Control Systems

Performance

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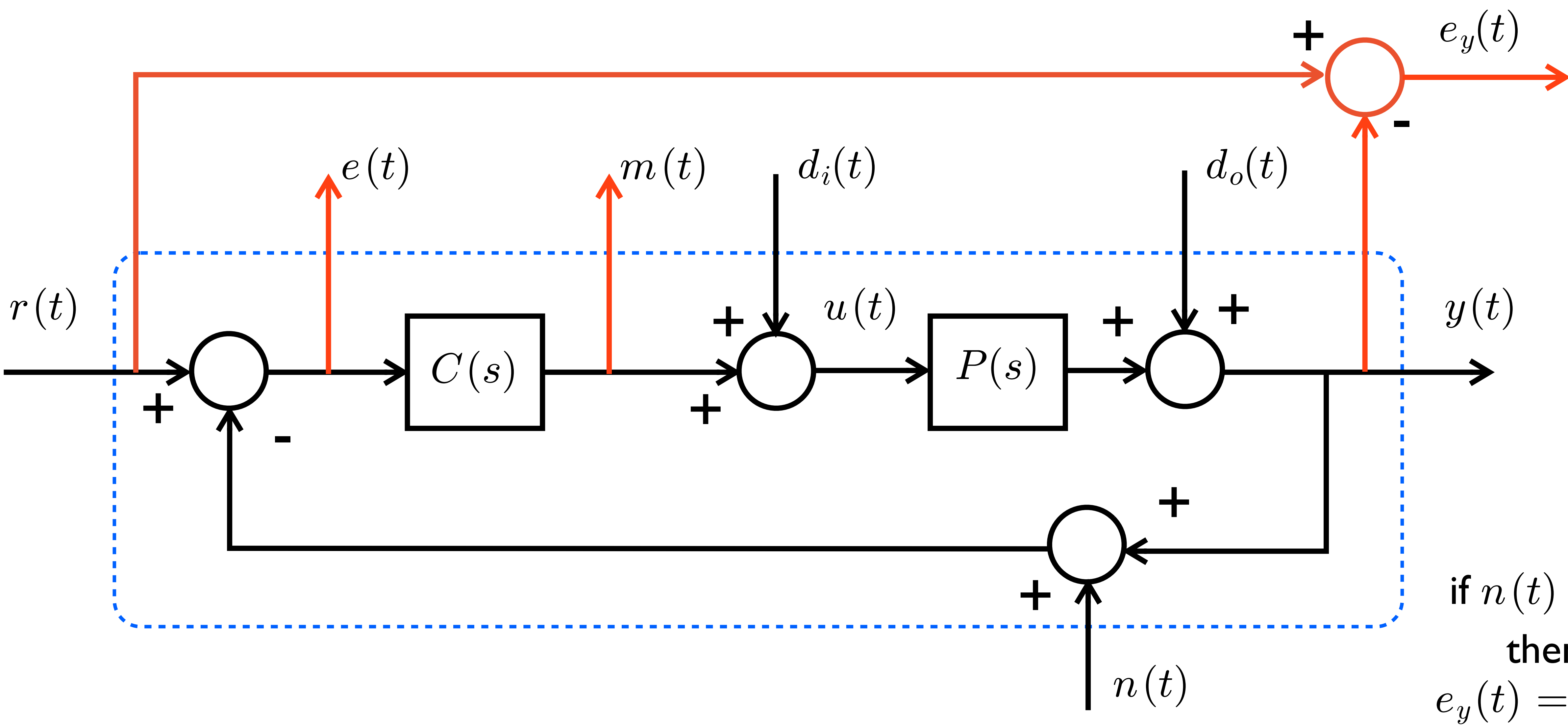
Outline

- loop function approximation
- sensitivity function approximation
- complementary sensitivity function approximation
- control effort and Parseval theorem
- control sensitivity function approximation
- constraints on the loop function
- the integrator

feedback control scheme

recall the general feedback scheme with its inputs and outputs of interest



if $n(t) = 0$
then
$e_y(t) = e(t)$

feedback control scheme

recall that in the general feedback scheme the outputs of interest are related to the inputs as

$$\begin{aligned} e_y(s) &= S(s)r(s) - S(s)P(s)d_i(s) - S(s)d_o(s) + T(s)n(s) \\ y(s) &= T(s)r(s) + S(s)P(s)d_i(s) + S(s)d_o(s) - T(s)n(s) \\ e(s) &= S(s)r(s) - S(s)P(s)d_i(s) - S(s)d_o(s) - S(s)n(s) \\ m(s) &= S_u(s)r(s) - T(s)d_i(s) - S_u(s)d_o(s) - S_u(s)n(s) \end{aligned}$$

and being $T(s) = S_u(s)P(s)$

$$m(s) = S_u(s)(r(s) - P(s)d_i(s) - d_o(s) - n(s))$$

The closed-loop system is therefore characterized by the three sensitivity functions

$$\boxed{T(s),\ S(s),\ S_u(s)}$$

By analyzing the magnitude of their frequency response, we can understand how the closed-loop system behaves w.r.t. sinusoidal inputs $r(t)$, $d(t)$ and $n(t)$

sensitivity functions

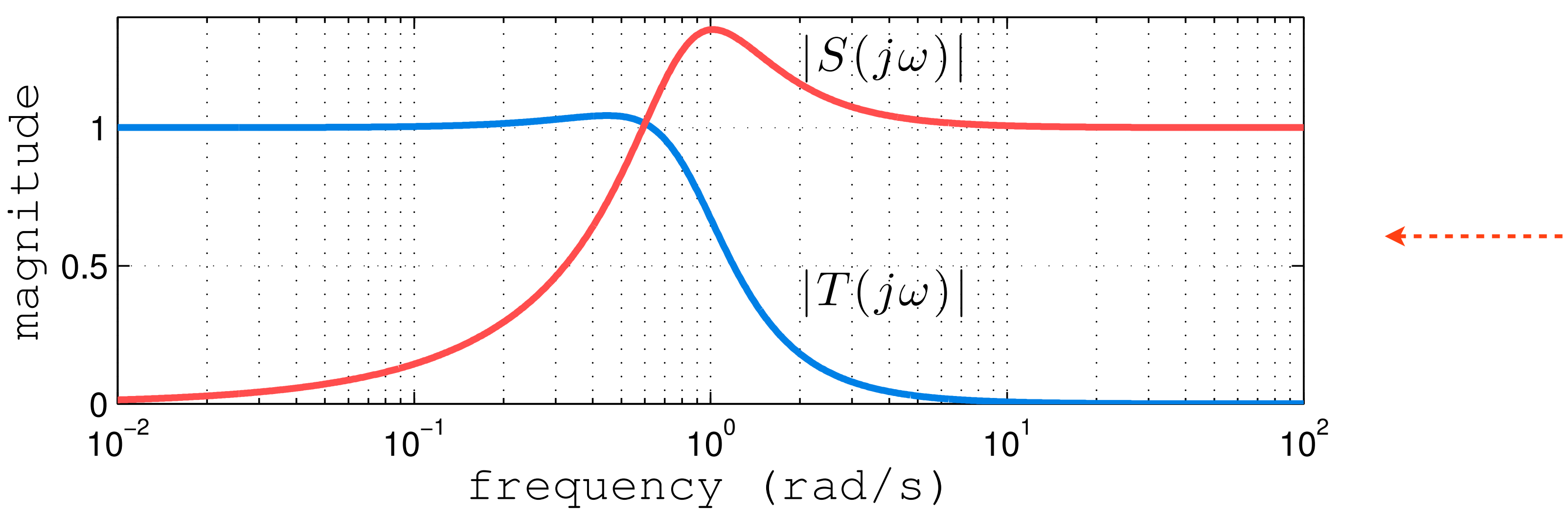
We previously defined the **loop function** $L(s) = C(s)P(s)$ and

$$S(s) = \frac{1}{1+L(s)}$$ **sensitivity function**

$$T(s) = \frac{L(s)}{1+L(s)}$$ **complementary sensitivity function**

$$S_u(s) = \frac{C(s)}{1+L(s)}$$ **control sensitivity function**

- Since $S(s) + T(s) = 1$ there is a clear trade-off between requirements on the reference or disturbances, $S(s)$, and the measurement noise, $T(s)$.
- Note , however, this however does not imply that the sum of the magnitudes is equal to 1 (although it is a good approximation at some frequencies)
- It can also be shown that $|S(j\omega)|$ and $|T(j\omega)|$ differ at most by 1 at the same frequency



$$|S(j\omega)| + |T(j\omega)| \neq 1$$

general considerations

$P(s)$ strictly proper

$C(s)$ strictly proper or proper

→ $L(s) = C(s)P(s)$ strictly proper

$$y(s) = T(s)r(s) + S(s)P(s)d_i(s) + S(s)d_o(s) - T(s)n(s)$$

reference

input disturbance

output disturbance

measurement noise

depends also on the filter properties of the plant (discuss)

we can solve the conflicting requirements between $r(t)$ and $n(t)$ by considering signals in different frequency intervals

low $|S(j\omega)|$

high $|T(j\omega)|$

low $|T(j\omega)|$

low frequency

high frequency

ω

Loop function

from some static requirements we have either

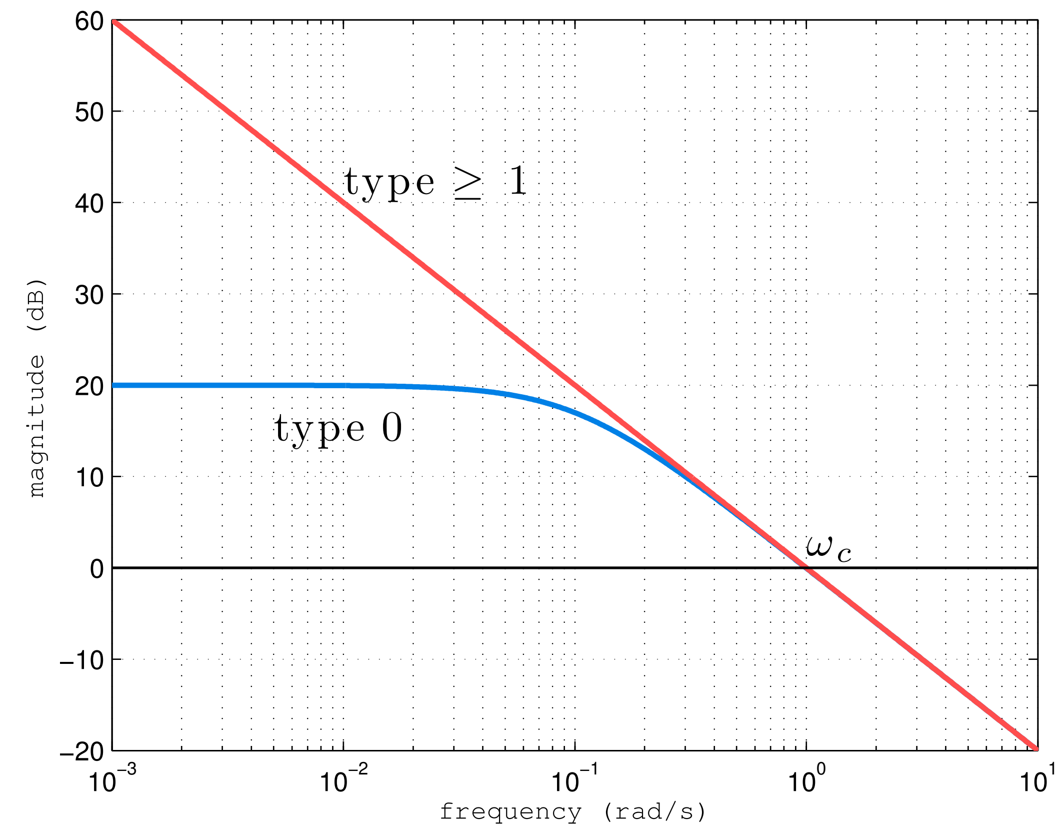
- poles in $s = 0$
- or for a type 0 system we require a small value of the error and therefore a high value of the loop gain K_L

↓

at low frequencies the magnitude is usually required to be large

(see also specifications on sinusoidal disturbances)

typical behavior of the loop function magnitude



approximation

$$|1 + L(j\omega)| \approx \begin{cases} |L(j\omega)| & \text{if} \quad \omega \leq \omega_c \\ 1 & \text{if} \quad \omega > \omega_c \end{cases}$$

bad approximation where $|L(j\omega)|$ is close to 1 (i.e., around ω_c)

in dB

$$|1 + L(j\omega)|_{dB} \approx \begin{cases} |L(j\omega)|_{dB} & \text{if} \quad \omega \leq \omega_c \\ 0\, dB & \text{if} \quad \omega > \omega_c \end{cases}$$

Sensitivity function

$$|S(j\omega)| = \frac{1}{|1 + L(j\omega)|} \approx |S(j\omega)|^{\text{approx}} = \begin{cases} \frac{1}{|L(j\omega)|} & \text{if} \quad \omega \le \omega_c \\ 1 & \text{if} \quad \omega > \omega_c \end{cases}$$

in dB $\begin{cases} -|L(j\omega)|_{dB} \\ 0\,dB \end{cases}$

⇒ the sensitivity function is usually similar to a **high-pass filter**

ok for low frequency reference signals

ok for low frequency disturbance signals

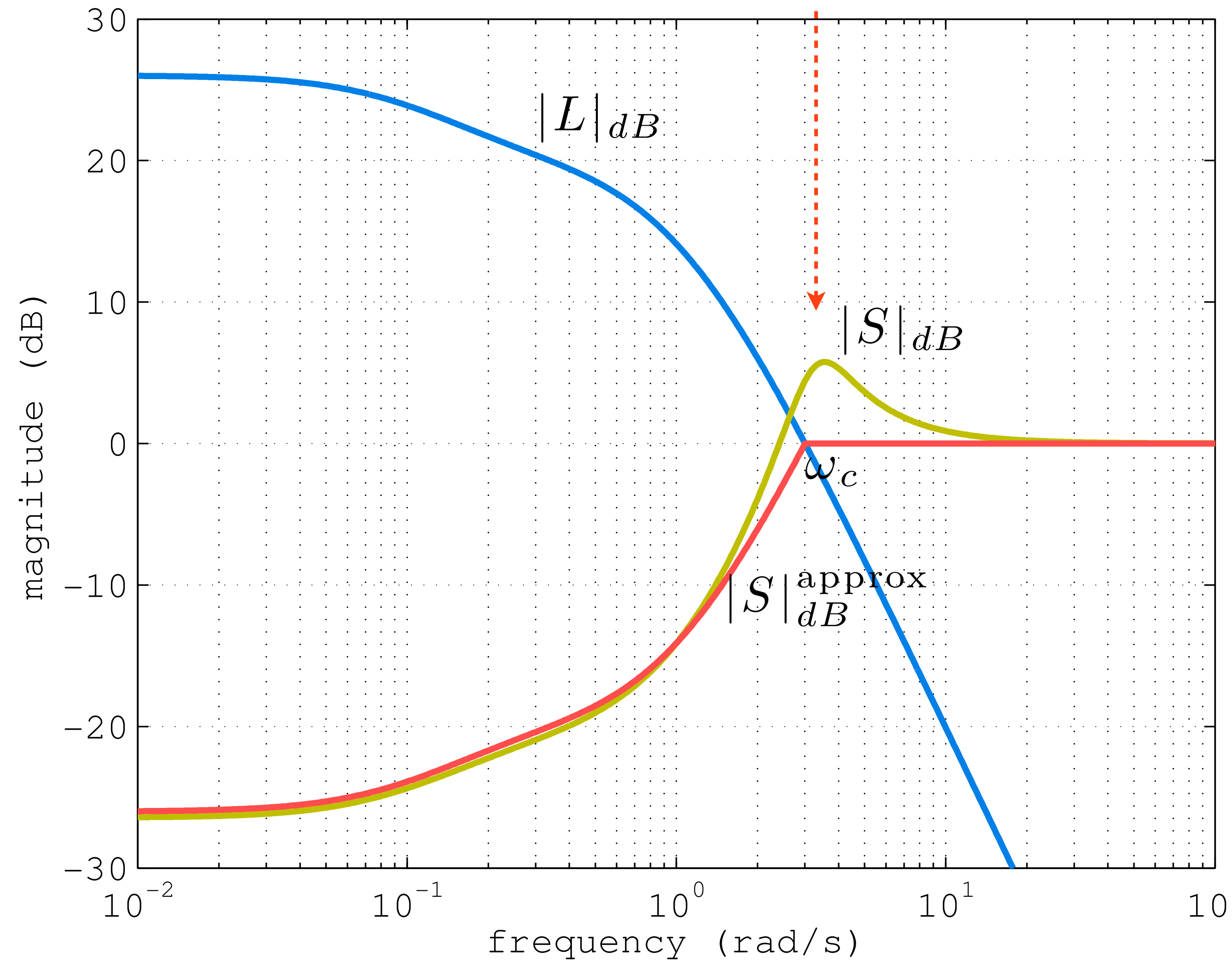
@ low frequency

low sensitivity magnitude

↕

high loop magnitude

bad approximation around ω_c



Complementary sensitivity function

$$|T(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \approx |T(j\omega)|^{\text{approx}} = \begin{cases} 1 & \text{if} \quad \omega \leq \omega_c \\ |L(j\omega)| & \text{if} \quad \omega > \omega_c \end{cases}$$

in dB $\begin{cases} 0\,dB \\ |L(j\omega)|_{dB} \end{cases}$

⇒ the complementary sensitivity function is usually similar to a **low-pass filter**

ok for low frequency reference signals

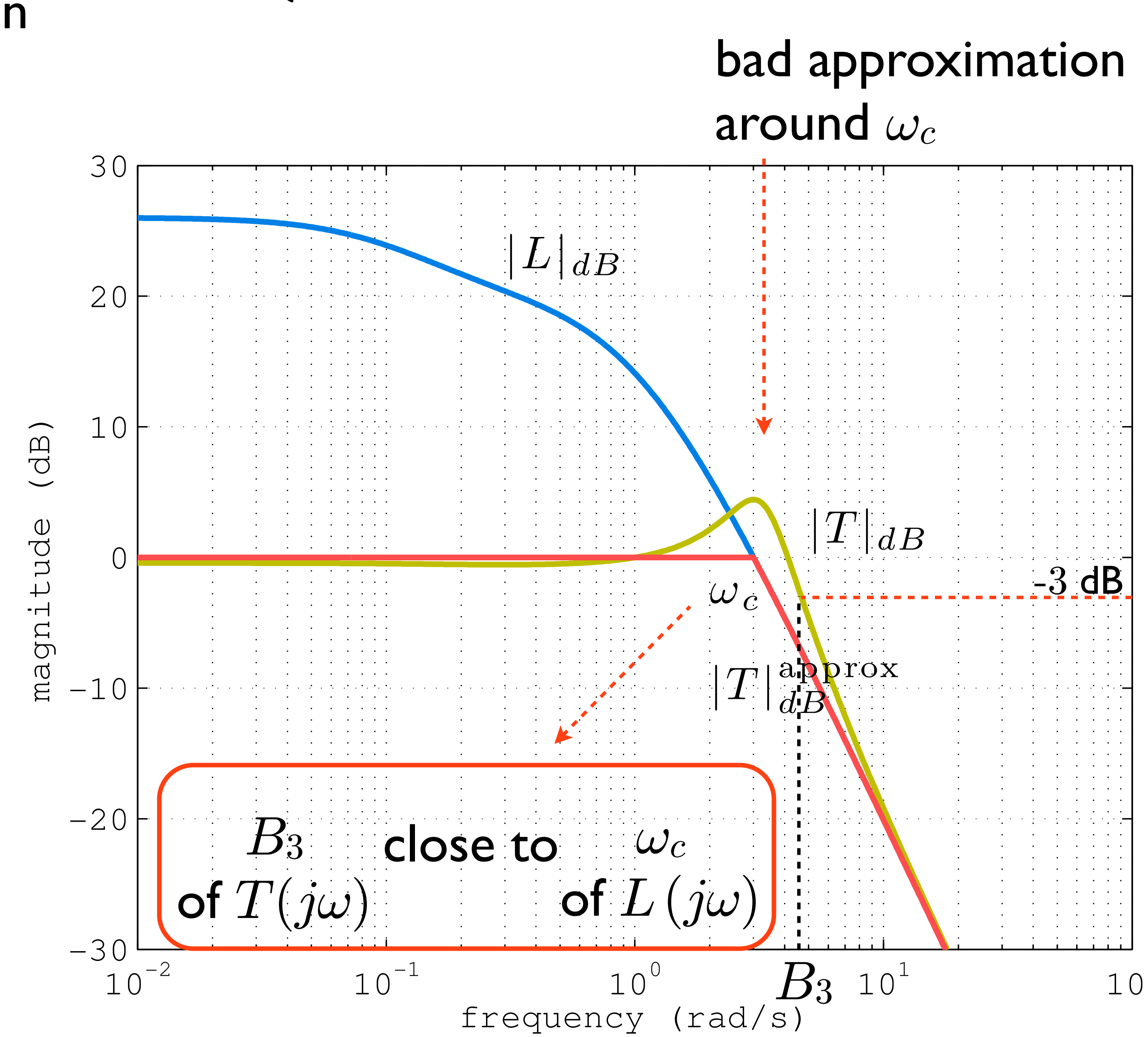
ok for high frequency measurement noise

@ high frequency

low complementary sensitivity magnitude

↕

low loop magnitude

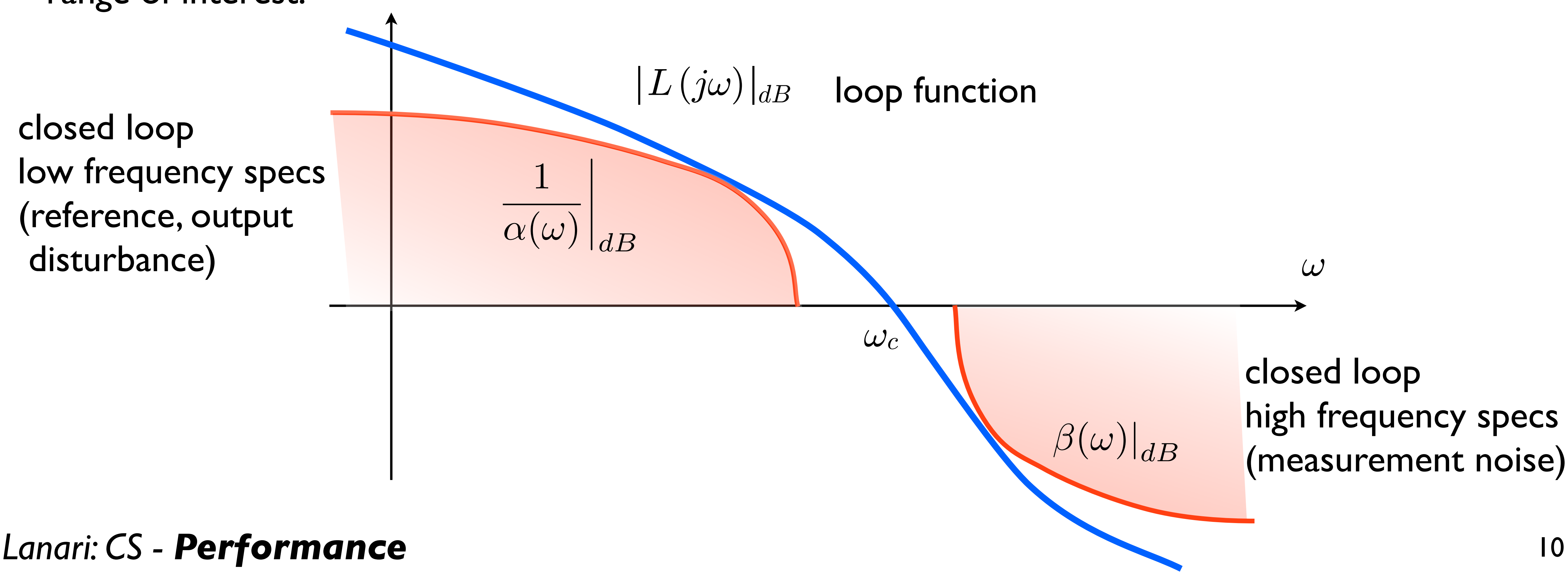


Constraints on the loop function

The previous approximations allow to transform **closed-loop specifications** in open-loop ones

closed-loop specification			open-loop specification (approximated)		
$\lvert S(j\omega)\rvert \leq \alpha(\omega)$	for	$\omega < \omega_c$	$\lvert L(j\omega)\rvert \geq \dfrac{1}{\alpha(\omega)}$	for	$\omega < \omega_c$
$\lvert T(j\omega)\rvert \leq \beta(\omega)$	for	$\omega > \omega_c$	$\lvert L(j\omega)\rvert \leq \beta(\omega)$	for	$\omega > \omega_c$

Since we want attenuation of the disturbances and of the measurement noise and also smaller than one steady state errors w.r.t. sinusoidal references, both α and β are < 1 in the frequency range of interest.



Control effort

- In the presence of a reference and/or disturbances, the control variable should not be in magnitude excessively large. We want to avoid saturation of the actuators, limit too rapid variations which can stress the actuator and reduce its lifespan, reduce the "energy" consumption and remain in the linear domain if we are approximating a nonlinear system around a working condition

- recall that in general, the frequency content of the reference signal $r(t)$ and the disturbances $d_i(t)$ and $d_o(t)$ is concentrated at low frequency while that of the measurement noise $n(t)$ is mostly at high frequency

- at steady state the output $m(t)$ corresponding to a sinusoidal reference $r(t) = \sin(\omega_r t)$ is

$$m_{ss}(t) = |S_u(j\omega_r)| \sin(\omega_r t + \angle S_u(j\omega_r))$$

  Similar contributions are due to $d_i(t)$, $d_o(t)$ and $n(t)$ since

$$m(s) = S_u(s)(r(s) - P(s)d_i(s) - d_o(s) - n(s))$$

Therefore since at steady state the response of $m(t)$ to these signals depends on the magnitude $|S_u(j\omega)|$ we want this magnitude to be as small as possible (while still solving all the specifications)

Control effort

moreover

- for signals $f(t)$ with $f(t) = 0$ for $t < 0$ which admit Fourier transform $F(j\omega)$ we have

$$\|f(t)\|_2 = \langle f(t), f(t) \rangle^{1/2} = \left(\int_0^\infty f(t)^2 dt \right)^{1/2}$$ square root of energy

$$\|F(s)\|_2 = \langle F(s), F(s) \rangle^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega \right)^{1/2}$$ 2-norm

and from the Parseval theorem for finite energy signals

$$\|f(t)\|_2 = \|F(s)\|_2$$

Therefore for finite energy signals $r(t), d_i(t),\ d_o(t)$ and $n(t)$ (with Fourier transforms indicated with capital letters) the contributions to $m(t)$

$$\left.\begin{array}{l} |S_u(j\omega)||R(j\omega)| \\ |S_u(j\omega)||P(j\omega)D_i(j\omega)| \\ |S_u(j\omega)||D_o(j\omega)| \\ |S_u(j\omega)||N(j\omega)| \end{array}\right] \longrightarrow |M(j\omega)|$$

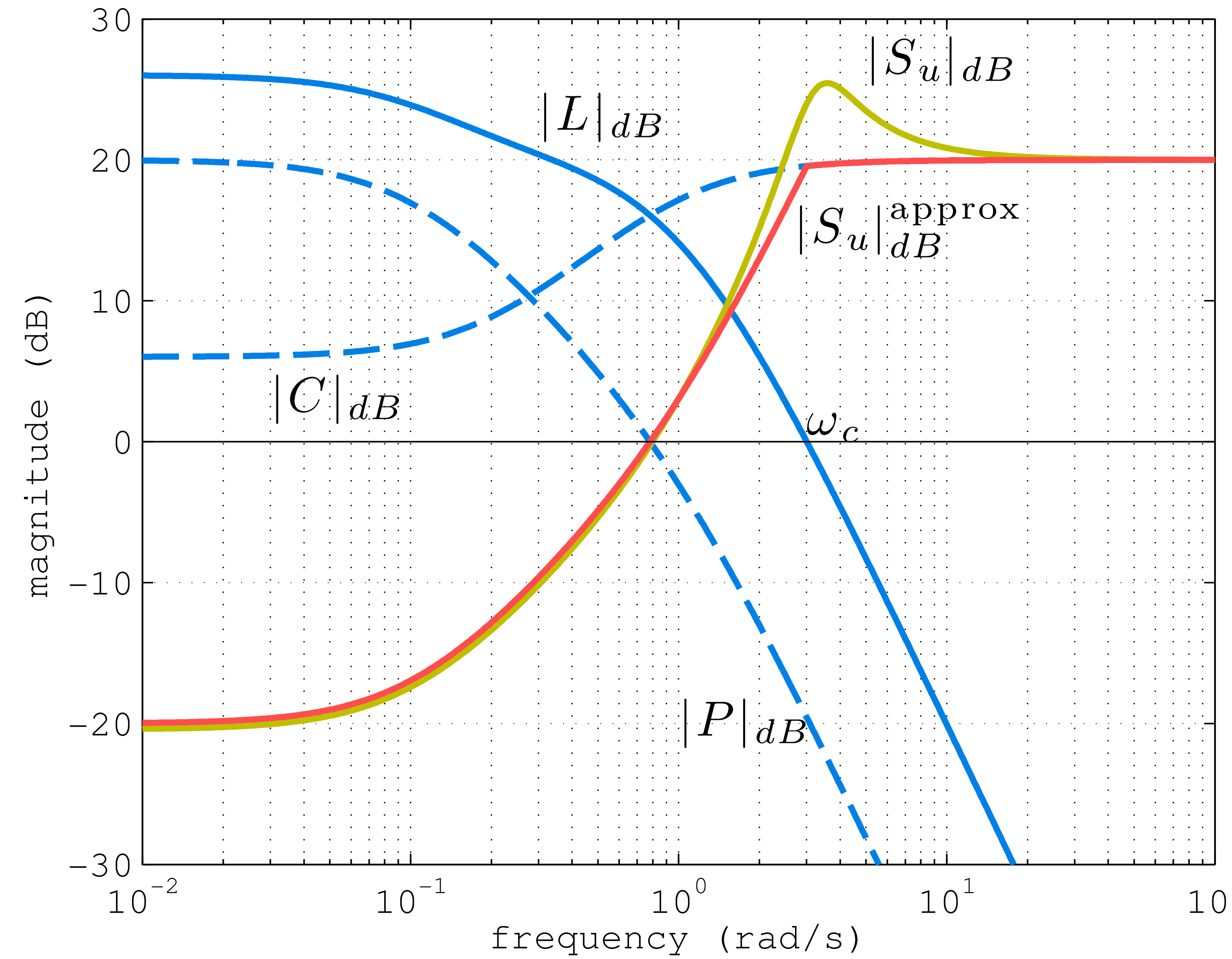
it is therefore useful to make $|S_u(j\omega)|$ as small as possible (while satisfying the specifications)

Control sensitivity function

$$|S_u(j\omega)| = \frac{|C(j\omega)|}{|1+L(j\omega)|} \approx |S_u(j\omega)|^{\text{approx}} = \begin{cases} \frac{1}{|P(j\omega)|} & \text{if} \quad \omega \leq \omega_c \\ |C(j\omega)| & \text{if} \quad \omega > \omega_c \end{cases}$$

in dB $\begin{cases} -|P(j\omega)|_{dB} \\ |C(j\omega)|_{dB} \end{cases}$

bad approximation around ω_c



at **low frequency**

independent from the controller

depends only on the plant

at **high frequency**

depends only on the controller

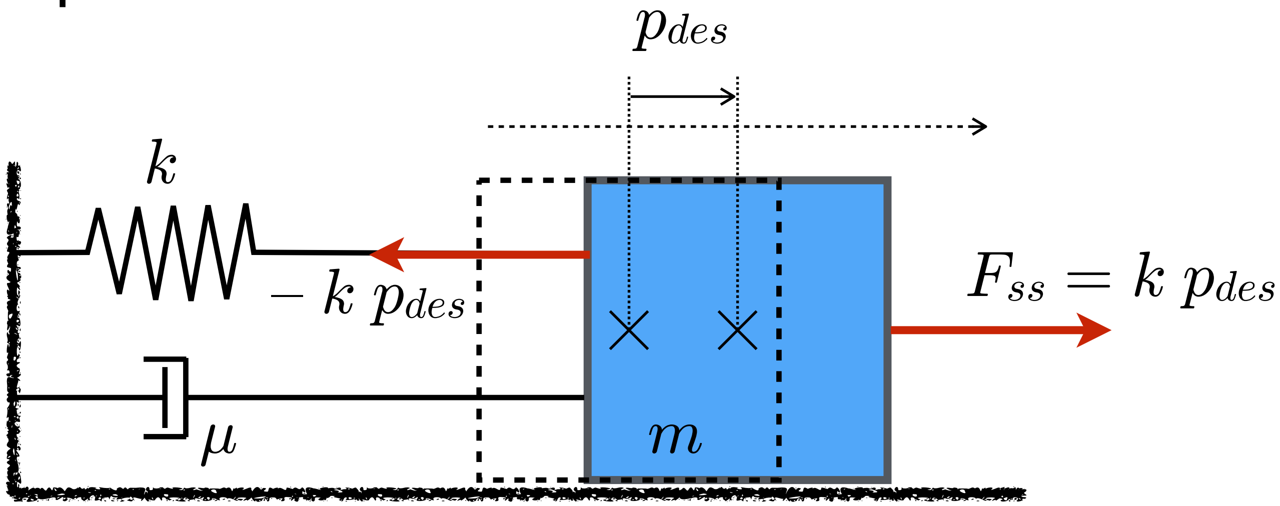
and not on the plant

example

consider the Mass-Spring-Damper (MSD) system with the transfer function from the traction force F to the mass position p being

$$P(s) = \frac{1}{m\, s^2 + \mu\, s + k}$$

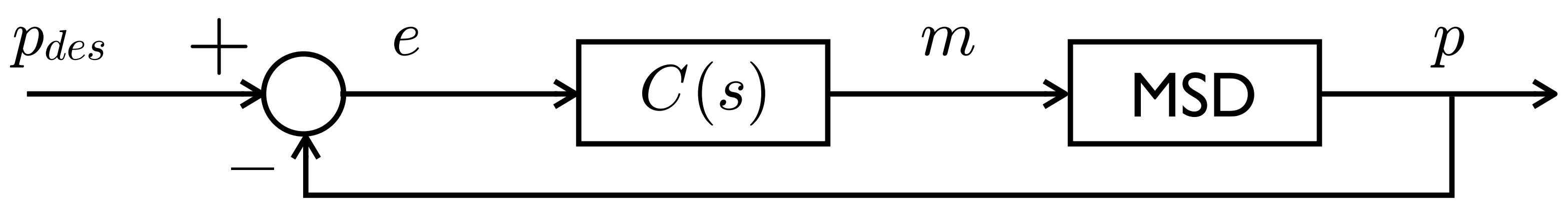
- clearly to keep the mass at a constant position p_{des} from the origin, we need to apply the constant force $F_{ss} = k\, p_{des}$ to counterbalance the force due to the spring. Note that this steady state force is independent from m and μ. These parameters instead, together with k, influence the transient.
- This would be an open loop solution.



- See also Matlab Live Script file `MSD_Control_Effort.mlx`

example (continued)

- In a feedback control system, we can analyze the control input at steady state through the control sensitivity function $S_u(s)$ which relates the reference r to the control input m.



any controller that guarantees 0 steady state error (type 1 system for a constant $r(t) = p_{des}$) will necessarily have a pole in $s = 0$ since the plant does not have any. In this case the approximation of $|S_u(j\omega)|$ at low frequency is valid (in fact $S_u(0)$ is exactly $1/P(0)$) and the control input is

$$m_{ss} = S_u(0)\; p_{des} = p_{des}\;/P(0) = k\; p_{des} = F_{ss}$$

the value of the control input m_{ss} at steady state depends only from the plant parameters (here only k) but do not get confused since it is the controller who provides this input.

If the control system is type 0, the control input at steady state is

$$m_{ss} = S_u(0)p_{des} = \left.\frac{C(s)}{1+C(s)P(s)}\right|_{s=0} p_{des} = \frac{K_c}{1+K_cK_p}p_{des} = \frac{1}{1/K_c + K_p}p_{des} = \frac{1}{1/K_c + k}p_{des}$$

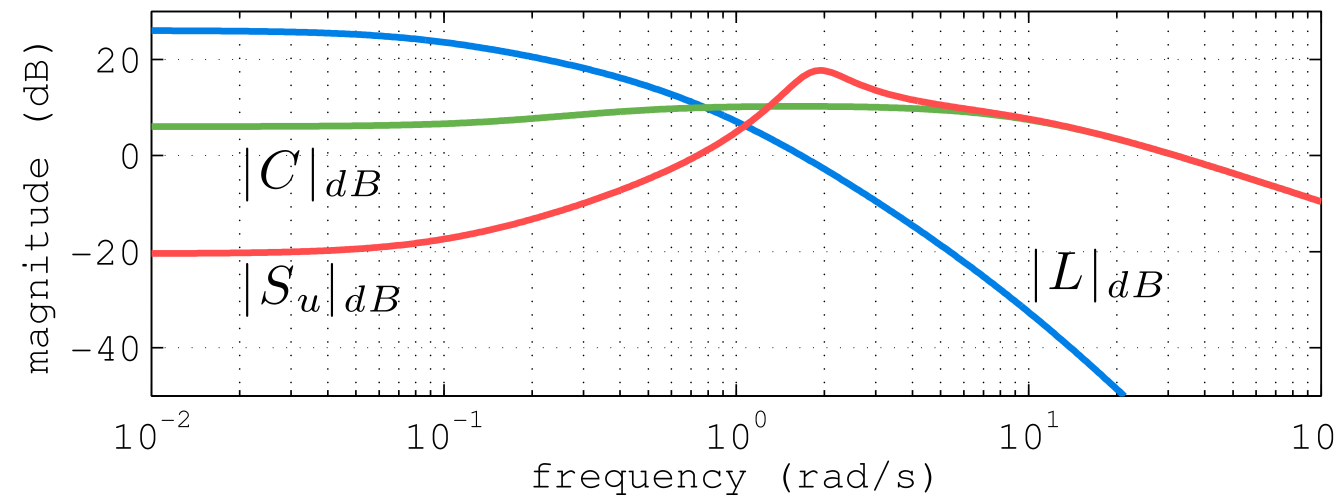
which, for high values of K_c (and therefore for high values of the loop frequency response magnitude $|L(j\omega)|$ at low frequency) tends to $k\; p_{des}$

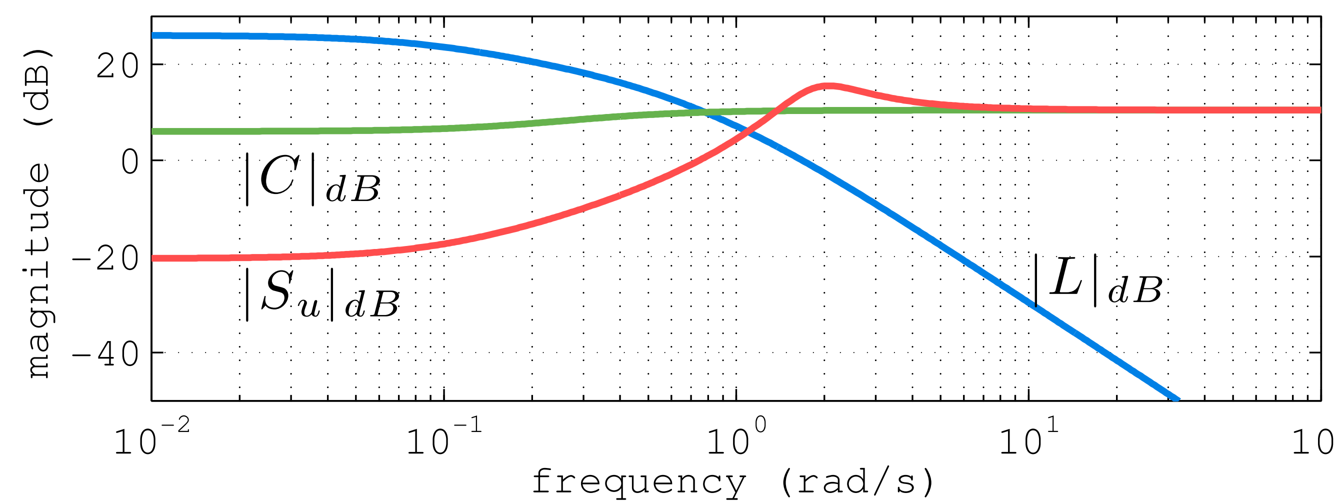
Control sensitivity function

If we consider the other specifications met, it is preferable to have low magnitude for the control sensitivity function and therefore it is better to avoid a proper controller when possible

Techniques that lead to proper controllers:

- PD (approximation)
- pole placement





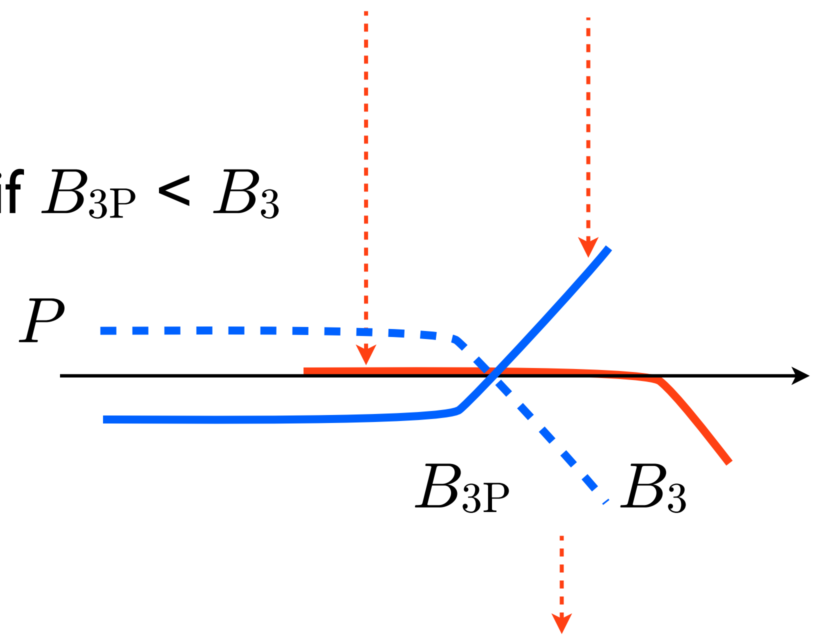
It always depends upon the frequency content of the signals involved

Control sensitivity function

$$S_u(s) = C(s)S(s) = T(s)P(s)^{-1}$$

$$|S_u(j\omega)| = |T(j\omega)||P(j\omega)|^{-1}$$

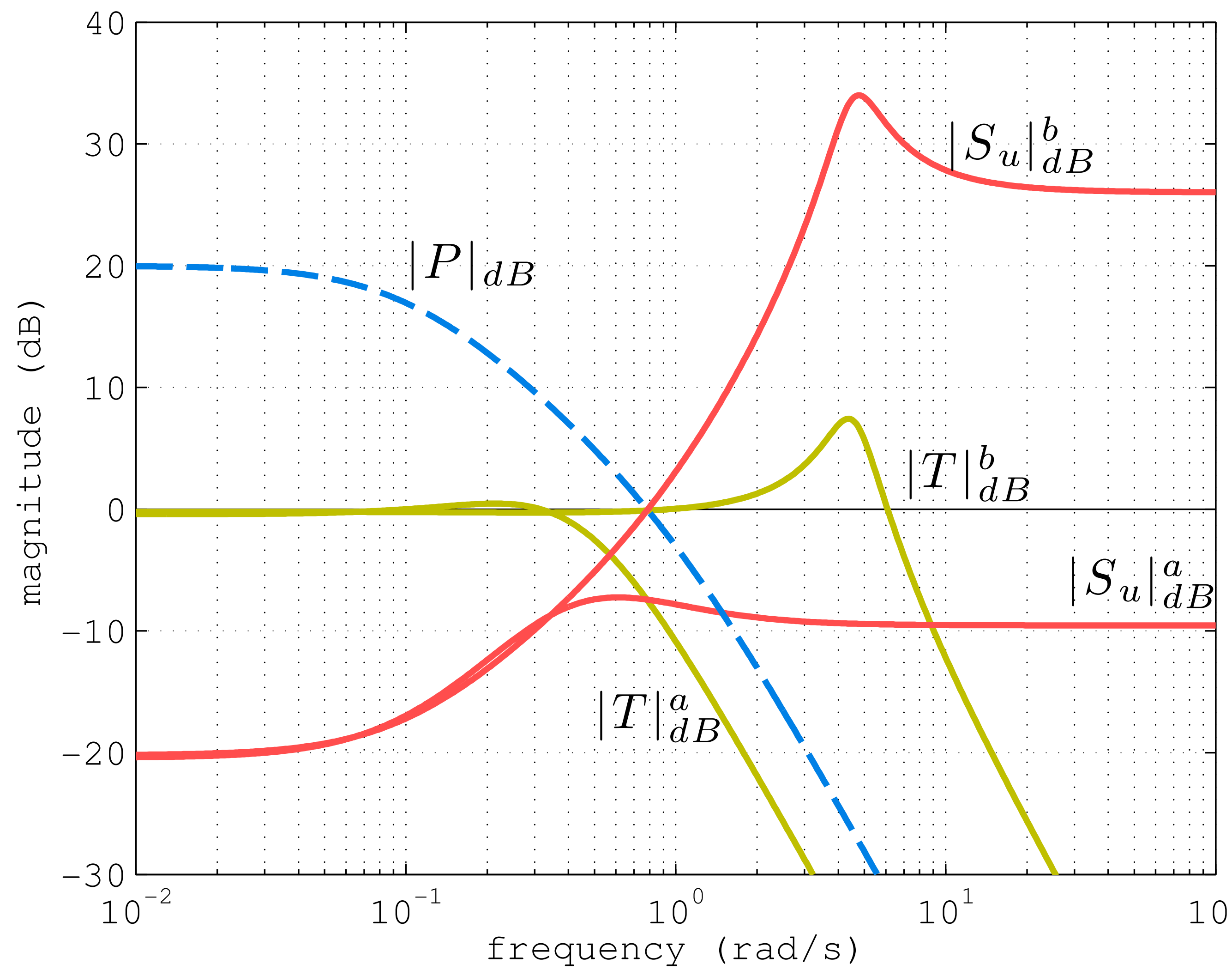
if $B_{3\mathrm{P}} < B_3$



magnitude increase in the interval $[B_{3\mathrm{P}},\ B_3]$

plant bandwidth $B_{3\mathrm{P}}$

closed-loop bandwidth B_3

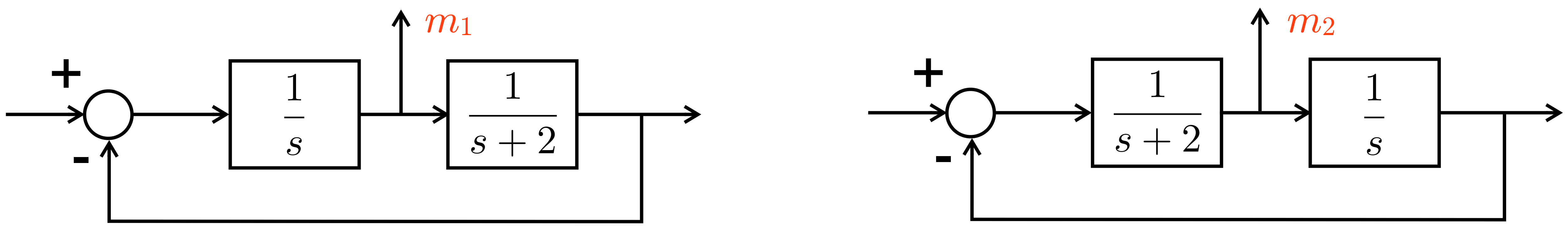


closed-loop bandwidth increase from $B_3{}^a$ to $B_3{}^b$

an increase of the closed-loop bandwidth B_3 w.r.t. the plant's bandwidth $B_{3\mathrm{P}}$ comes at the expense of an increase of the control effort

Control sensitivity function

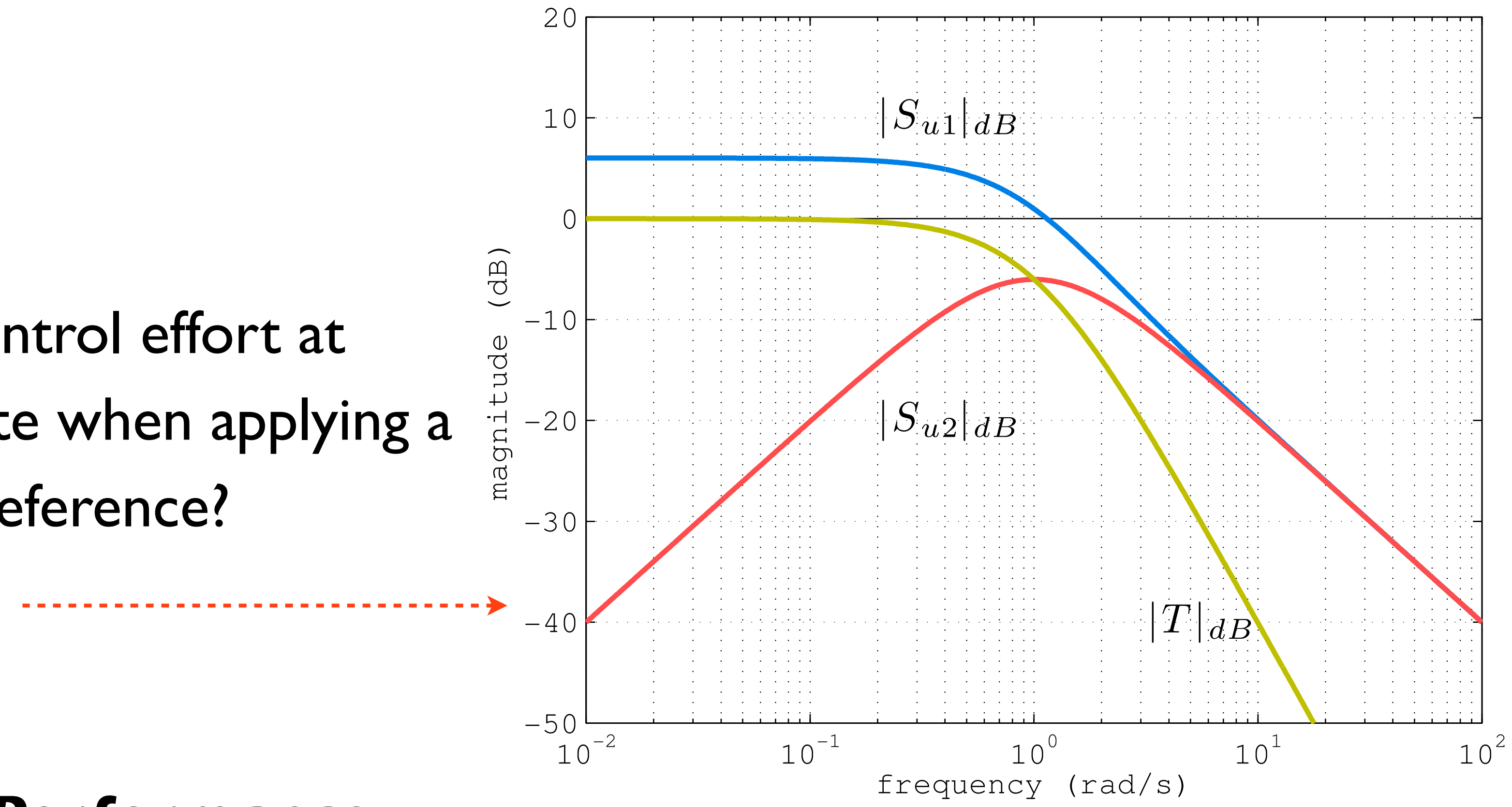
Effect of an integrator on control effort

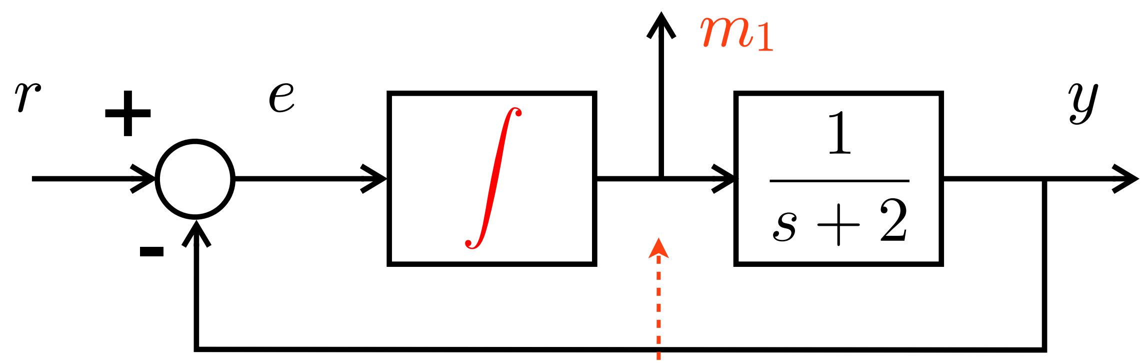


same $L(s)$ then same $T(s)$ and $S(s)$

$$S_{u1}(s) = \frac{s+2}{(s+1)^2} \qquad\qquad S_{u2}(s) = \frac{s}{(s+1)^2}$$

why no control effort at steady-state when applying a constant reference?





integral of e

at steady-state:

y is equal to the constant r

then the error e is null

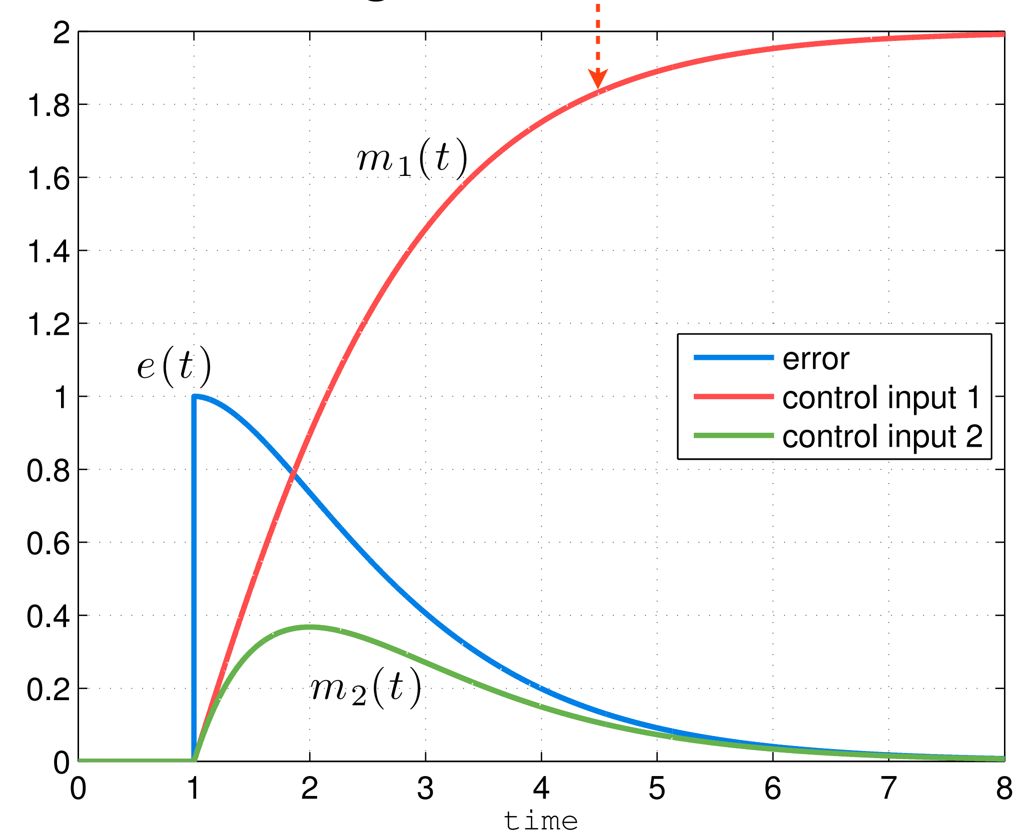
m_1 is the accumulated

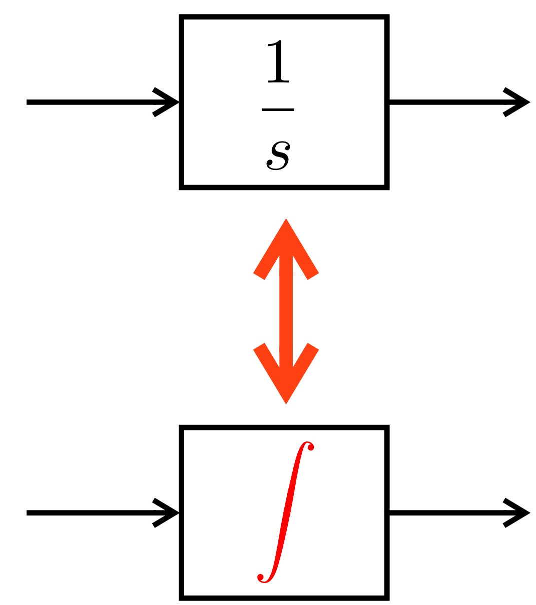
error and drives $1/s{+}2$

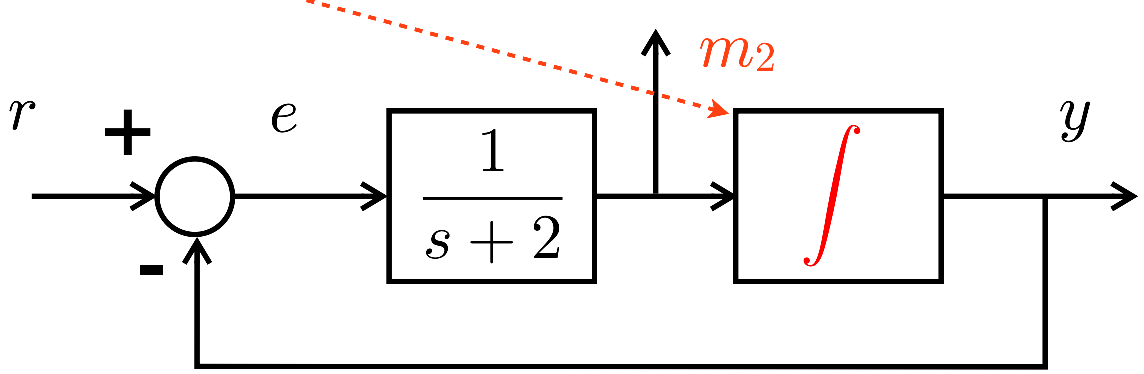
while m_2 is zero but the

output is non-zero due to

the integrator







Vocabulary

English	Italiano
sensitivity function	funzione di sensitività
complementary sensitivity function	funzione di sensitività complementare
control sensitivity function	funzione di sensitività del controllo