_1)

(S_2) is autonomous (no inputs)



the state evolution of this subsystem is **not influenced**, either directly or indirectly through z_1, by the input u

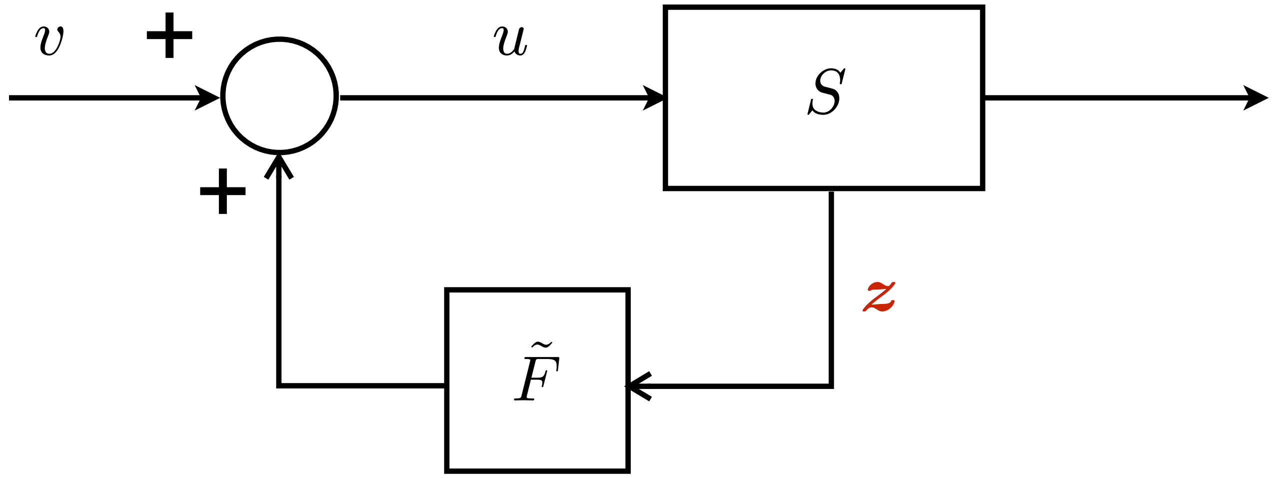
Kalman decomposition w.r.t. controllability



- effect of a state feedback
  (system is no more fully controllable so the previous result on eigenvalue assignment is not applicable)

$$u = \tilde{F}z + v = \tilde{F}Tx + v = \underbrace{\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \end{bmatrix}}_{\text{partitioned accordingly to } z} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + v$$

closed-loop system



effect is evident in
the new coordinates z

$$\dot{z} = \tilde{A}z + \tilde{B}u = (\tilde{A} + \tilde{B}\tilde{F})z + \tilde{B}v$$

$$\begin{aligned}(\tilde{A} + \tilde{B}\tilde{F}) &= \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1\tilde{F}_1 & \tilde{A}_{12} + \tilde{B}_1\tilde{F}_2 \\ 0 & \tilde{A}_{22} \end{bmatrix}\end{aligned}$$

$$\text{eig}\left\{\tilde{A} + \tilde{B}\tilde{F}\right\} = \text{eig}\left\{\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1\right\} \bigcup \text{eig}\left\{\tilde{A}_{22}\right\}$$

we can set $\tilde{F}_2 = 0$ since it does not affect the eigenvalues the eigenvalues of the uncontrollable subsystem are **fixed**, no state-feedback is able to move them

$(\tilde{A}_{11}, \tilde{B}_1)$ controllable $\longrightarrow$ applying the result for controllable systems, we can arbitrarily assign the m eigenvalues of the controllable subsystem through a state feedback (for example with the Ackermann formula)

$$\tilde{F}_1 = -\tilde{\gamma}_1 p_1^*(\tilde{A}_{11})$$

in the original coordinates $$F = \tilde{F}T = \begin{bmatrix} \underbrace{-\tilde{\gamma}_1 \, p_1^*(\tilde{A}_{11})}_{\tilde{F}_1} & \underbrace{0}_{\tilde{F}_2} \end{bmatrix} T$$

$\longrightarrow$ if we want to make the closed-loop system asymptotically stable (or stabilize the closed loop system) with a **state feedback**, either the eigenvalues of the uncontrollable subsystem (if any) are already with negative real part (the system is said to be **stabilizable**) or the problem has no solution alternatively we can say that a system (or a pair (A,B)) is **stabilizable with state feedback** if every eigenvalue with positive or null real part is controllable i.e.

$$\text{rank}\left(\begin{array}{c|c} A - \lambda_i I & B \end{array} \right) = n \qquad \forall \lambda_i \quad / \quad \text{Re}[\lambda_i] \geq 0$$

note that stabilizability is a weaker requirement w.r.t. controllability (we need controllability only for those eigenvalues which do not have negative real part)

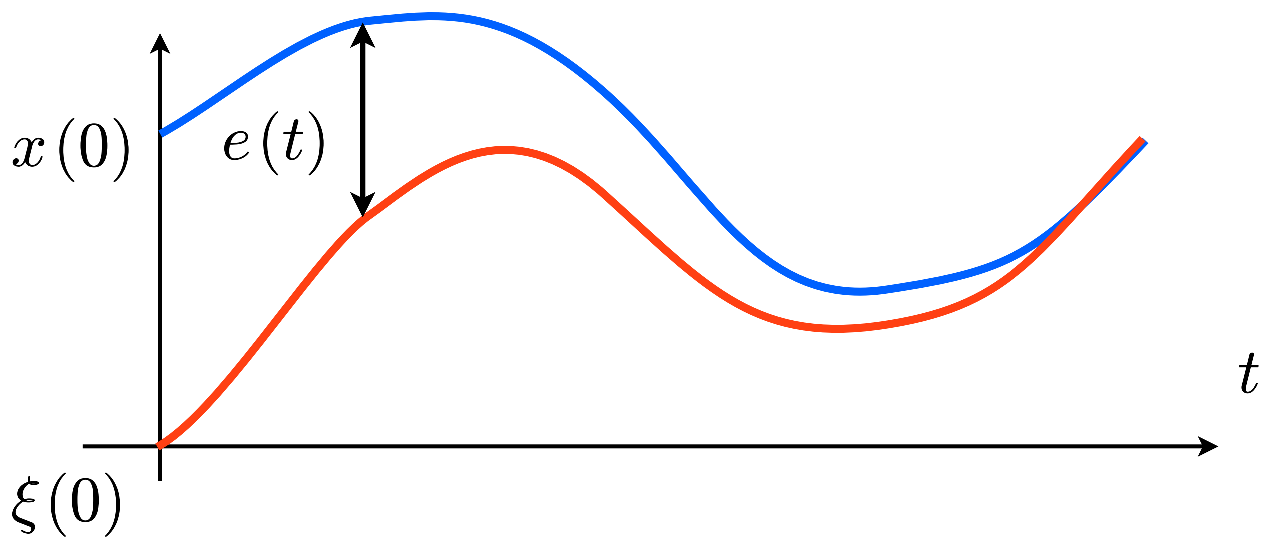
Observability

Given a system S with state x described by (A,B,C) we want to find a dynamical system S_o with state ξ which reconstructs asymptotically the system's state x that is, defining the error as $e = \xi$ - x, we want

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [\xi(t) - x(t)] = 0$$

unknown plant's initial condition

initial condition for S_o



Plant
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

First idea: since we know (A,B,C) we just use a copy of the plant driven by the same input

Plant $\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}$ Copy $\dot{\xi} = A\xi + Bu$

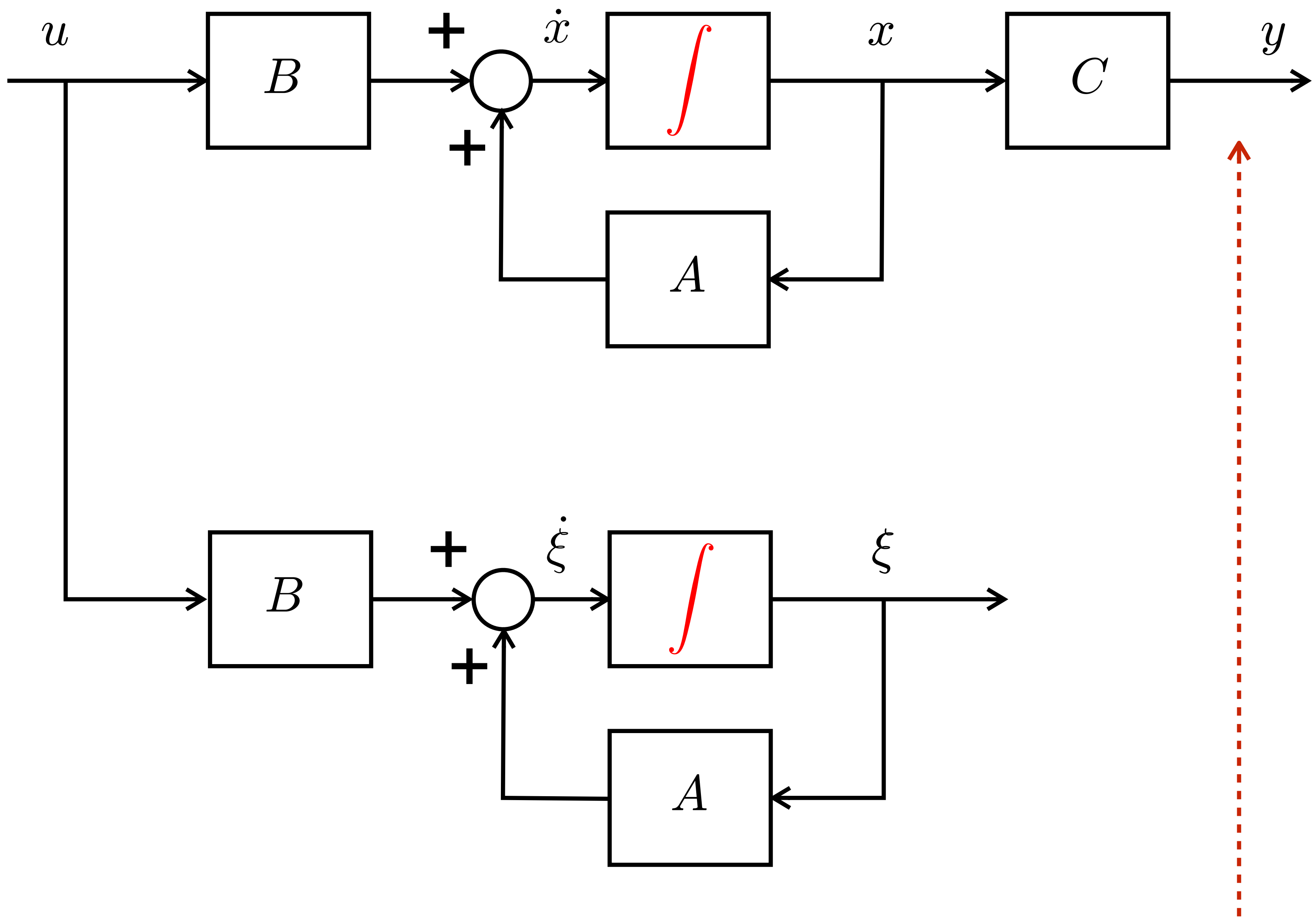
How does the **reconstruction error** $e = \xi - x$ change in time?

$$\dot{e} = \dot{\xi} - \dot{x} = A\xi + Bu - Ax - Bu = A(\xi - x) = Ae$$

- either I know the initial state (too strong assumption) of the plant and I initialize the copy in $\xi(0) = x(0)$ so that $e(0) = 0$
- or the plant is asymptotically stable and therefore the error tends to zero with a convergence rate that depends only on the system (also a restrictive assumption)

unsatisfactory solution

We assume to know the system model (A,B,C,D) the input u and the output y



Plant

Copy

in our previous idea we did not use the measured output y yet we add a term which is proportional to the reconstructed output error $(Cx$ - $C\xi)$

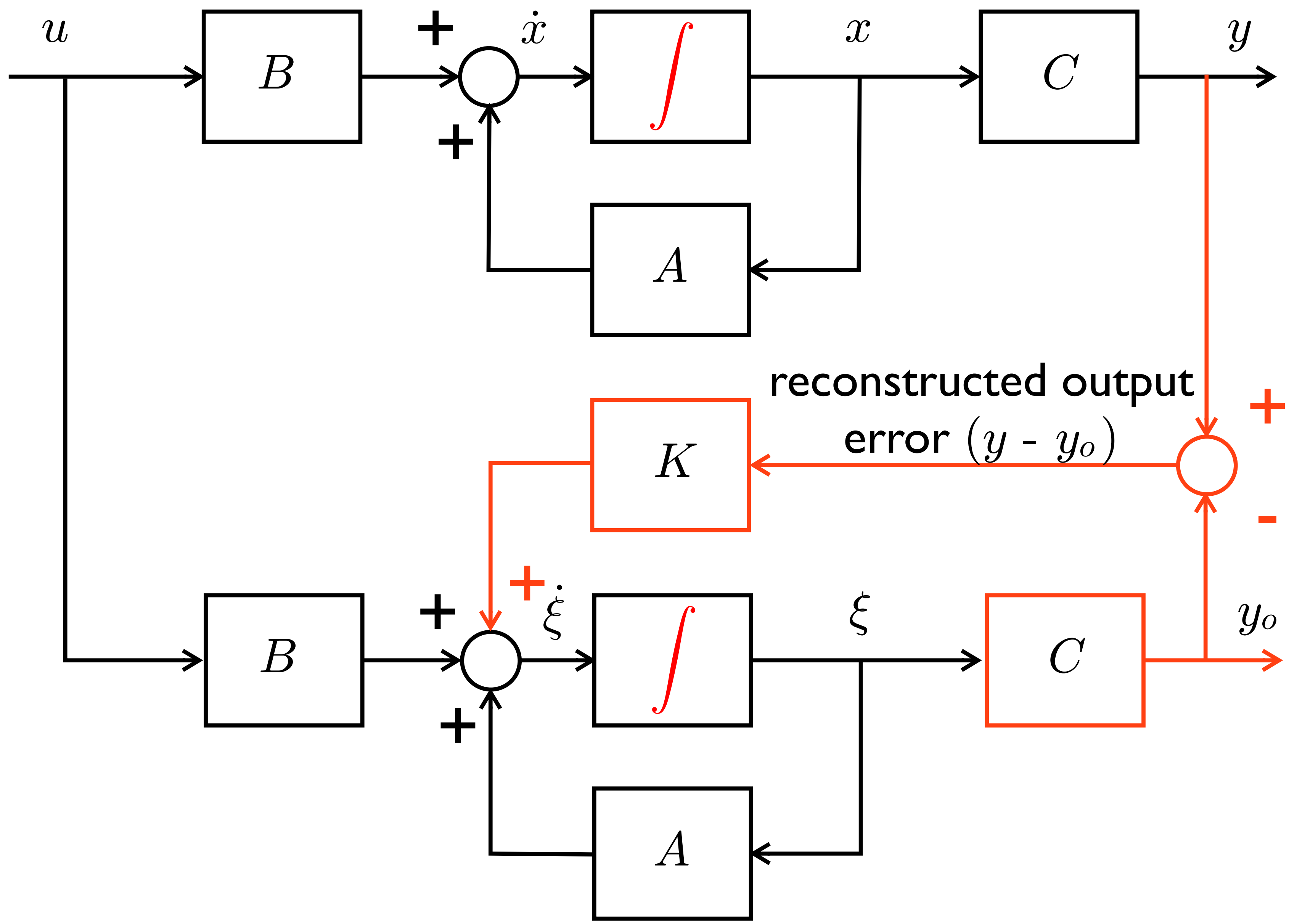
true output (measurable)

asymptotic observer

$$\begin{cases} \dot{\xi} = A\xi + Bu + K(Cx - C\xi) \\ y_o = C\xi \end{cases}$$

$K : (n \times 1)$

reconstructed output $y_o = C\xi$



the ***error*** now evolves as

$$\begin{aligned}\dot{e} &= \dot{\xi} - \dot{x} = A\xi + Bu + K(Cx - C\xi) - Ax - Bu \\ &= (A - KC)(\xi - x) = (A - KC)e\end{aligned}$$

and therefore the error converges asymptotically to zero if and only if all the eigenvalues of the matrix $(A$ - $KC)$ have negative real part

- either the plant S is asymptotically stable and we can choose $K = 0$ but then the convergence rate is fixed
- or we can use K (n x 1 vector) to assign the eigenvalues of $(A$ - $KC)$ and therefore choose the convergence rate at which the error tends to zero. We need to understand under which conditions this is possible and then how to choose K to assign the convergence rate arbitrarily, i.e., the eigenvalues of the matrix $(A$ - $KC)$

$$\exists ?\, K \;\; / \;\; \mathrm{eig}(A - KC) = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$$

remember that for a generic square matrix M

$$\mathrm{eig}(M) = \mathrm{eig}(M^T)$$

therefore

$$\mathrm{eig}((A - KC)) = \mathrm{eig}((A - KC)^T) = \mathrm{eig}(A^T - C^T K^T)$$

and defining

$$\bar{A} = A^T \qquad \bar{B} = C^T \qquad \bar{F} = -K^T$$

$$\mathrm{eig}(A - KC) = \mathrm{eig}(\bar{A} + \bar{B}\bar{F})$$

therefore the question

$$\exists ?\, K \;\; / \;\; \mathrm{eig}(A - KC) = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$$

is equivalent to requiring if

$$\Leftrightarrow \quad \exists ?\, \bar{F} \;\; / \;\; \mathrm{eig}(\bar{A} + \bar{B}\bar{F}) = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$$

we know that we can assign n arbitrarily eigenvalues to $\bar{A} + \bar{B}\bar{F}$ iff the pair $(\bar{A}, \bar{B})$ is controllable since in general

$$\mathrm{rank}(M) = \mathrm{rank}(M^T)$$

$(\bar{A}, \bar{B})$ controllable is equivalent to

$$\begin{aligned}\mathrm{rank}\begin{bmatrix}\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B}\end{bmatrix} &= \mathrm{rank}\begin{bmatrix}C^T & A^TC^T & \cdots & A^{(n-1)T}C^T\end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix}C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} = \mathrm{rank}\begin{bmatrix}\mathcal{O}\end{bmatrix} = n\end{aligned}$$

$$\exists\, K \;\; / \;\; \mathrm{eig}(A - KC) = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$$

if and only if

(A, C) is **observable** or equivalently the plant is observable

$\mathcal{O}$ is the **observability matrix** how do we choose K s.t. $\mathrm{eig}(A - KC) = \{\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*\}$

$$\bar{F} = -\bar{\gamma} p^*(\bar{A})$$

- first we have that $p^*(\bar{A}) = p^*(A^T) = [p^*(A)]^T$
- therefore $\quad \bar{F} = -K^T \quad \Rightarrow \quad K = -\bar{F}^T = \left[p^*(\bar{A})\right]^T \bar{\gamma}^T = p^*(A)\bar{\gamma}^T$
- moreover, being $\bar{\gamma}$ the last row of

$$\left[\bar{B} \quad \bar{A}\bar{B} \quad \dots \bar{A}^{n-1}\bar{B}\right]^{-1} = \left[C^T \quad A^T C^T \quad \dots (A^T)^{n-1} C^T\right]^{-1} = \left[\mathcal{O}^T\right]^{-1}$$

and since for any invertible matrix $\left[\mathcal{O}^T\right]^{-1} = \left[\mathcal{O}^{-1}\right]^T$

to assign n desired eigenvalues, solutions of the desired polynomial $p_A^*(\lambda)$, to the matrix A-KC which governs the observation error dynamics, the design matrix K has to chosen as

$$K = p^*(A)\bar{\gamma}^T$$

where $\bar{\gamma}^T$ is the last column of the observability matrix inverse $\mathcal{O}^{-1}$

We can build an **asymptotic observer** S_o

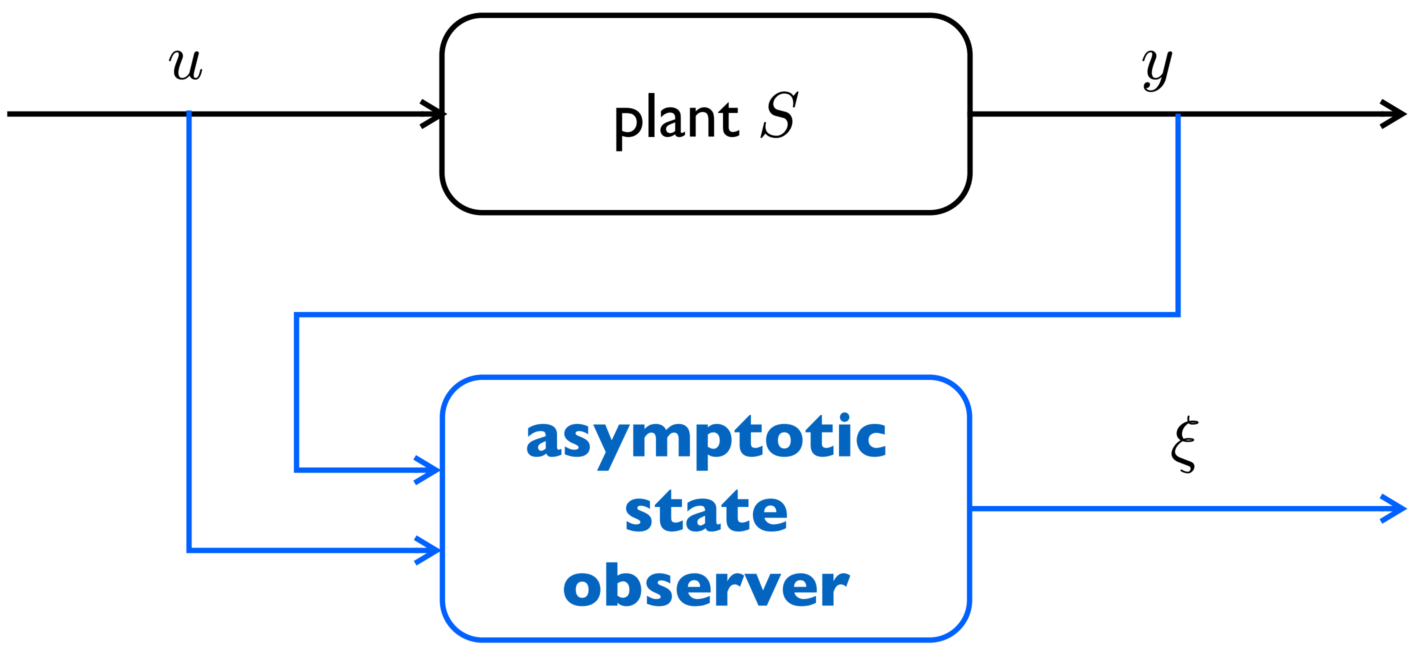
$$\dot{\xi} = A\xi + Bu + K(Cx - C\xi) = (A - KC)\xi + Bu + Ky$$

such that the observation error $e = \xi$ - x decays exponentially to zero with assigned rate given by the set of desired eigenvalues $\{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$

if and only if the plant S is **observable**

The matrix K is given by $$K = p^*(A)\bar{\gamma}^T$$

$\bar{\gamma}^T$ being the last column of the observability matrix inverse $\mathcal{O}^{-1}$



What happens if the plant S is **not observable**?

S is not observable $\longrightarrow$ the observability matrix is singular $\rightarrow$ $\operatorname{rank}(\mathcal{O}) = m < n$

the kernel or nullspace has dimension n - m

recall that $\ker(M) = \{v \mid Mv = 0\}$

$\longrightarrow$ $\ker(\mathcal{O}) = \operatorname{gen}\{v_1, v_2, \ldots, v_{n-m}\}$ linear subspace

$\longrightarrow$ choose T such that

$$T^{-1} = \begin{bmatrix} \underbrace{w_1 \quad \cdots \quad w_m}_{\text{completion}} & \underbrace{v_1 \quad v_2 \quad \cdots \quad v_{n-m}}_{\text{base of } \ker(\mathcal{O})} \end{bmatrix}$$

(such that all the columns are linearly independent)

Kalman decomposition w.r.t. observability

under this change of coordinates $z = Tx$

$$\tilde{A} = TAT^{-1} = \begin{array}{cc} m & n - m \\ \left[\begin{array}{cc} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \end{array} \begin{array}{c} m \\ n - m \end{array} \qquad \tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$$

no special structure

$$\tilde{C} = CT^{-1} = \begin{array}{cc} \left[\tilde{C}_1 \right. & \left. 0 \right] \\ m & n - m \end{array}$$

$$\text{rank} \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_1 \tilde{A}_{11} \\ \dots \\ \tilde{C}_1 \tilde{A}_{11}^{m-1} \end{bmatrix} = m \qquad \Leftrightarrow \qquad (\tilde{A}_{11}, \tilde{C}_1)$$ observable

$$\text{eig}\left\{\tilde{A}\right\} = \text{eig}\left\{\tilde{A}_{11}\right\} \bigcup \text{eig}\left\{\tilde{A}_{22}\right\}$$

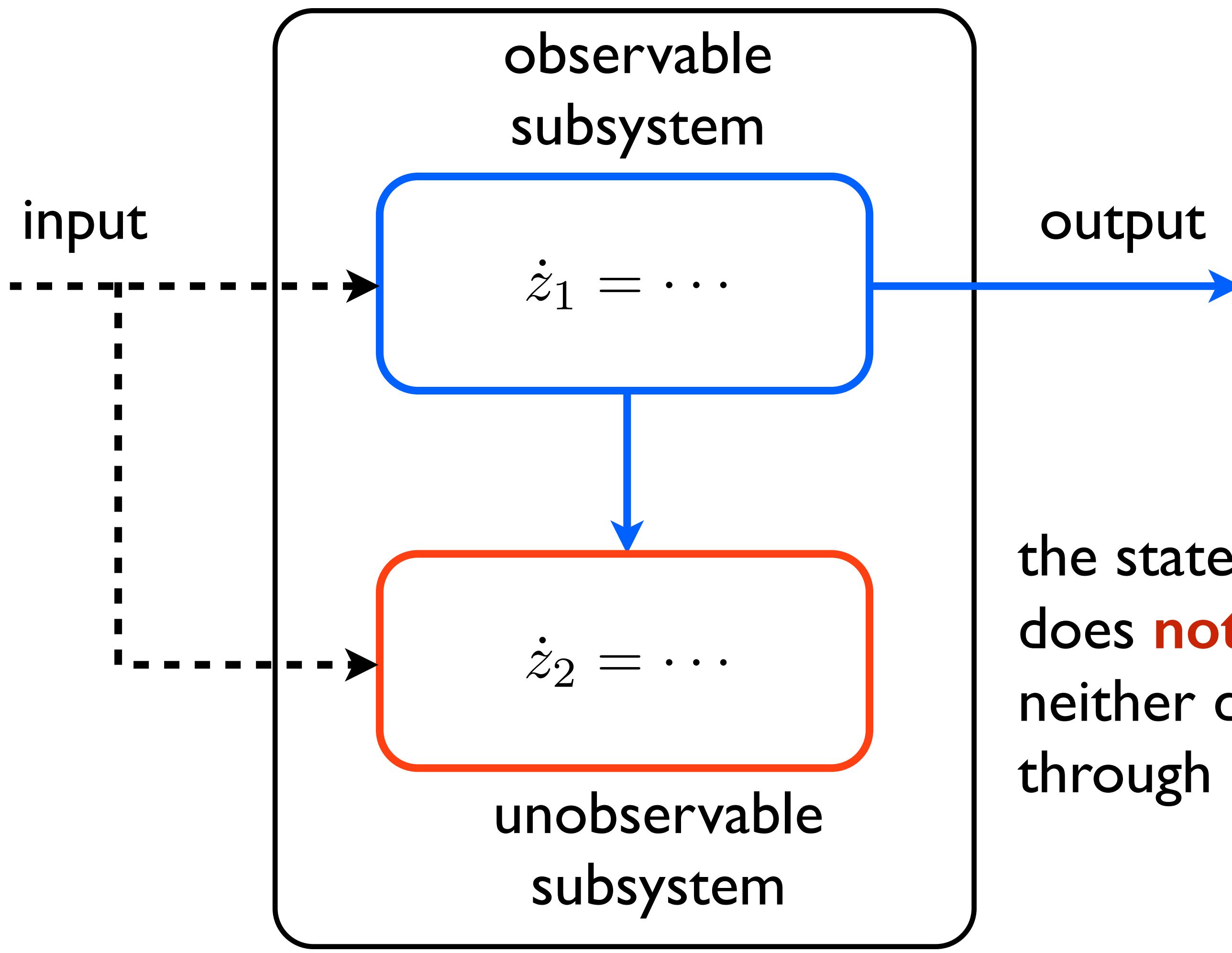
Kalman decomposition w.r.t. observability

partition z accordingly

$$z = Tx = \begin{array}{cc} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} & \begin{array}{c} m \\ n - m \end{array} \end{array}$$

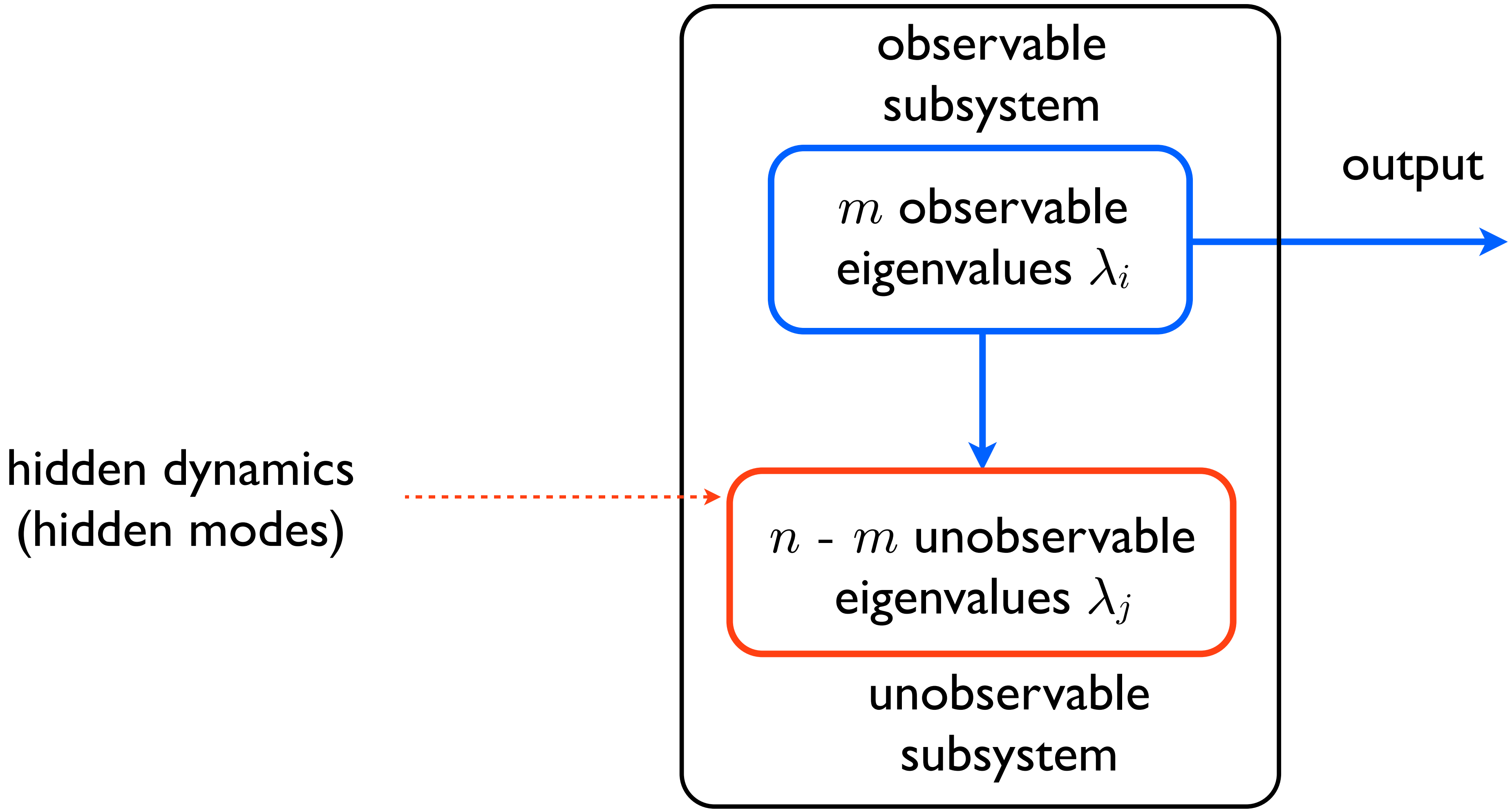
two sets of equations can be viewed as **two subsystems**, (S_1) with state z_1 and (S_2) with state z_2

$$\begin{cases} \dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{B}_1 u \\ \dot{z}_2 = \tilde{A}_{21} z_1 + \tilde{A}_{22} z_2 + \tilde{B}_2 u \\ y = \tilde{C}_1 z_1 \end{cases}$$



the state evolution of this subsystem does **not influence** the output y, neither directly nor indirectly through z_1

Kalman decomposition w.r.t. observability



what happens if we try to built an observer?

first note that since now we are in the z coordinates, the observer built with the matrices obtained in this decomposition will try to reconstruct z so the new error will be

$$\tilde{e} = \tilde{\xi} - z$$

Kalman decomposition w.r.t. observability

tentative observer

$$\begin{aligned} \dot{\tilde{\xi}} &= \tilde{A}\tilde{\xi} + \tilde{B}u + \tilde{K}(\tilde{C}z - \tilde{C}\tilde{\xi}) = (\tilde{A} - \tilde{K}\tilde{C})\tilde{\xi} + \tilde{B}u + \tilde{K}y \\ y_o &= \tilde{C}\tilde{\xi} \end{aligned}$$

the reconstruction error (in the new coordinates) evolves as

$$\dot{\tilde{e}} = \cdots = (\tilde{A} - \tilde{K}\tilde{C})\tilde{e}$$

with the new dynamic matrix

$$\begin{aligned} (\tilde{A} - \tilde{K}\tilde{C}) &= \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}_{11} - \tilde{K}_1\tilde{C}_1 & 0 \\ \tilde{A}_{21} - \tilde{K}_2\tilde{C}_1 & \boxed{\tilde{A}_{22}} \end{bmatrix} \end{aligned}$$

unaffected by $\tilde{K}$

$$\mathrm{eig}\left\{\tilde{A} - \tilde{K}\tilde{C}\right\} = \mathrm{eig}\left\{\tilde{A}_{11} - \tilde{K}_1\tilde{C}_1\right\} \bigcup \mathrm{eig}\left\{\tilde{A}_{22}\right\}$$

we can set $\tilde{K}_2 = 0$ since it does not affect the eigenvalues

Observer design - unobservable case

- the eigenvalues of the **unobservable subsystem** are **fixed** in the observer

$$\dot{\tilde{\xi}} = \tilde{A}\tilde{\xi} + \tilde{B}u + \tilde{K}(\tilde{C}z - \tilde{C}\tilde{\xi}) = (\tilde{A} - \tilde{K}\tilde{C})\tilde{\xi} + \tilde{B}u + \tilde{K}y$$

- since $(\tilde{A}_{11}, \tilde{C}_1)$ observable we can apply the result for observable systems and arbitrarily assign the m eigenvalues of the matrix $\tilde{A}_{11} - \tilde{K}_1\tilde{C}_1$ by choosing $\tilde{K}_1 = p^*(\tilde{A}_{11})\bar{\gamma}_1^T$ where $\bar{\gamma}_1^T$ is the last column of the inverse of the observability matrix associated to the observable subsystem, i.e., to $(\tilde{A}_{11}, \tilde{C}_1)$

- if we want to reconstruct asymptotically the state of the plant, either the eigenvalues of the unobservable subsystem (if any) are already with negative real part (the system is said to be **detectable**) or the problem has no solution

- alternatively we can say that a system (or a pair (A,C)) is detectable if every eigenvalue with positive or null real part is observable i.e.

$$\text{rank}\begin{pmatrix} A - \lambda_i I \\ C \end{pmatrix} = n \qquad \forall \lambda_i \quad / \quad \text{Re}[\lambda_i] \geq 0$$

note that detectability is a weaker condition w.r.t. observability

Observer design - unobservable case

if the system is **detectable** we can therefore build an observer such that

- the reconstruction error relative to the **observable subsystem** can be made decaying to zero with arbitrary rate of convergence
- the reconstruction error relative to the **unobservable subsystem** has a rate of convergence fixed and given by the unobservable eigenvalues (which are all with negative real part since the system is detectable)

in the original coordinates, since $z = Tx$, we also have $\tilde{\xi} = T\xi$

$$\begin{aligned}\dot{\xi} &= T^{-1}\dot{\tilde{\xi}} = T^{-1}(TAT^{-1} - \tilde{K}CT^{-1})T\xi + T^{-1}TBu + T^{-1}\tilde{K}y \\ &= (A - T^{-1}\tilde{K}C)\xi + Bu + T^{-1}\tilde{K}y \\ &= (A - KC)\xi + Bu + Ky\end{aligned}$$

with $K = T^{-1}\tilde{K}$ and therefore

$$K = T^{-1}\begin{bmatrix}\tilde{p}^*(\tilde{A}_{11})\bar{\gamma}_1^T \\ 0\end{bmatrix}$$

where, being $(\tilde{A}_{11}, \tilde{C}_1)$ observable, $\tilde{K}_1 = p^*(\tilde{A}_{11})\bar{\gamma}_1^T$ assigns m desired eigenvalues to $\tilde{A}_{11} - \tilde{K}_1\tilde{C}_1$

Separation principle

We have seen that for a **controllable** and **observable** system we can

- design a **state feedback** $u = F\,x$ which assigns the eigenvalues to $A + BF$ i.e. we want $x(t)$ to tend to 0 asymptotically
- design a dynamic system, the **observer**, which asymptotically reconstructs the plant's state (i.e. the state ξ tends to x or, equivalently, the error $e = \xi - x$ tends to 0) at a desired rate since we can assign the eigenvalues to the matrix $A - KC$ which governs the observation error dynamics

the state is usually not measurable (at least not all its components) and therefore not available for the feedback $u = F\,x$. We could try to **use the state estimate ξ instead of the real state** in the feedback law, that is

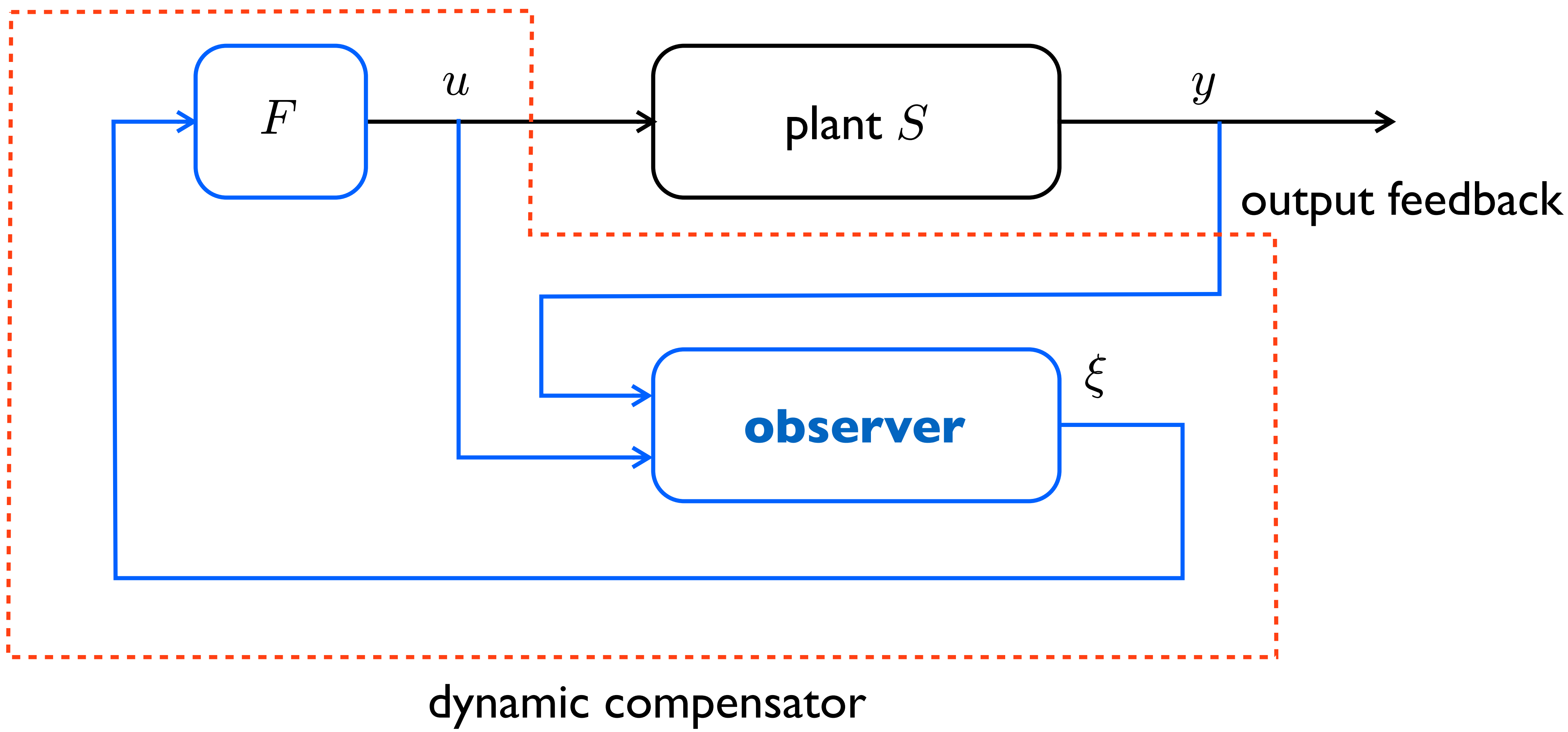
$$\boxed{u = F\,\xi} \qquad \text{instead of} \qquad u = F\,x$$

with

$$\dot{\xi} = (A - KC)\xi + Bu + Ky$$

What happens?

Separation principle



closed-loop system
$$\left[\begin{array}{lcl} \dot{x} & = & Ax + BF\xi \\ \dot{\xi} & = & KCx + (A + BF - KC)\xi \end{array}\right.$$

dynamic matrix $\begin{bmatrix} A & BF \\ KC & A + BF - KC \end{bmatrix}$ no helpful structure

Separation principle

we do a change of coordinates, from $(x,\ \xi)$ to $(x,\ e)$ with $e = \xi$ - x

$$
\begin{aligned}
\dot{x} &= Ax + BF(x+e) \\
\dot{e} &= \dot{\xi} - \dot{x} = KCx + (A + BF - KC)(x+e) - [Ax + BF(x+e)]
\end{aligned}
$$

closed-loop system $\longrightarrow$
$$
\left[\begin{aligned}
\dot{x} &= (A+BF)x + BFe \\
\dot{e} &= (A-KC)e
\end{aligned}\right.
$$

dynamic matrix $\longrightarrow$
$$
\begin{bmatrix} A+BF & BF \\ 0 & A-KC \end{bmatrix}
$$

$$
\mathrm{eig}(A+BF) \bigcup \mathrm{eig}(A-KC)
$$

the dynamic output feedback controller can be designed by separately choosing F as if we were solving a state feedback stabilization problem and K to make the estimation error decay with a prescribed rate: this is the **separation principle**

Separation principle

Analysing the closed-loop system

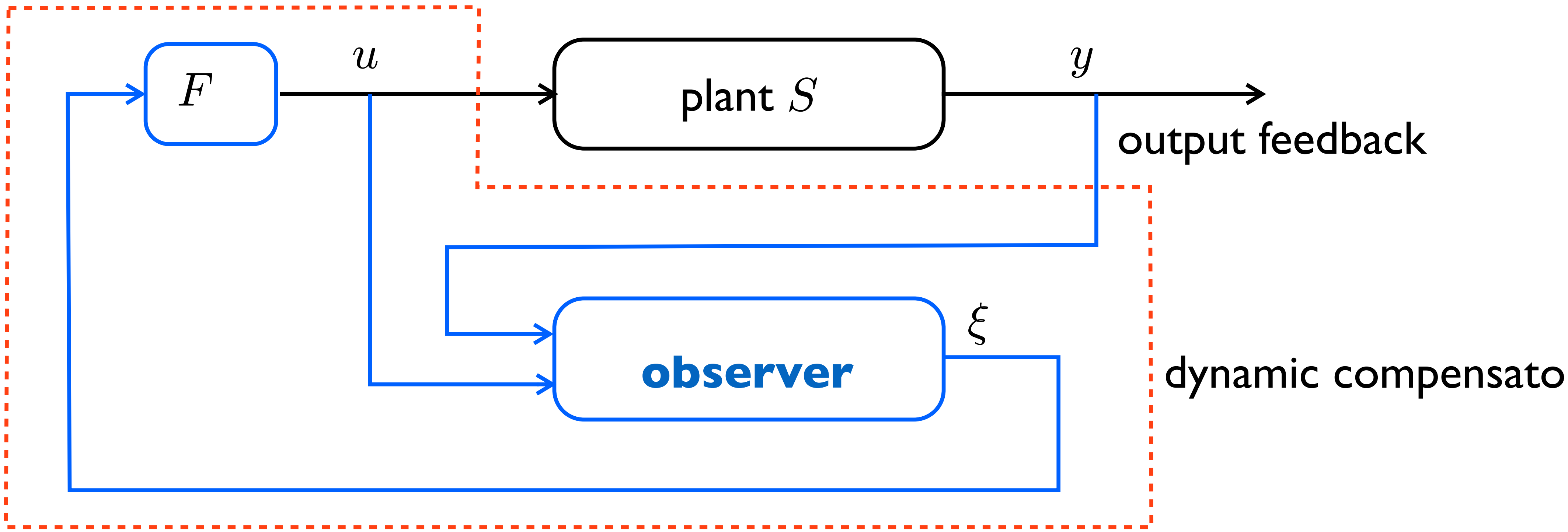
$$\left[\begin{array}{lcl} \dot{x} & = & (A+BF)x + BFe \\ \dot{e} & = & (A-KC)e \end{array}\right.$$

we have obtained that

- the reconstruction error e goes to zero asymptotically with assigned rate of convergence
- the evolution of the state x is governed by an asymptotically stable matrix $A{+}BF$ with assigned eigenvalues and is forced by an input e which tends to zero asymptotically: the ZIR tends to zero as well as the ZSR and therefore the state x will tend to zero asymptotically with assigned rate of convergence plus a forced response which tends to 0

typically one chooses the desired eigenvalues of A-KC ten times "faster" than those of $A{+}BF$

Separation principle



if we rewrite the asymptotic observer system as $\dot{\xi} = (A + BF - KC)\xi + Ky$

and note that the output of the dynamic compensator is $u = F\,\xi$ we can compute the transfer function from y to u of the controller (dynamic compensator) as

$$C(s) = F\left[sI - (A + BF - KC)\right]^{-1} K$$

extras (not published)

- uncontrollable eigenvalues are not changed by a state feedback (see `inv-sub.pdf` p. 6.20-6.21). This is a dual point of view w.r.t. "if a system is controllable, a state feedback $u = Fx + v$ does not alter controllability" in the sense that

$$(A,B) \text{ controllable iff } (A+BF,B) \text{ controllable for every } F$$

(proof with PBH)

$$\mathrm{rk}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = \mathrm{rk}\left\{\begin{bmatrix} A-\lambda I & B \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}\right\} = \mathrm{rk}\begin{bmatrix} A+BF-\lambda I & B \end{bmatrix}$$

- observability is affected by state feedback (choose an assigned eigenvalues coincident with a system zero)