Control Systems

Control Design: Loop shaping

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Outline

- specifications
- open-loop shaping principle
- lead and lag controllers
- 4 basic situations
- sketch of PID controllers

Specifications (closed-loop)

Static or at steady state

- desired behavior w.r.t. order k inputs in terms of a maximum allowed absolute error
- desired attenuation level w.r.t. constant disturbances acting on the forward loop
- tracking of a sinusoidal reference
- attenuation of sinusoidal disturbances and measurement noise

Dynamic

- location of eigenvalues/poles in the complex plane
- time domain specifications on the step response (mainly on the reference to output behavior)
- frequency domain specifications through the resonance peak and the bandwidth (mainly on the reference to output behavior)

Stability

- location of eigenvalues/poles in the complex plane
- robustness in terms of gain and/or phase margin

Specifications

CLOSED-LOOP System

OPEN-LOOP System

Static or steady state equivalent to

presence of a sufficient number of poles in $s = 0$ and/or at the reference/disturbance angular frequency + sufficiently high gain (in absolute value)

Necessary conditions require the following structure in the controller

$$\frac{K_c}{s^h}\left(\frac{1}{s^2+\bar{\omega}^2}\right)$$

for sinusoidal reference or disturbance

Dynamic

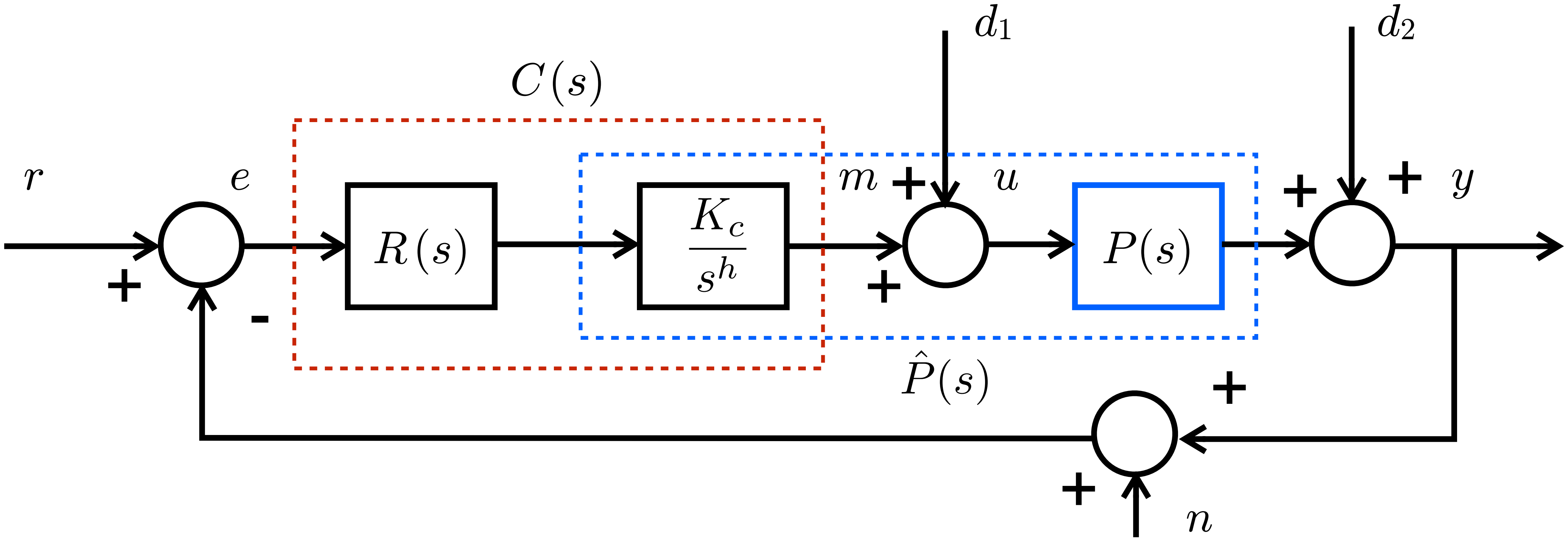
bandwidth B_3 (and rise time t_r) → ω_c crossover frequency

resonant peak M_r (and overshoot M_p) → PM phase margin

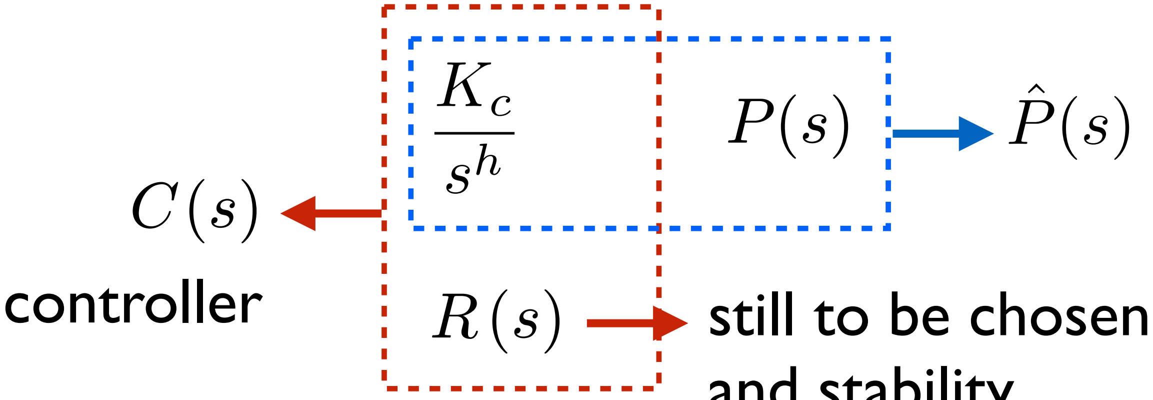
taken care by $R(s)$

generic structure of the controller provided $R(s)$ does not alter the satisfaction of the steady-state requirements (may have extra imaginary poles for sinusoidal reference)

$$\boxed{C(s) = \frac{K_c}{s^h}R(s)}$$



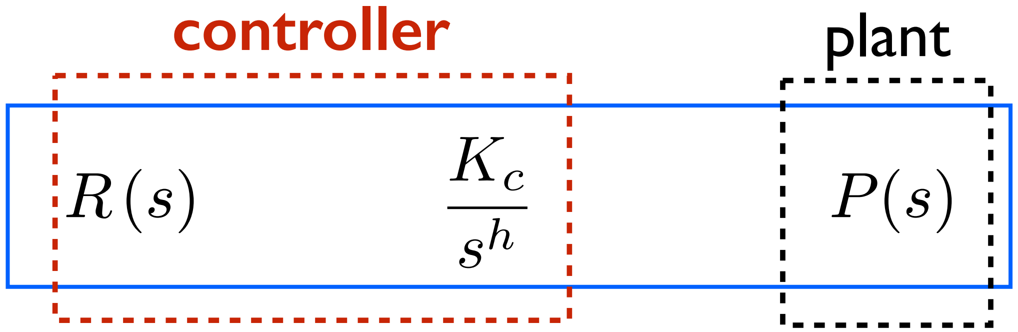
necessary part of the controller



$\frac{K_c}{s^h}$ $P(s)$ → $\hat{P}(s)$ extended/modified plant

$C(s)$ ← controller

$R(s)$ → still to be chosen in order to satisfy the dynamic specifications and stability



controller

plant

Loop function $L(s) = R(s) \frac{K_c}{s^h} P(s)$

dynamic specifications + stability

static specifications

open-loop shaping: basic idea

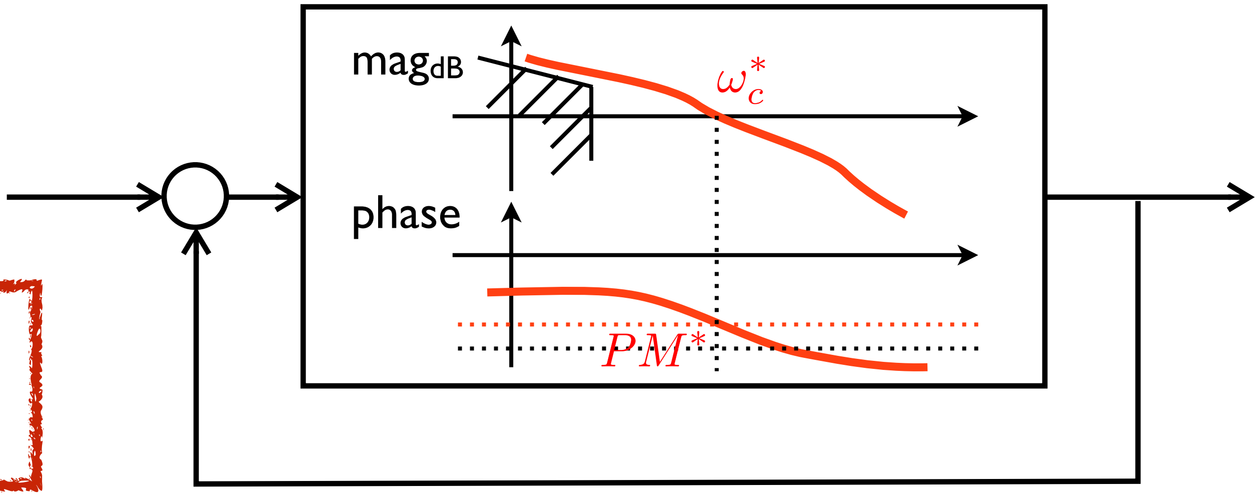
Basic idea: being the plant $P(s)$ fixed, choose the controller $C(s)$ to **shape** the **loop function** frequency response $L(j\omega)$ such that the closed-loop satisfies the specifications

$C(s)$

such that

closed-loop stability will be guaranteed by the **Bode stability theorem** provided it can be applied

$L(j\omega)$ with some desired characteristics



Loop shaping will be applied only to plants which do not have poles with Re [.] > 0

- for the same reason, we will not introduce poles with Re [.] > 0 through the controller
- a part of the desired shape of the loop function is determined by the necessary steady-state requirements (poles in $s = 0$, minimum value for the loop gain, …)

$C(s) = C_1(s)R(s)$ with

$C_1(s)$ necessary part for static specs → $\dfrac{K_c}{s^h}$ either $|K_c|$ free or $|K_c| \geq K_{c,min}$

$R(s)$ dynamic performance specs + stability → to be defined (lead/lag compensators)

open-loop shaping: steady-state specifications

Loop function $L(s) = R(s)\,\dfrac{K_c}{s^h}\,P(s)$ $\longrightarrow$ Loop (generalized) gain $K_L = K_r K_c K_p$

the (static) gain K_r of $R(s)$ can be:

- if a steady state specification requires a sufficiently high loop gain $|K_L| \geq K_{L,min}$

  which is guaranteed choosing $|K_c| \geq K_{c,min}$

  then K_r can only be greater equal than 1, for example to achieve some amplification

- if the steady state specifications do not imply a specific requirement for the loop gain (except for its sign) then we have a unique controller gain $K_c K_r$ which can be chosen (in magnitude) freely, for example in order to achieve (if necessary) some attenuation

open-loop shaping: steady-state specifications

hyp: the necessary part of the controller $C(s)$ has been already determined so we have the **extended/modified plant**

$$\hat{P}(s) = C_1(s)P(s) = \frac{K_c}{s^h}P(s)$$

we need to determine $C(s)$, and therefore $R(s)$, so to satisfy also the **dynamic specifications** and ensure **closed loop stability**

+ dynamic specifications

$\omega_c{}^*$ desired crossover frequency at which we want to have a phase margin of at least PM^*

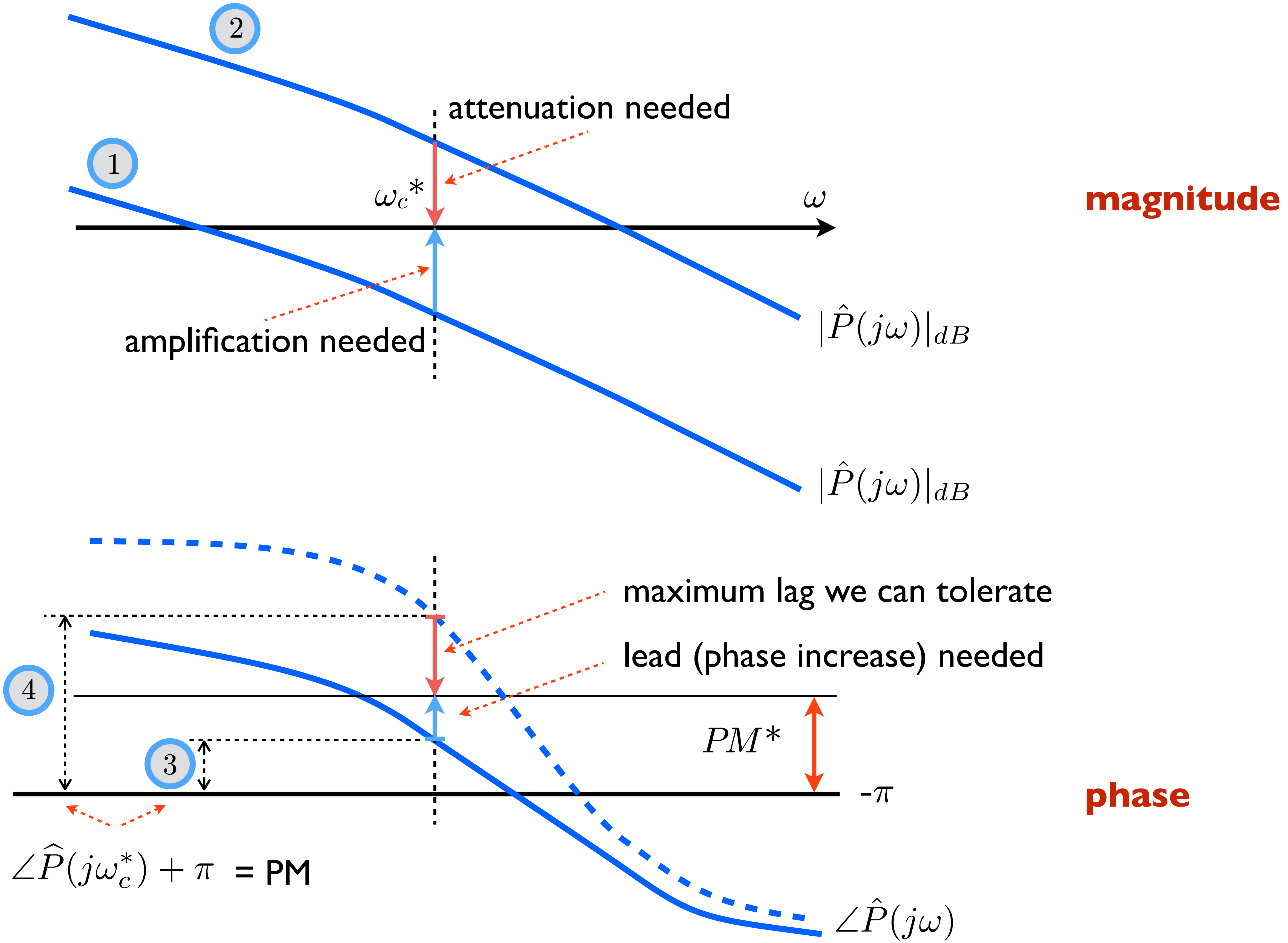
from the extended plant frequency response we need to check which action is necessary both in terms of magnitude and phase by comparing the actual value of the magnitude and phase at the future crossover frequency $\omega_c{}^*$

Magnitude
- **Amplification**: we need to increase the magnitude at some frequency
- **Attenuation**: we need to decrease the magnitude at some frequency

Phase
- **Lead**: we need to increase the phase at some frequency
- **Lag**: phase can be decreased at some frequency if necessary

we want in $\omega_c{}^*$ $\quad PM \geq PM^*$ $\quad$ so if we have extra phase we can keep it

4 possible actions in terms of magnitude and phase at some frequency (typically at the desired crossover frequency $\omega_c{}^*$)



$\angle\widehat{P}(j\omega_c^*) + \pi$ = PM

summary

① • if $\left|\widehat{P}(j\omega_c^*)\right|_{dB} < 0$ we have to **amplify** at $\omega_c{}^*$ by exactly $-\left|\widehat{P}(j\omega_c^*)\right|_{dB}$

therefore $R(s)$ needs to be such that $|R(j\omega_c^*)|_{dB} = -\left|\widehat{P}(j\omega_c^*)\right|_{dB} > 0$

② • if $\left|\widehat{P}(j\omega_c^*)\right|_{dB} > 0$ we have to **attenuate** at $\omega_c{}^*$ by exactly $-\left|\widehat{P}(j\omega_c^*)\right|_{dB} < 0$

therefore $R(s)$ needs to be such that $|R(j\omega_c^*)|_{dB} = -\left|\widehat{P}(j\omega_c^*)\right|_{dB} < 0$

③ • if $\angle\widehat{P}(j\omega_c^*) + \pi < PM^*$

we have to **increase the phase** at $\omega_c{}^*$ by at least $PM^* - \angle\widehat{P}(j\omega_c^*) - \pi > 0$

therefore $R(s)$ needs to be such that $\angle R(j\omega_c^*) \geq PM^* - \angle\widehat{P}(j\omega_c^*) - \pi > 0$

④ • if $\angle\widehat{P}(j\omega_c^*) + \pi > PM^*$

we can **tolerate to decrease the phase** at $\omega_c{}^*$ by at most $PM^* - \angle\widehat{P}(j\omega_c^*) - \pi < 0$

therefore $R(s)$ needs to be such that $\angle R(j\omega_c^*) \geq PM^* - \angle\widehat{P}(j\omega_c^*) - \pi < 0$

in both cases phase margin requirement is $\angle R(j\omega_c^*) + \angle\widehat{P}(j\omega_c^*) + \pi \geq PM^*$

but magnitude and phase are not independent (except for gain term)

Remember that the static specifications have already been met (if we ensure stability) and therefore we do not want to alter this first step.

Therefore, in general, we are **not** going to use a

- zero in $s = 0$ to obtain a phase lead
- pole in $s = 0$ to attenuate at some frequency
- a gain smaller than 1 in magnitude to attenuate if we have a constraint on the loop gain of the form $|K_L| \geq K_L^{min}$ from the static requirements (while we may use a gain greater than 1 to amplify)

elementary functions that can provide these magnitude and phase contributions

lead compensator

$$R_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s} \qquad \begin{array}{l} \tau_a > 0 \\ m_a > 1 \end{array}$$

lag compensator

$$R_i(s) = \frac{1 + \frac{\tau_i}{m_i} s}{1 + \tau_i s} \qquad \begin{array}{l} \tau_i > 0 \\ m_i > 1 \end{array}$$

both have **unit gain** so no magnitude change in $\omega = 0$

Names come from their effect on the phase

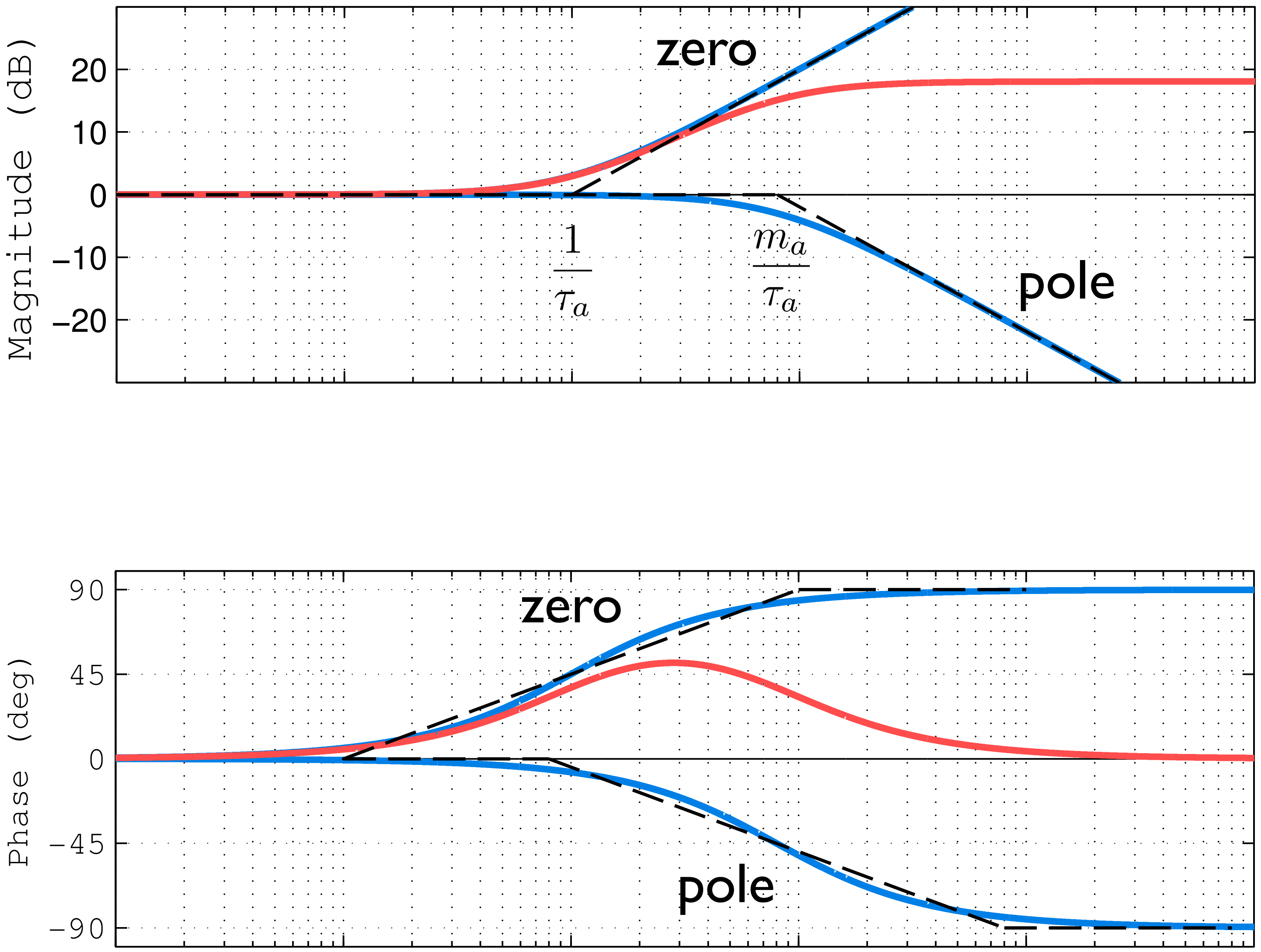
Lead compensator

magnitude and phase contributions of a lead compensator (or function)

$$R_a(s) = \frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s} \qquad \begin{aligned}\tau_a &> 0 \\ m_a &> 1\end{aligned}$$

zero cut-off frequency $\frac{1}{\tau_a}$

pole cut-off frequency $\frac{m_a}{\tau_a}$ to the right of the zero



amplification

↑

in a frequency range

↓

phase lead

here
$m_a = 8$

Lag compensator

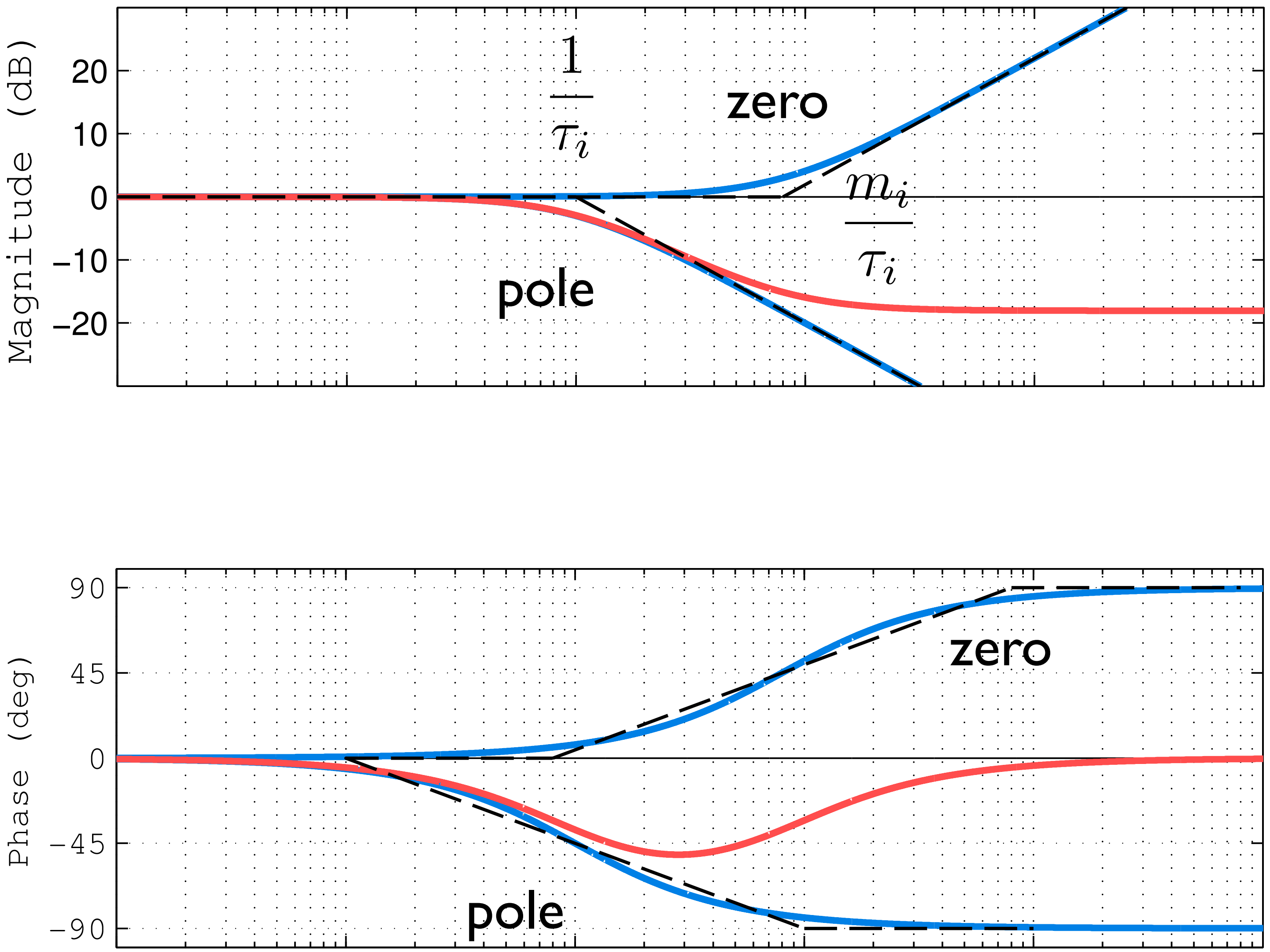
magnitude and phase contributions of a lag compensator (or function)

zero in

$$R_i(s) = \frac{1 + \frac{\tau_i}{m_i} s}{1 + \tau_i s} \qquad \begin{array}{l} \tau_i > 0 \\ m_i > 1 \end{array}$$

zero cut-off frequency $\frac{m_i}{\tau_i}$ to the right of the pole

pole cut-off frequency $\frac{1}{\tau_i}$



attenuation

↑

in a frequency range

↓

phase lag

here

$m_i = 8$

Choice of $R(s)$

we assume $C_1(s)$ (static specs) has already been chosen. Therefore we need to

- by evaluating the actual values of the extended plant magnitude and phase **at the desired crossover** frequency $\omega_c{}^*$, understand what **action** needs to be undertaken:
  - **amplification** or **attenuation** at $\omega_c{}^*$
  - **phase lead** or **maximum allowed phase lag** at $\omega_c{}^*$
- choose the elementary function(s) $R_a(s)$ and/or $R_i(s)$ (since multiple actions can be combined) needed
  - choose m_a (or m_i) and the normalized frequency $\omega\tau$
  - deciding to obtain the desired action at $\omega_c{}^*$ choose τ_a (or τ_i) so that

$$\omega_c{}^*\tau_a = \boxed{\omega\tau}$$

(or $\omega_c{}^*\tau_i = \omega\tau$)

frequency
at which we
want to obtain
the desired action

chosen

$\rightarrow$ we solve
in τ_a

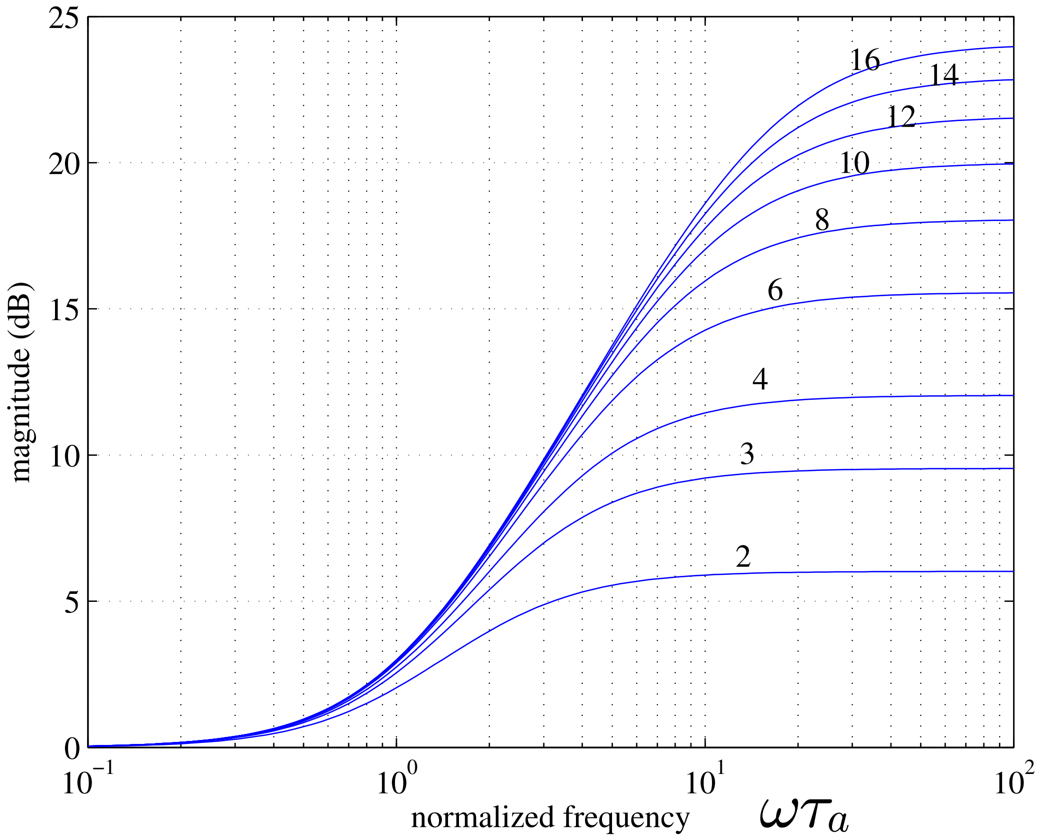
(or in τ_i)

Universal diagrams

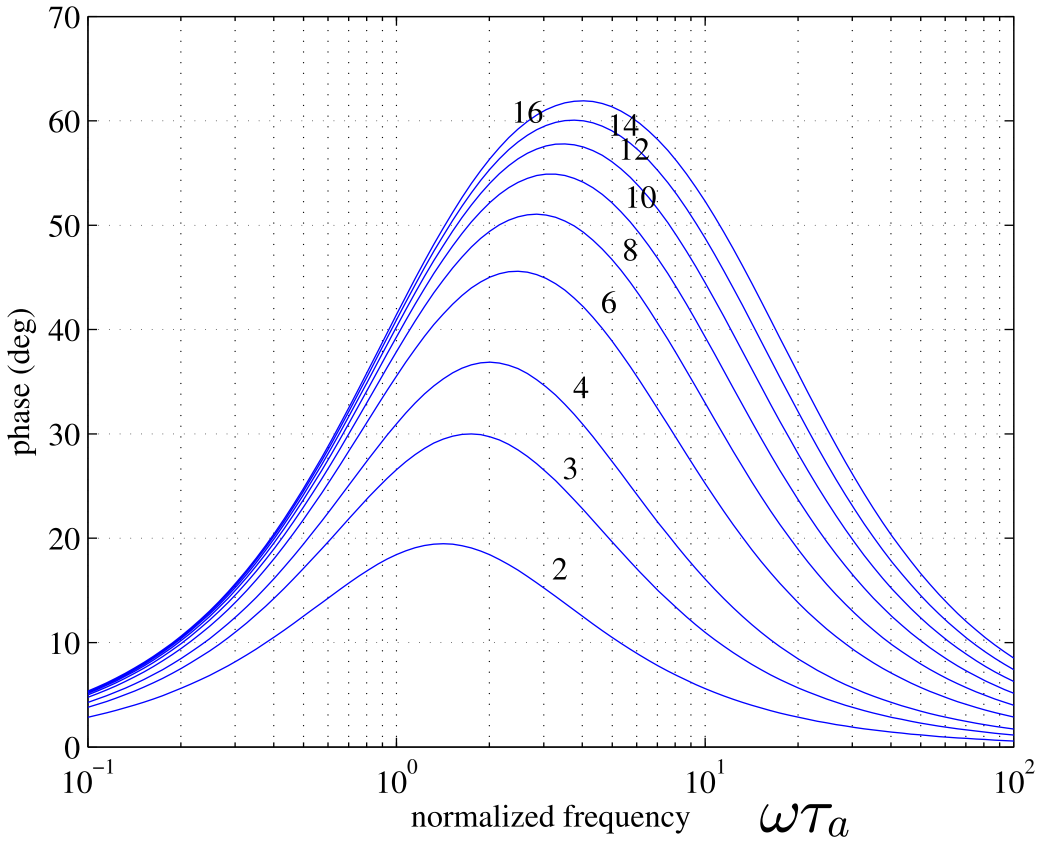
$$R_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}$$

$$\tau_a > 0$$

$$m_a > 1$$

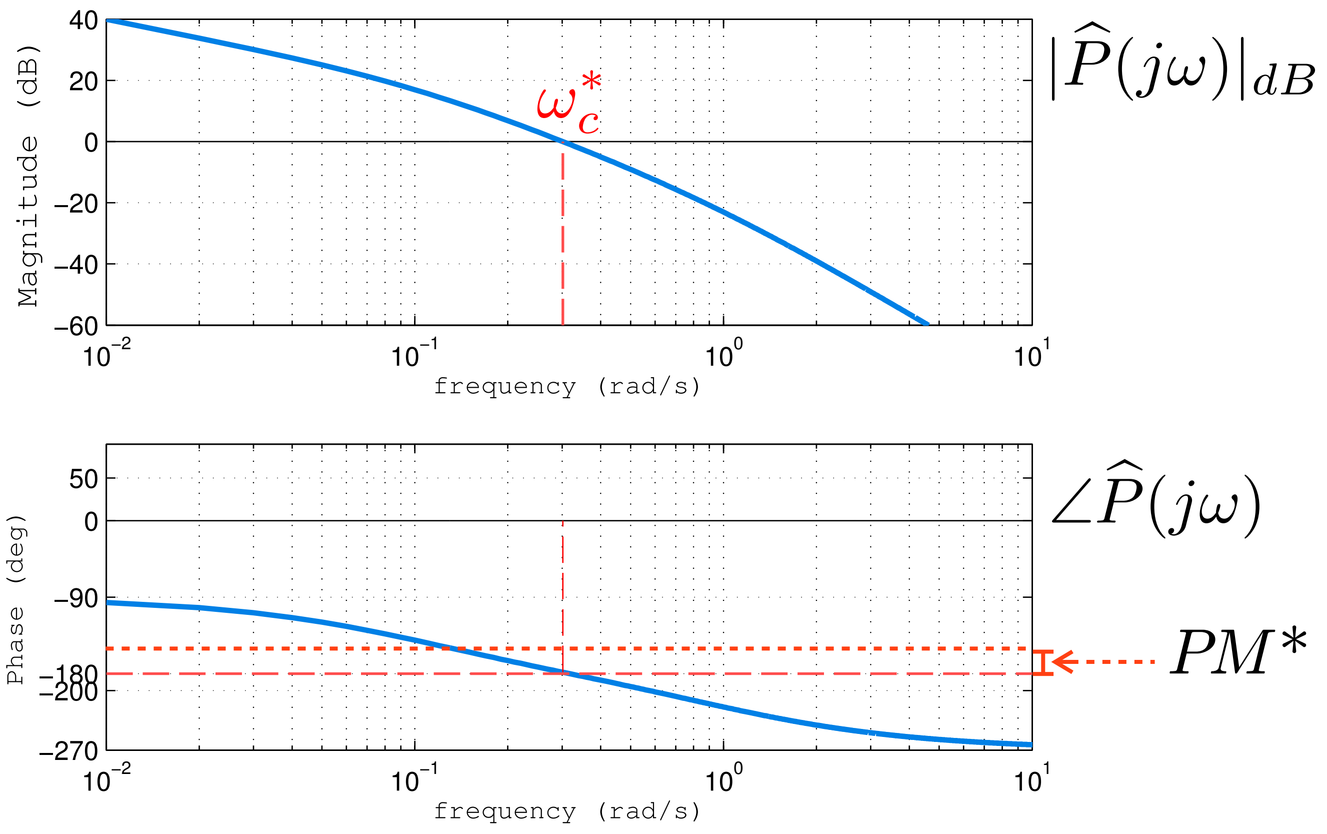


for different values of m_a



for $R_i(s)$ just change sign to the ordinates

Case I



specifications $\quad {\omega_c}^* = \omega_c$

$PM \geq PM^*$

actions needed:

- **magnitude**: as small amplification as possible in order not to move the current crossover frequency which coincides with the desired one
- **phase**: increase the phase, in this case of exactly PM^*

provided by a **lead function**

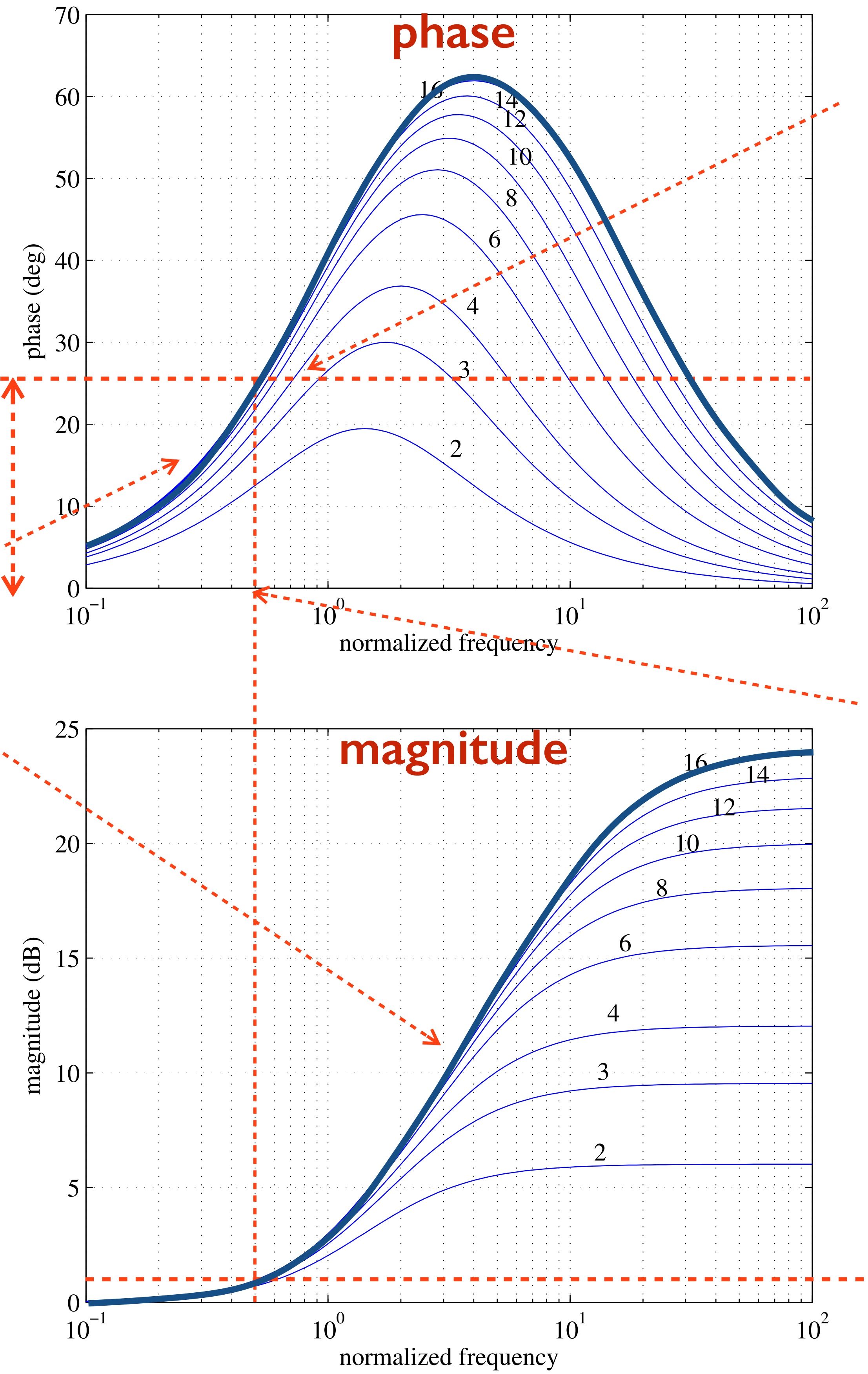
which, as a by-product also gives some amplification, therefore we need to choose the pair $(m_a, \omega T)$ such that the lead function provides the desired phase lead but almost no amplification

case I example: imagine we need a phase lead of 25° with the smallest amplification possible

step

① 25°

desired phase lead



several choices of m_a are possible (at different normalized frequencies), we need to find one which is compatible with the magnitude requirement

0.5 is the smallest (for these plots) normalized frequency at which we obtain the desired phase lead (for $m_a = 16$)

②

smallest amplification obtainable (around 1 dB)

1 dB

we have chosen

$m_a = 16$

$\omega\tau = 0.5$

therefore to obtain

this lead of 25° together

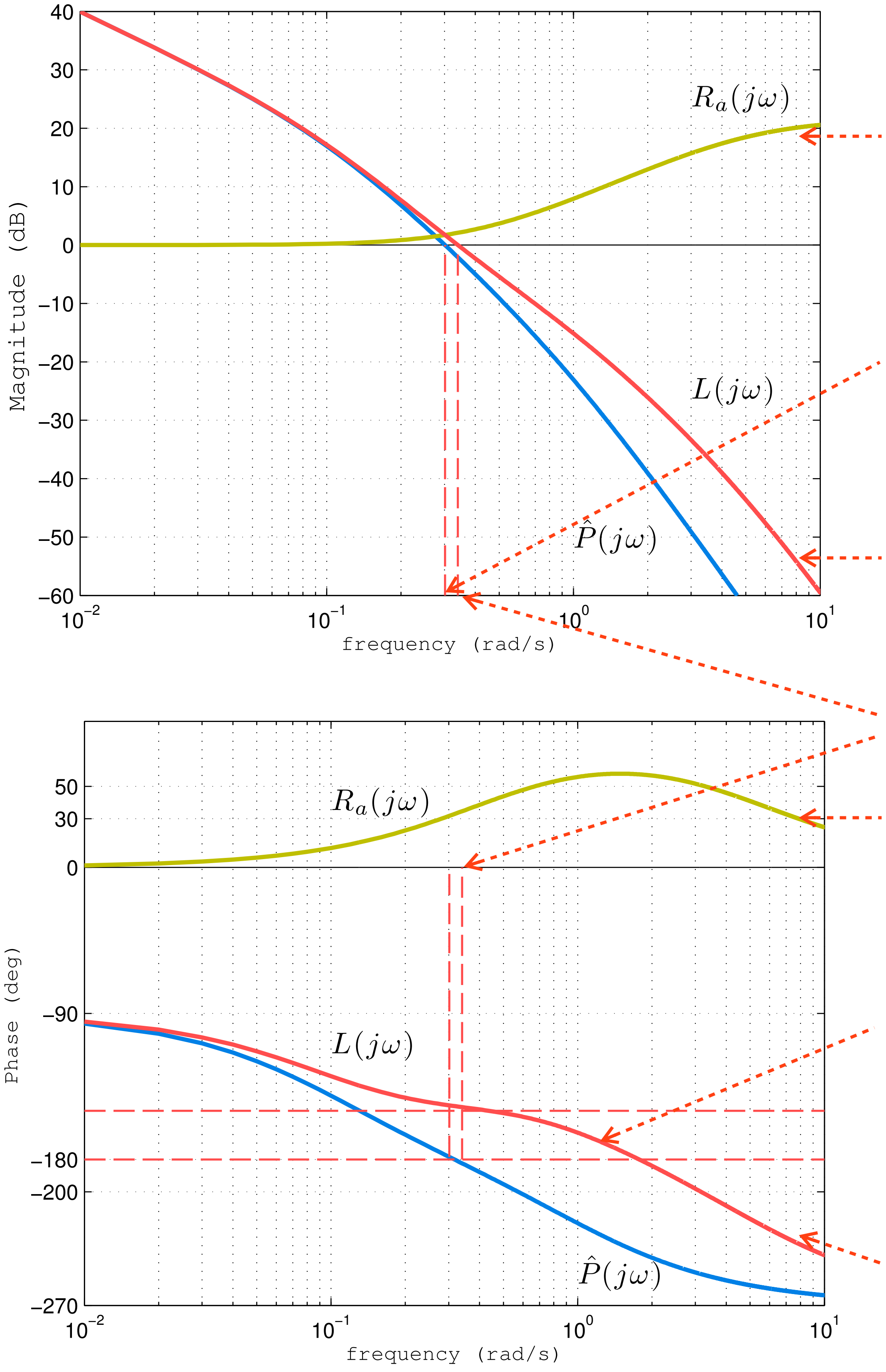
with the amplification of

1 dB at $\omega_c{}^* = 0.3$ rad/s

③ we choose τ_a as

$\tau_a = 0.5/0.3$

case I example



chosen lead function

desired crossover frequency

resulting
loop function

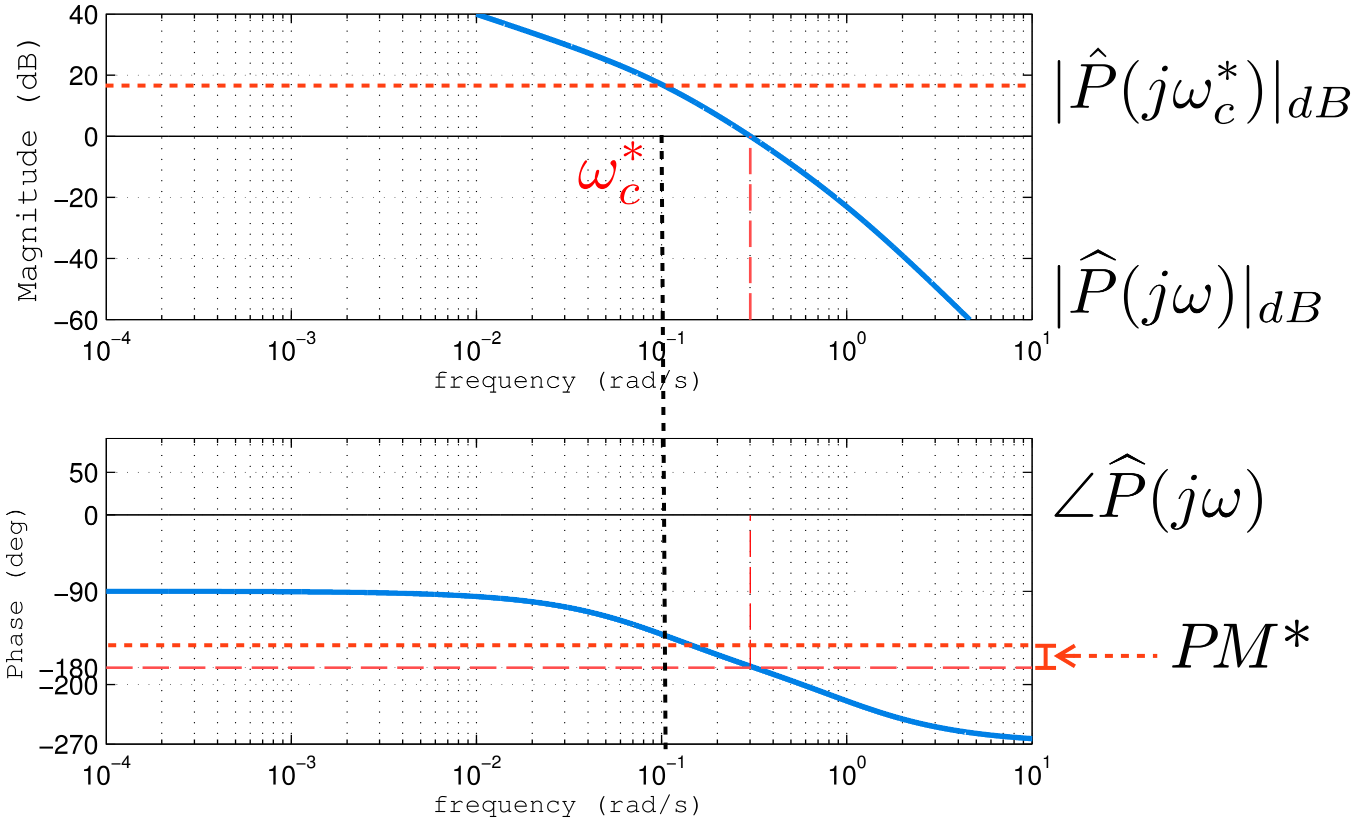
new crossover frequency, slightly larger than the desired one due to the small amplification introduced by $R_a(s)$

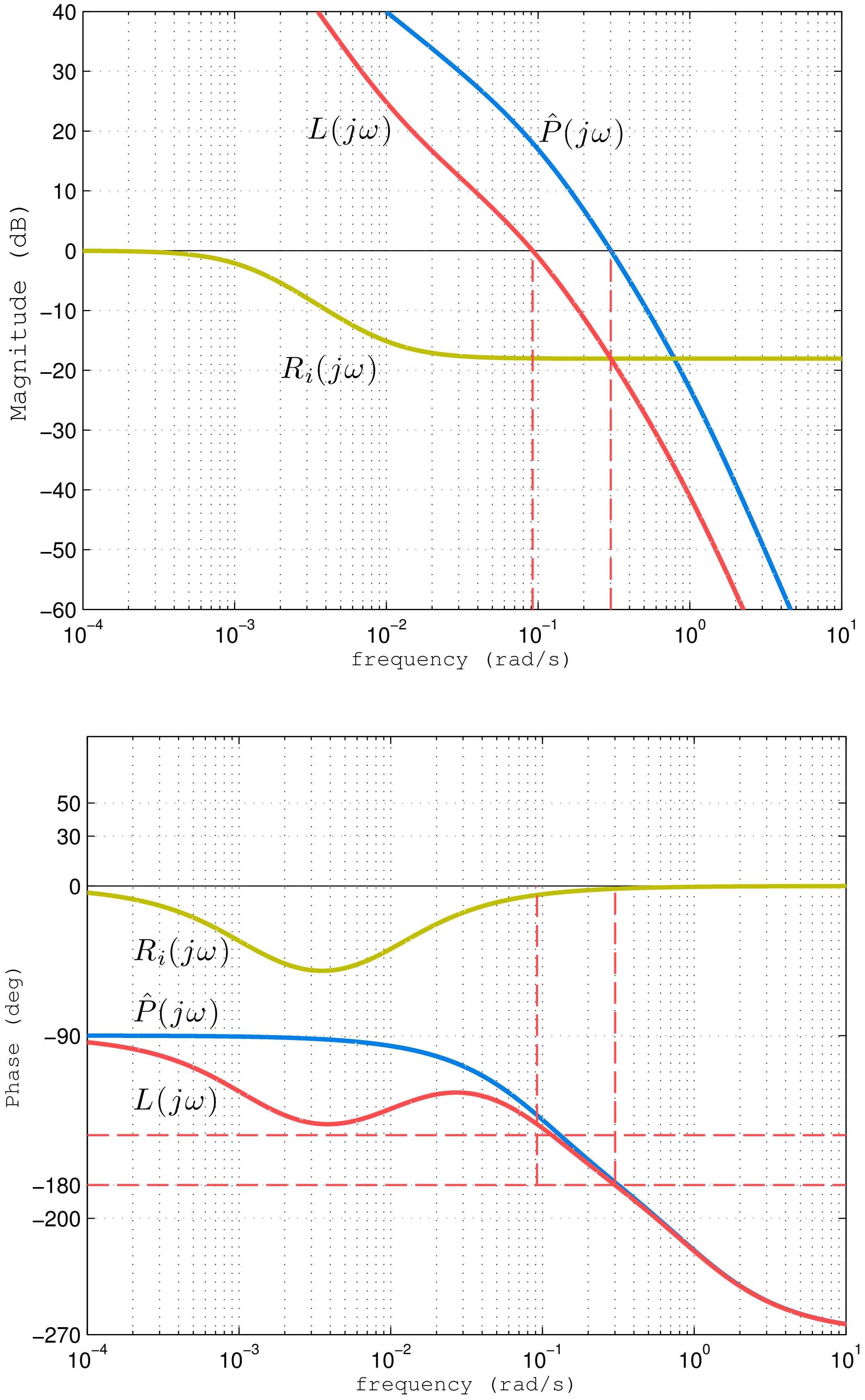
chosen lead function

having chosen the normalized frequency to the left of the “bell-shaped” phase, leads to an increase in phase lead for higher values of the frequency which partially compensates the typical decreasing phase plot of the modified plant thus achieving robustness w.r.t. the crossover frequency

resulting loop function: the phase margin needs to be verified at the final crossover frequency

Case II





specifications $\omega_c{}^*$ and $PM \geq PM^*$

actions needed:

- **magnitude**: attenuation of $|\hat P(j\omega_c^*)|_{dB}$
- **phase**: since

$$\angle\hat P(j\omega_c^*) + \pi > PM^*$$

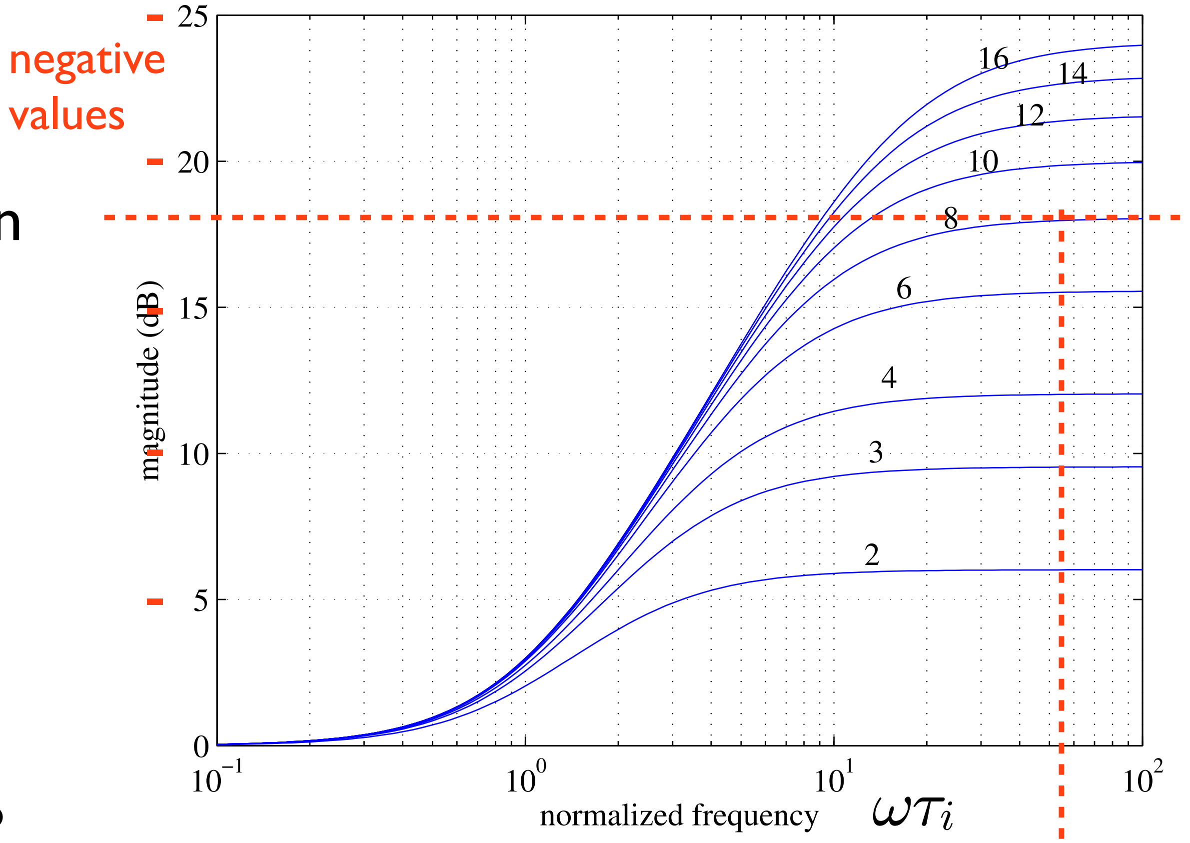
we can tolerate at most a lag of

$$\angle\hat P(j\omega_c^*) + \pi - PM^*$$

case II example: imagine we need an attenuation of 17 dB and can tolerate a maximum lag of $7°$

step

① required attenuation



several choices of m_i are possible (at different normalized frequencies), we need to find one which is compatible with the phase requirement

we have chosen

$m_i = 8$

$\omega\tau = 60$

therefore to obtain

this attenuation of 17 dB

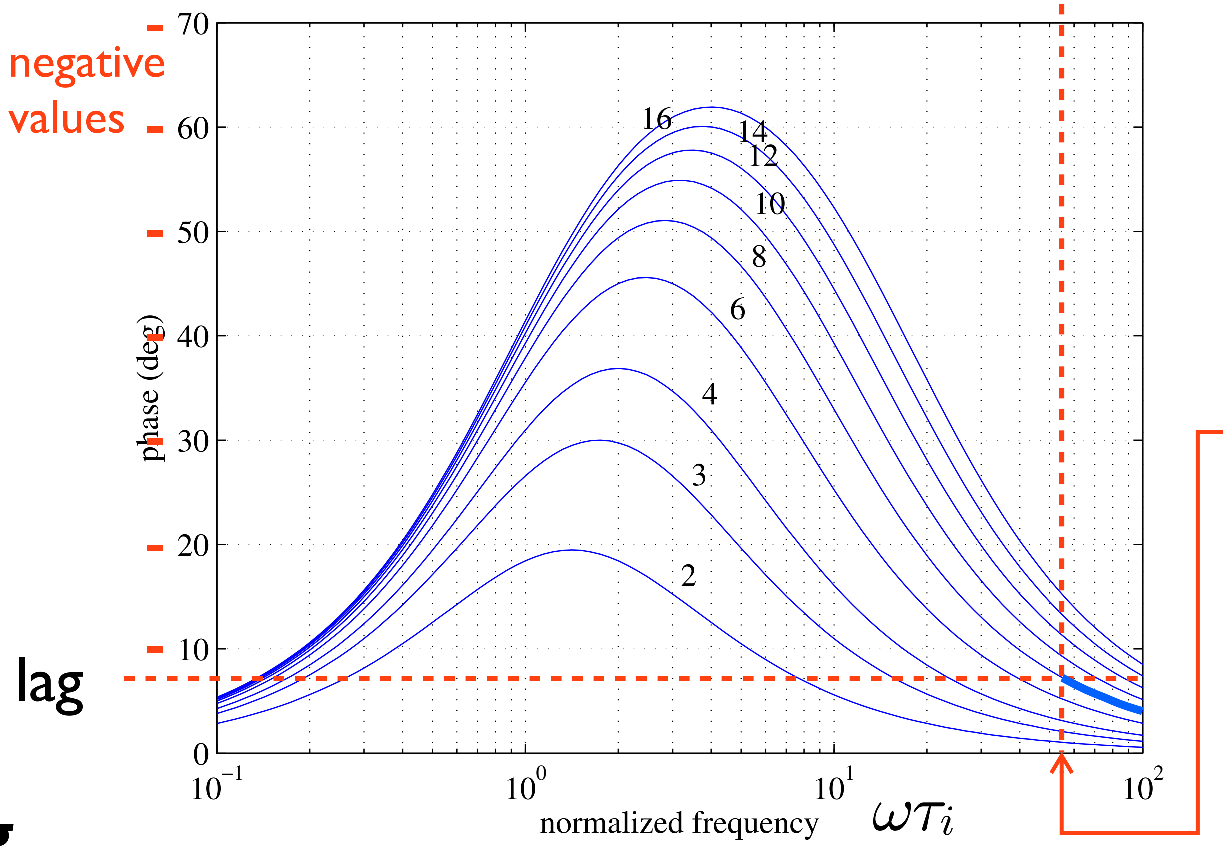
together with a lag

smaller than 7°

at $\omega_c{}^* = 0.1$ rad/s

we choose τ_i as

$\tau_i = 60/0.1$ ③

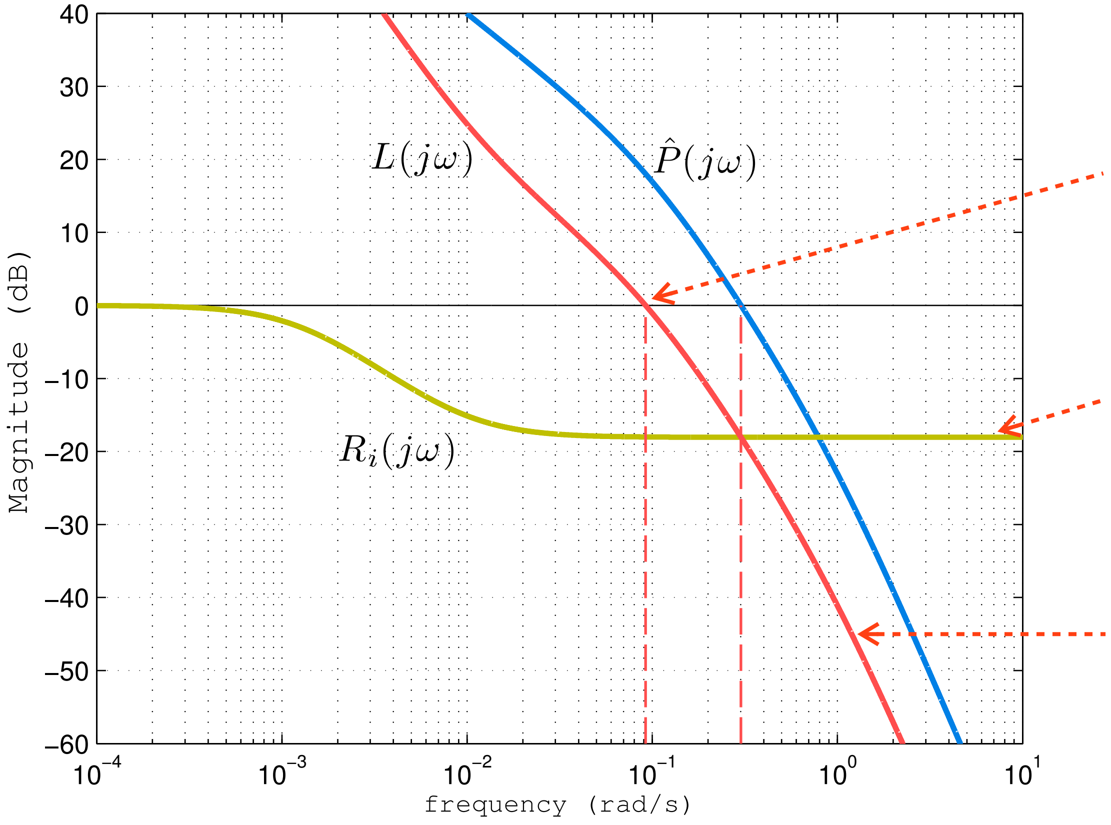


maximum allowed lag

②

60 is the smallest (for these plots) normalized frequency at which we obtain the desired attenuation (for $m_i = 8$)

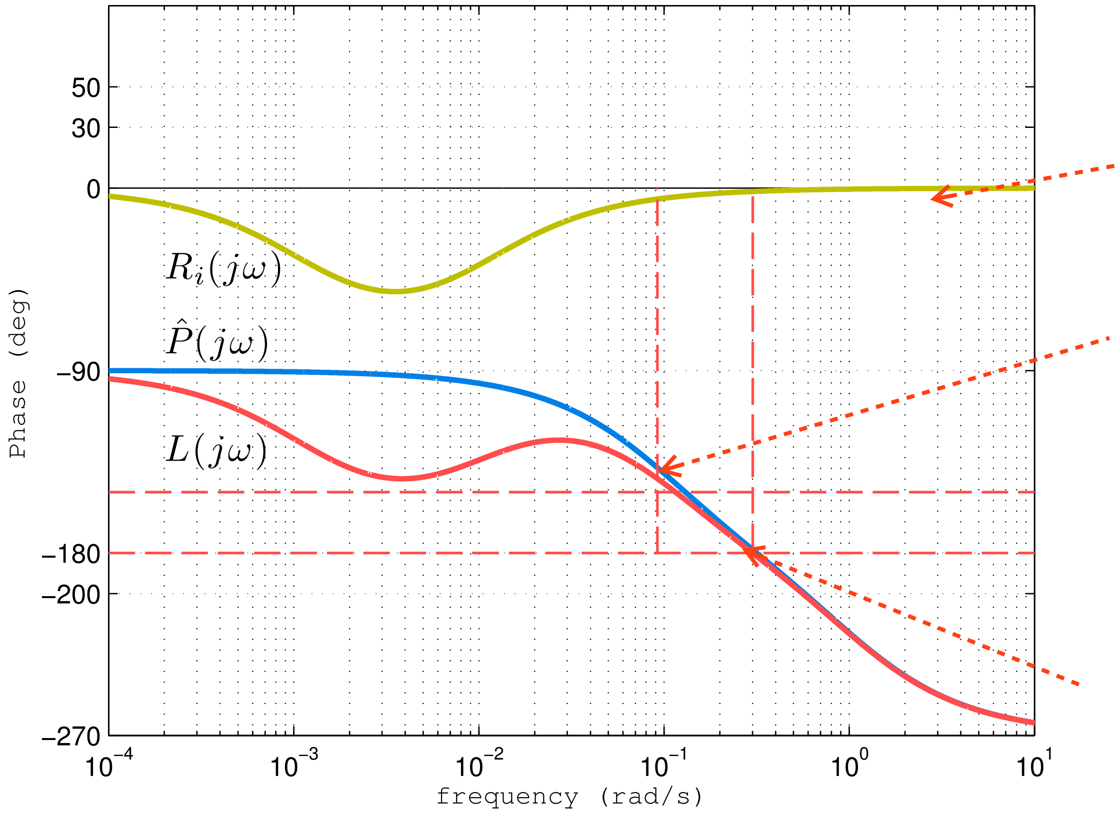
case II example



desired crossover frequency

chosen lag function

resulting loop function achieves exactly the desired crossover frequency



chosen lag function

the lag introduced by the chosen lag function in $\omega_c{}^*$ is compatible with the desired phase margin, i.e., the phase of $R_i(j\omega)$ at the desired crossover frequency is greater than the maximum allowed lag

the phase at the crossover frequency of the modified plant has no interest

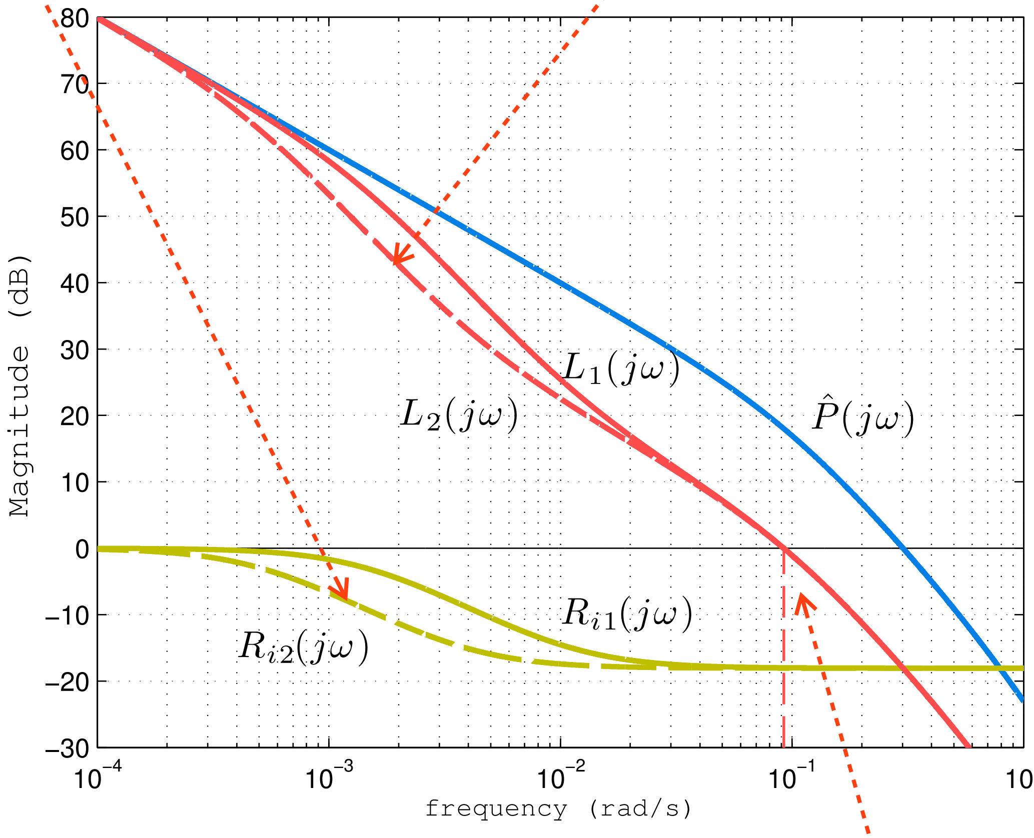
case II: on the choice of the normalized frequency

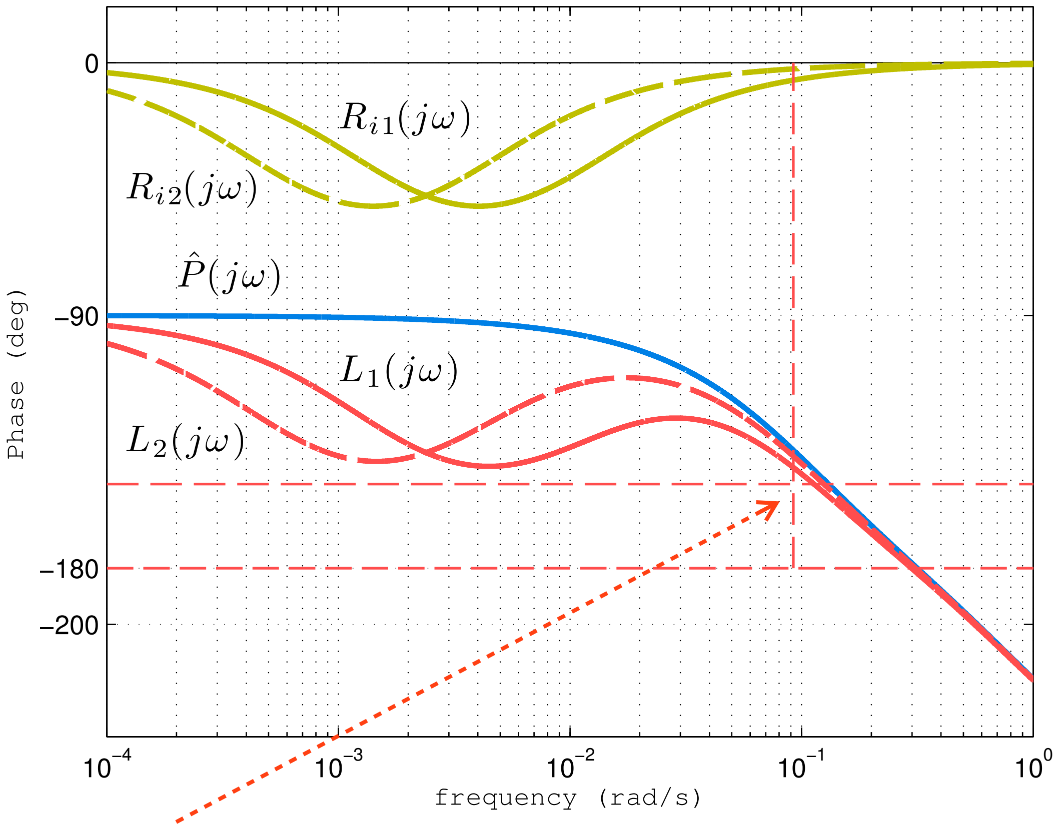
basic consideration (see sensitivity functions): we usually want open-loop high gain at low frequency, therefore anything that gives an unnecessary attenuation at low frequency should be avoided if possible

case II with two alternative choices of the $\omega\tau$ (both give the same attenuation at the desired frequency)

$R_{i1}(s) \qquad m_i = 8 \quad \omega\tau = 70$

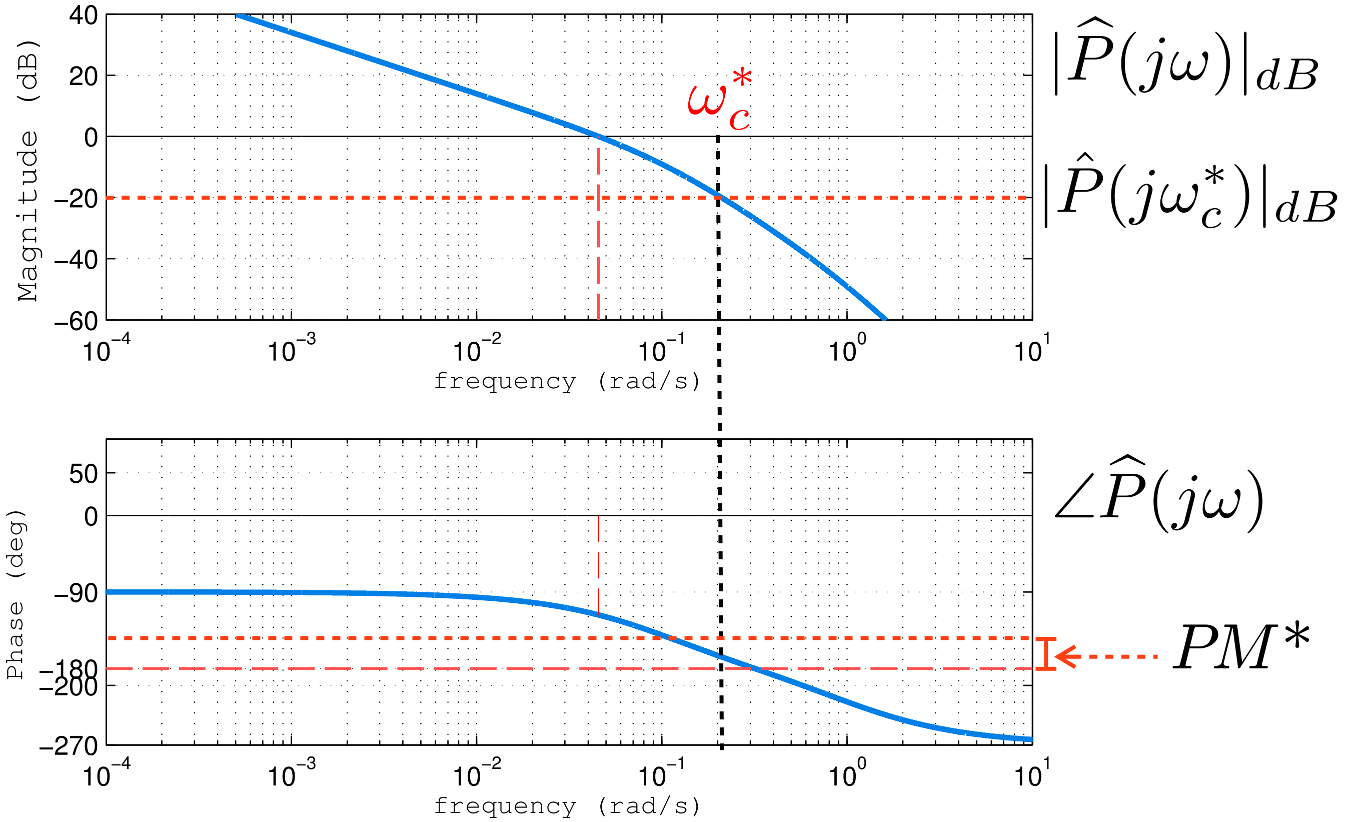
$R_{i2}(s) \qquad m_i = 8 \quad \omega\tau = 200$ starts attenuating before strictly needed

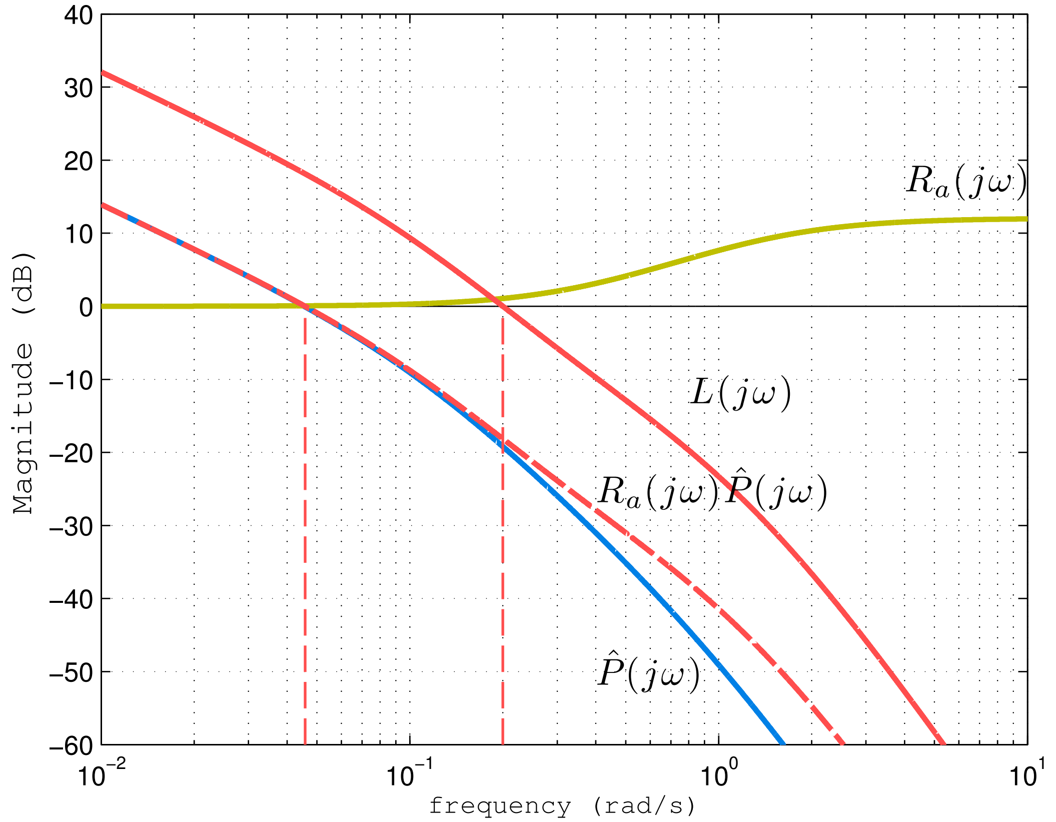




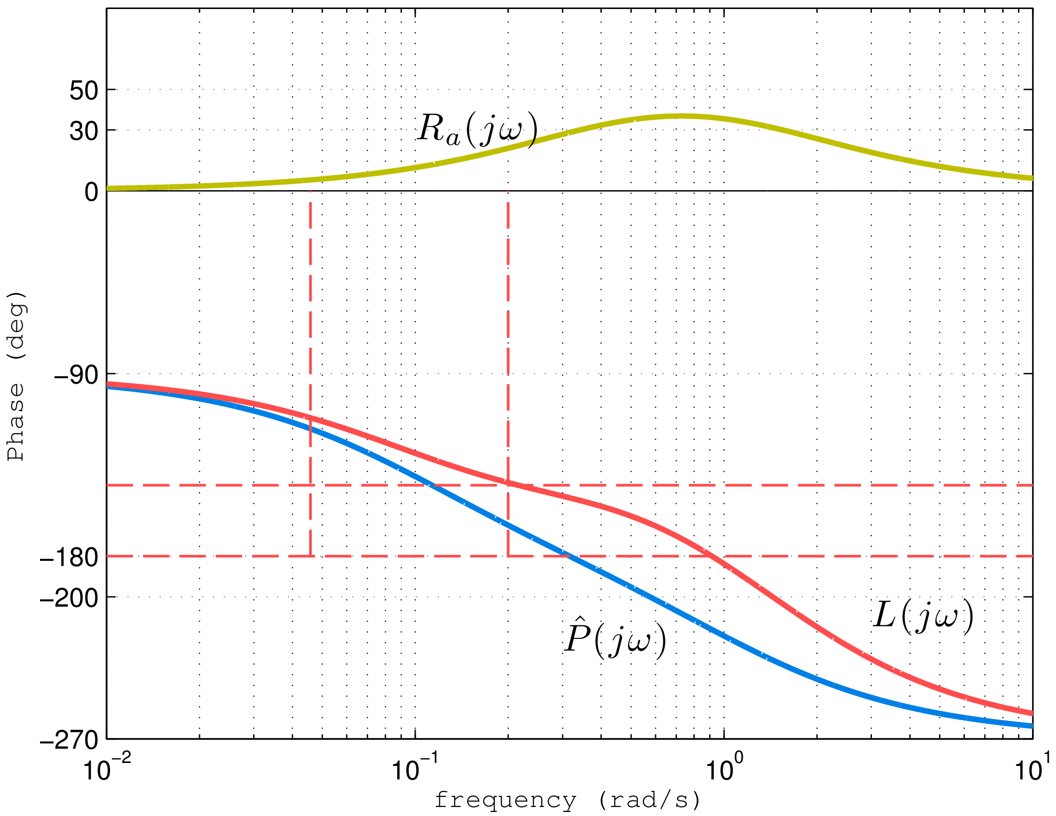
both compensators solve the specifications

Case III





specifications $\omega_c{}^*$ and $PM \geq PM^*$



actions needed:

- **magnitude**: amplification of $-|\hat{P}(j\omega_c^*)|_{dB}$
- **phase**: since

$$\angle\hat{P}(j\omega_c^*) + \pi < PM^*$$

we need to obtain a phase lead of

$$PM^* - \left(\angle\hat{P}(j\omega_c^*) + \pi\right)$$

example imagine we need an amplification of 20 dB and a phase lead of at least $25°$

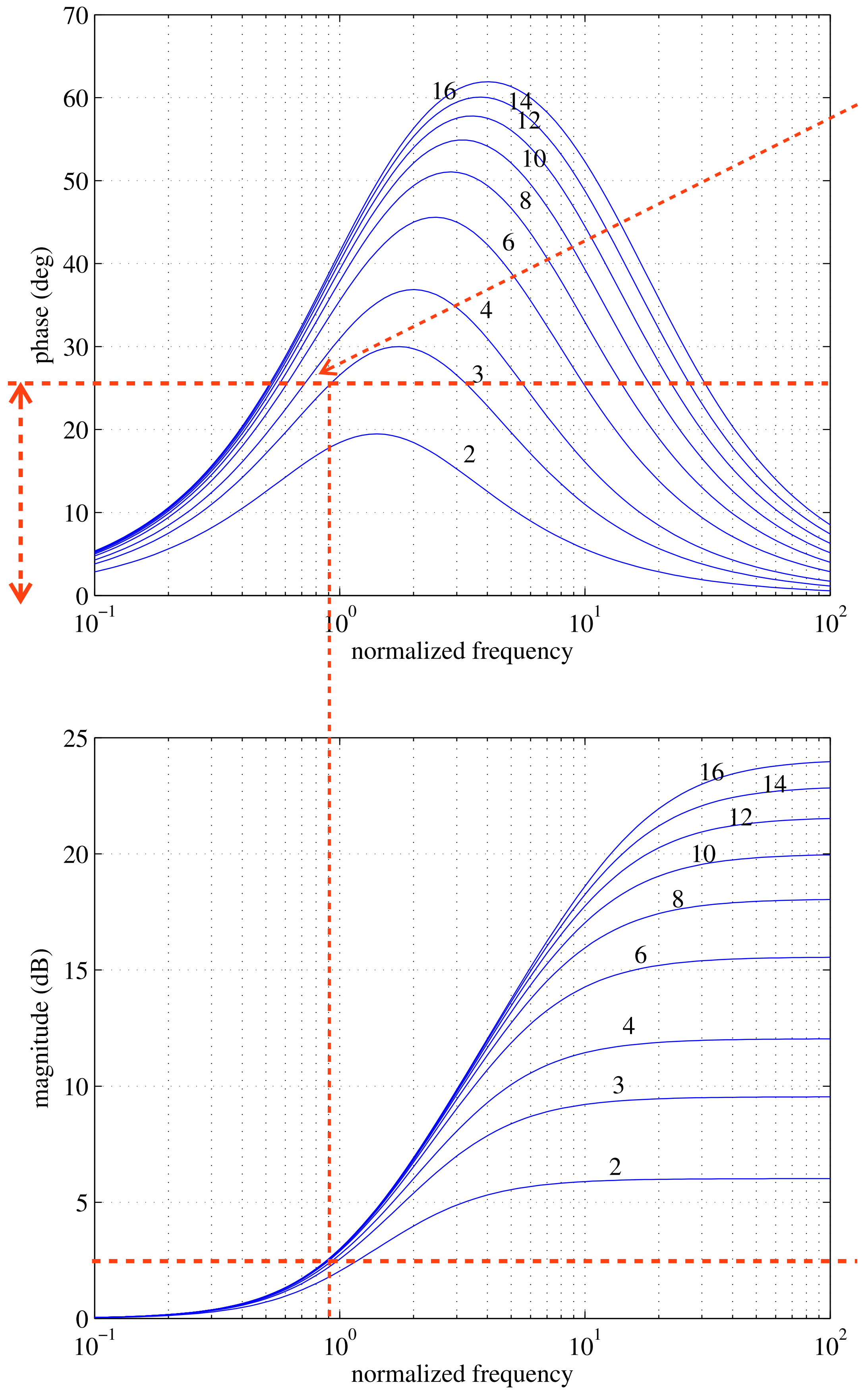
two choices

- find a lead compensator that will give simultaneously the required amplification and phase lead
  - usually requires a choice of the normalized frequency on the right-hand side of the phase "bell-shape" which gives poor robustness w.r.t. increases in the crossover frequency since the phase of both the extended plant and the compensator are decreasing at the chosen frequency
  - may be not easy to find
- find a lead compensator that gives the required lead and gives some amplification, integrate the required amplification with an additional gain K_{c2} greater that 1.

  This is usually possible since the static requirement (if any) asks for a loop gain sufficiently high.

example imagine we need an amplification of 20 dB and a phase lead of at least 25°

$K_{c2}R_a(s)$



25°

desired phase lead

2.5 dB

several choices of m_a are possible (at different normalized frequencies), we just keep the choice of the normalized frequency on the left of the “bell-shape”

we can choose

$m_a = 3$

$\omega\tau = 0.9$

therefore to obtain

this lead of 25° together

with the amplification of

2.5 dB at $\omega_c{}^* = 0.2$ rad/s

we choose τ_a as

$\tau_a = 0.9/0.2$

K_{c2} /

$[K_{c2}\ ]_{\rm dB} = 20$ - 2.5

we can obtain an amplification of 2.5 dB

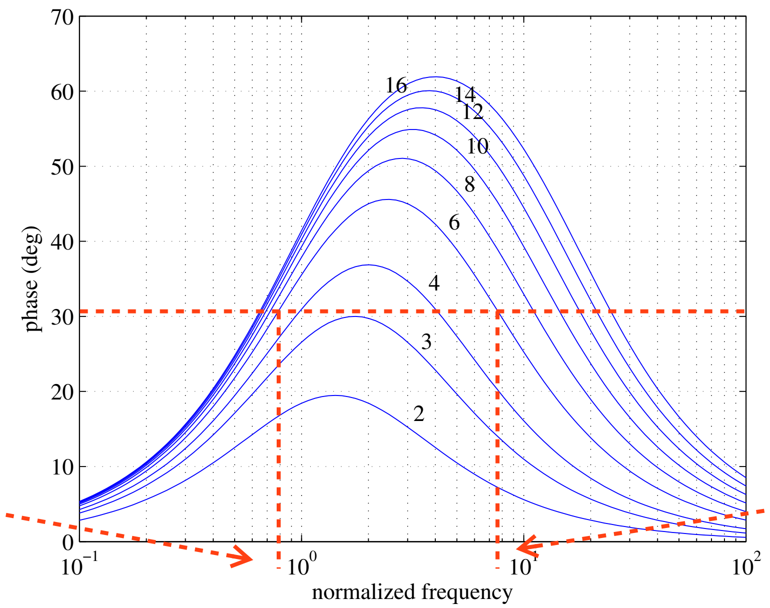
about the robustness issue

case III revisited with two alternative solutions for the choice of the normalized frequency $\omega\tau$ (the gains have been chosen appropriately)

$K_{c1}R_{a1}(j\omega)$

$m_a = 6$
$\omega\tau = 0.8$

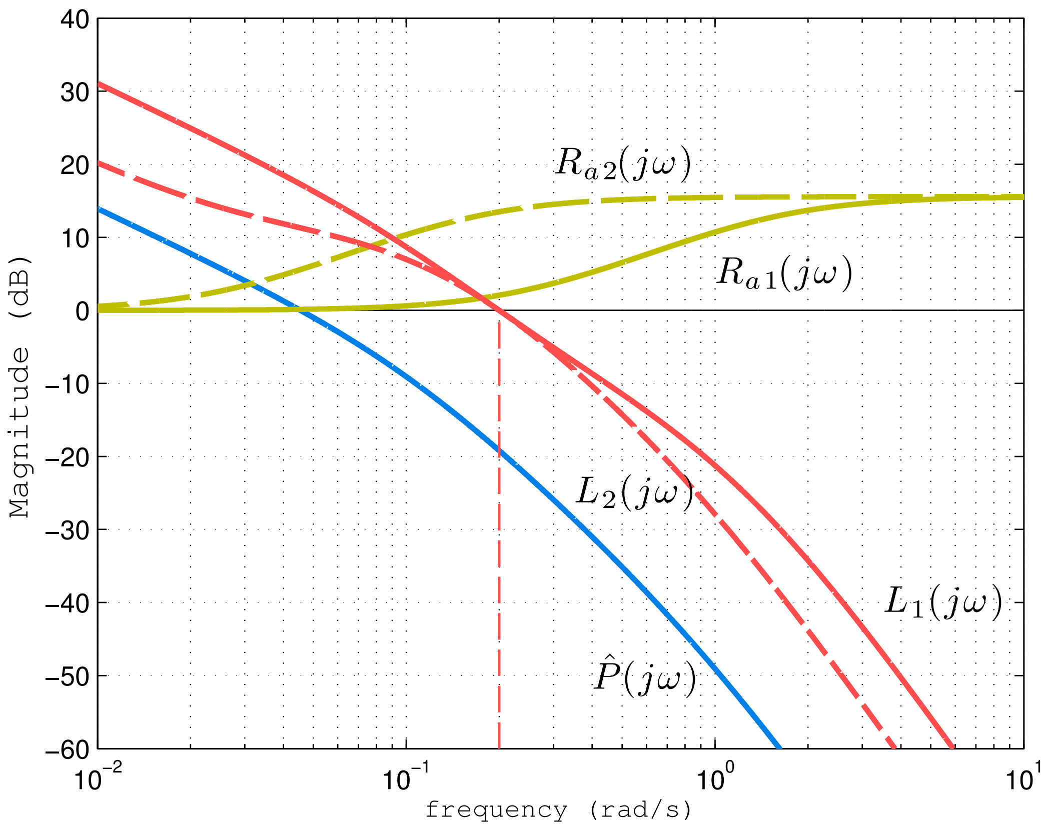
left-hand side of the bell (more robust w.r.t. variations in ω_c)

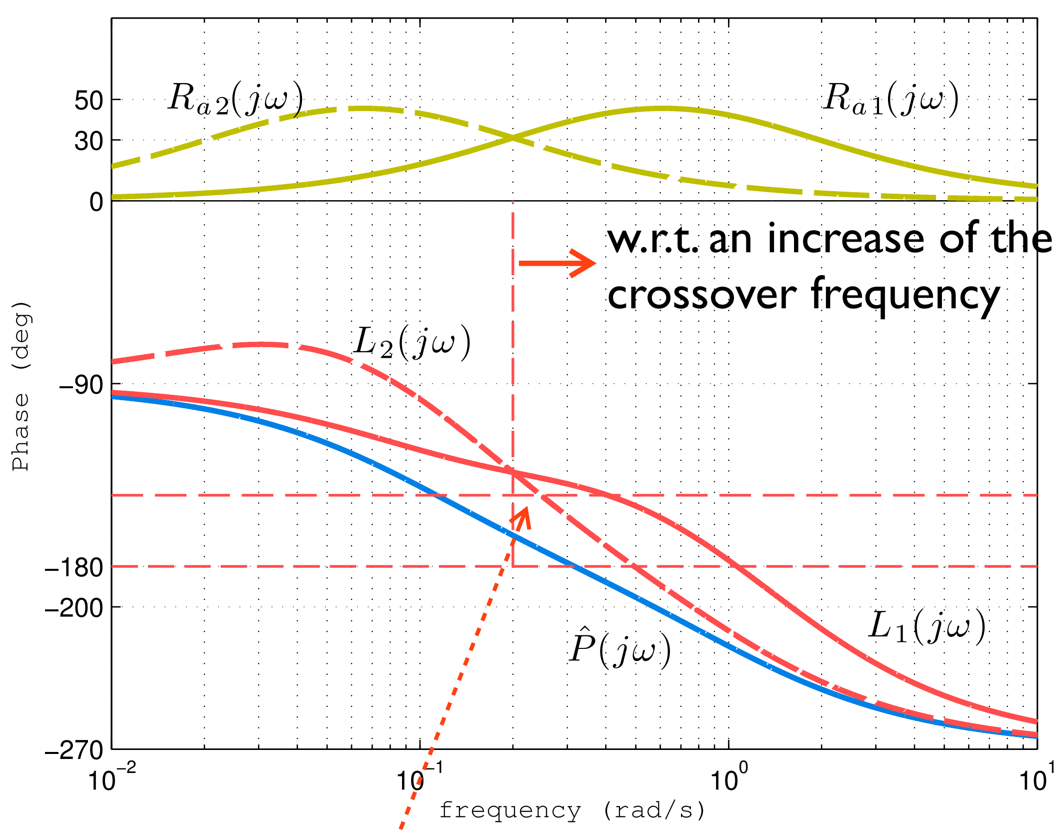


$K_{c2}R_{a2}(j\omega)$

$m_a = 6$
$\omega\tau = 7.5$

right-hand side of the bell (less robust w.r.t. variations in ω_c)

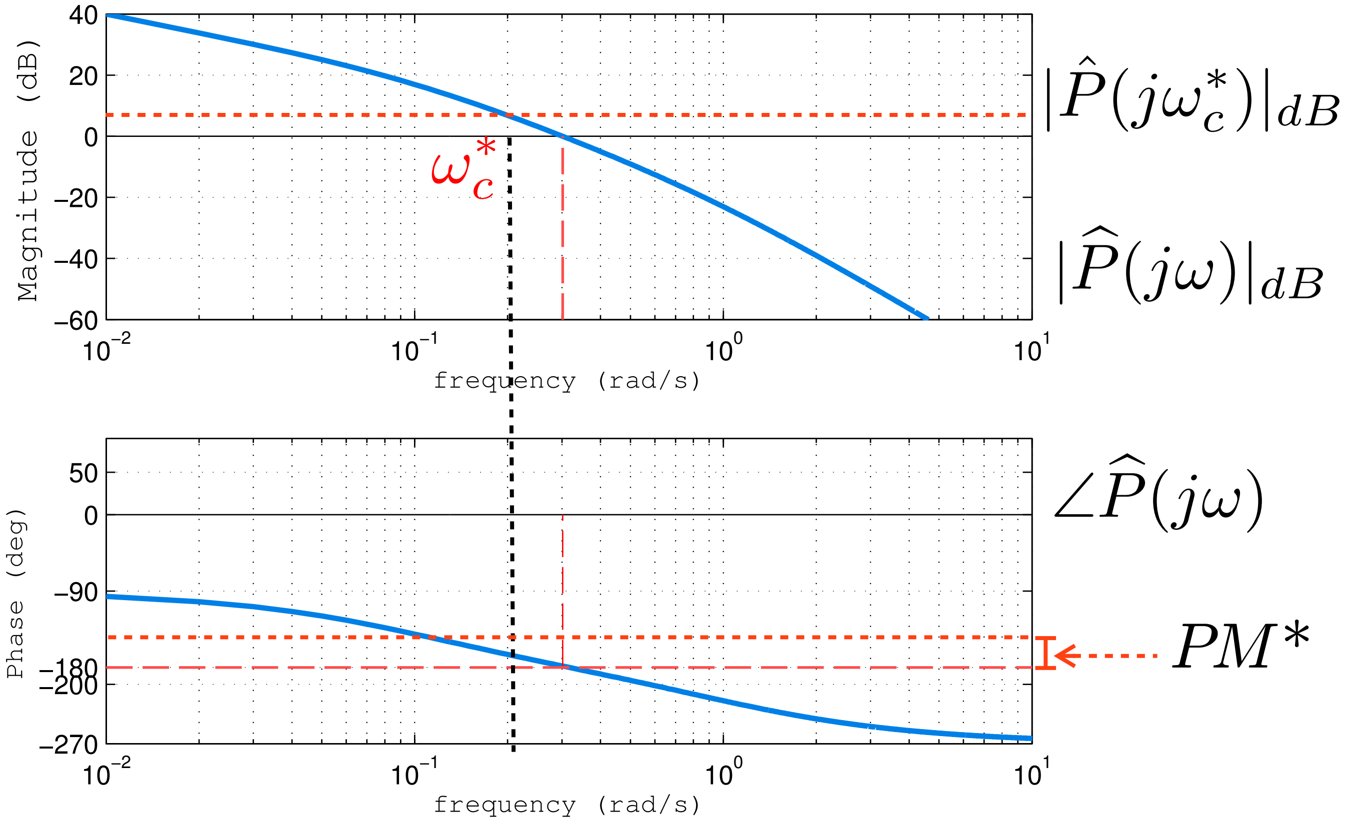


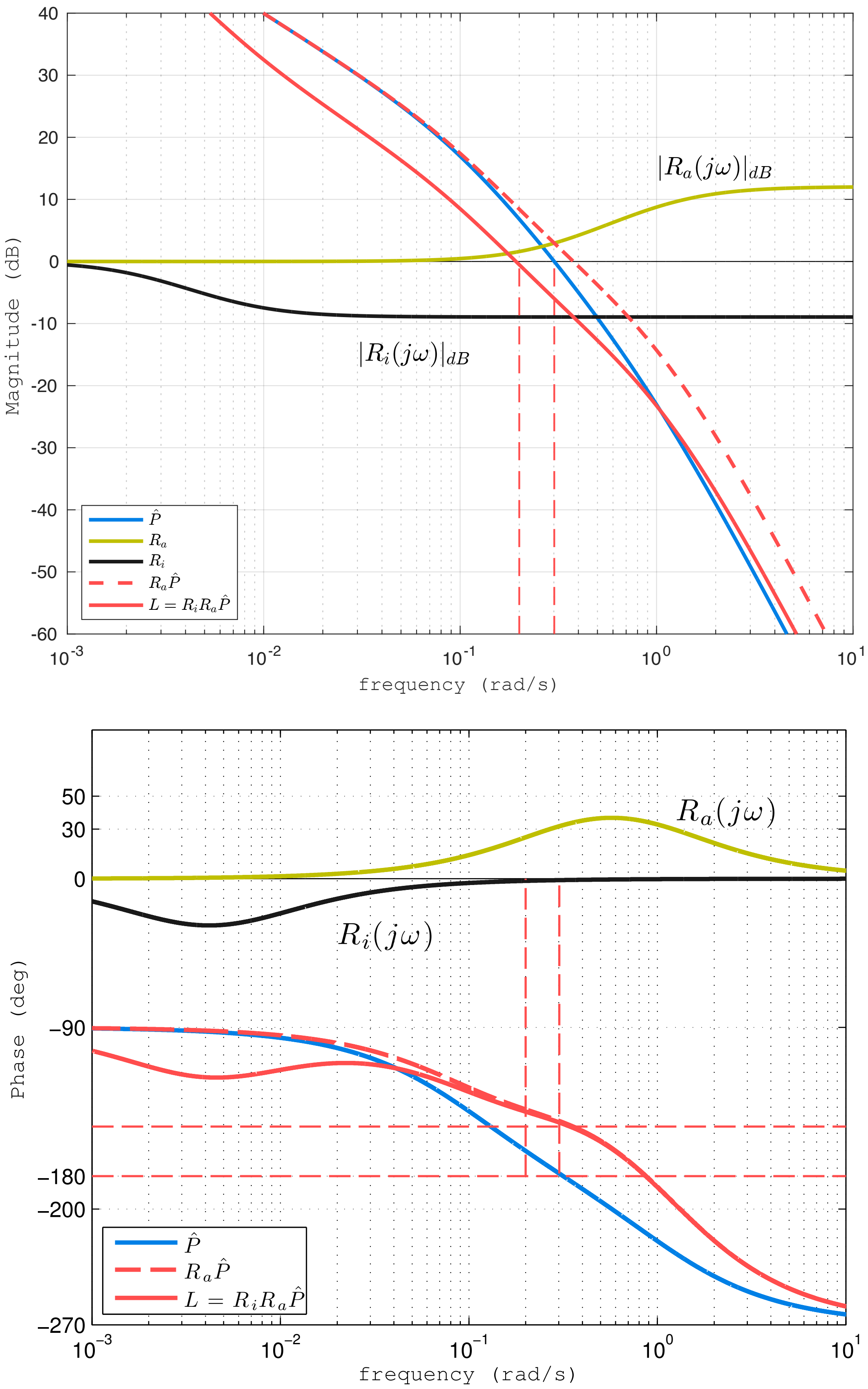


w.r.t. an increase of the crossover frequency

solution 1 is more robust w.r.t. uncertainties in the ω_c

Case IV





specifications $\omega_c{}^*$ and $PM \geq PM^*$

actions needed:

- **magnitude**: attenuation of $|\hat{P}(j\omega_c^*)|_{dB}$
- **phase**: since

$$\angle\hat{P}(j\omega_c^*) + \pi < PM^*$$

we need to obtain a phase lead of at least

$$PM^* - \left(\angle\hat{P}(j\omega_c^*) + \pi\right)$$

example imagine we need an attenuation of $8\ \mathrm{dB}$ and a phase lead of at least $25°$

we need to use both lead and a lag compensators but in the proper order

- we choose the **lead** compensator first in such a way to obtain a **phase increase** of the required $25°$ plus an extra (for example of $8°$) in order to compensate the lag that will be introduced by the lag compensator
  This lead function will also introduce, at the chosen frequency, an amplification of exactly

$$|\hat{R}_a(j\omega_c^*)|_{dB}$$

- the **lag** compensator will be chosen so to introduce an **attenuation** of

$$8\,dB + |\hat{R}_a(j\omega_c^*)|_{dB}$$

  and a lag smaller than the extra $8°$ previously introduced

$m_a = 8$	$\dashrightarrow$	$34°$	$m_i = 3.2$	$\dashrightarrow$	-10.5 dB
$\omega\tau = 0.8$		2.5 dB	$\omega\tau = 20$		$< 8°$

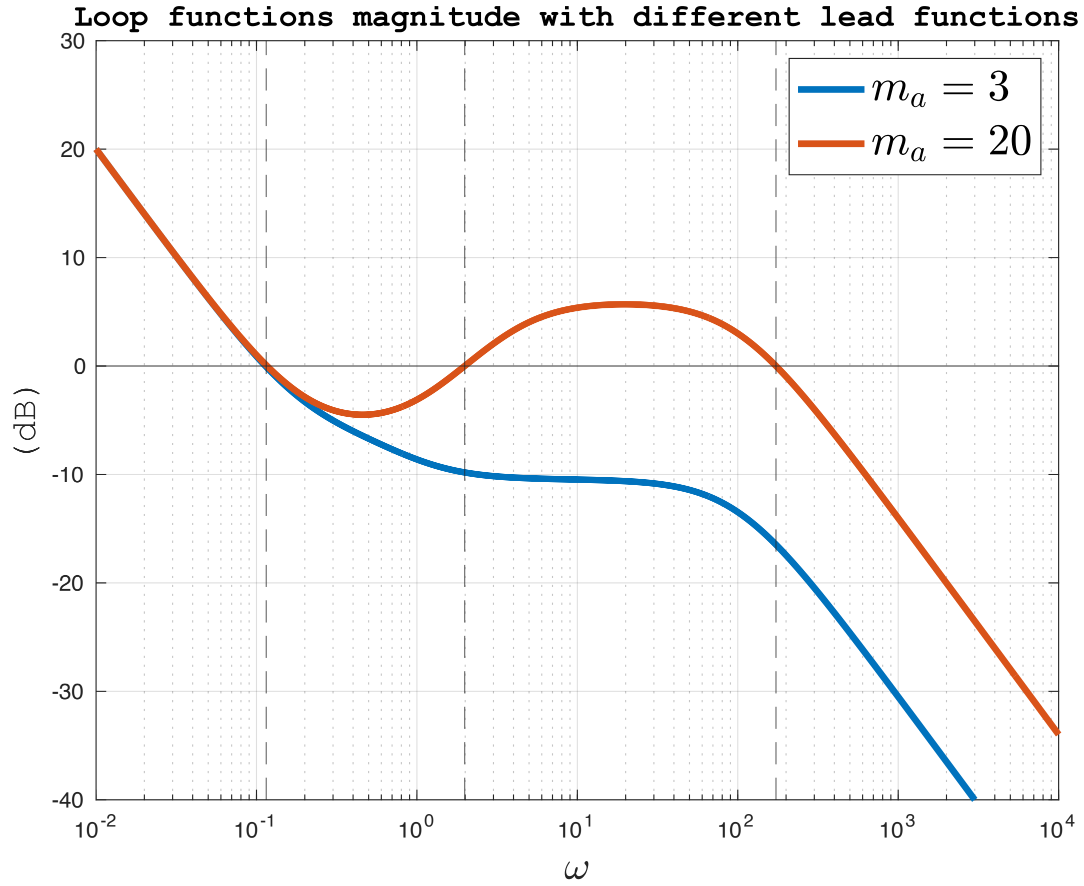
(these numbers are just for illustration purposes and have not been verified)

some remarks

- we may be tempted when requiring a high phase increase (lead) to choose a lead function with very high value of m_a but this may create multiple crossover frequencies (so that Bode's stability theorem cannot be used anymore and therefore ensuring closed loop stability becomes more complicated)

  this plot shows how a lead function with large m_a may cause multiple crossover frequencies compared with a small m_a (of course the phase contribution is different)

  we may try to obtain the same phase increase with multiple lead functions and smaller m_a and without generating multiple crossover frequencies)



- if we have the presence of some necessary poles in the controller (either in 0 or purely imaginary), we could alternatively use just a negative zero (that is with positive τ) to increase the phase. The resulting controller needs to be at most proper (not improper)

example: $$C(s) = \frac{\boxed{1 + 0.1s}}{\boxed{s^2}}$$

----> this zero gives $45°$ phase lead at 10 rad/s

----> necessary part for steady state requirements

PID controllers

3 basic heuristic actions

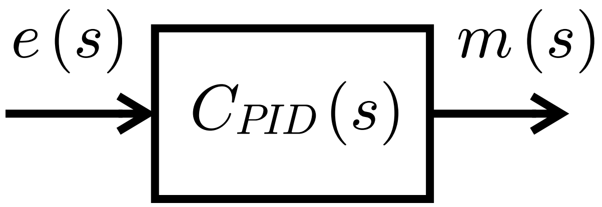
Proportional: the control action is set to be directly proportional to the system error (present)

Integral: the control action is set to be proportional to the system error integral (past)

Derivative: the control action is set to be proportional to the system error derivative (future)

output of the controller

$$m(t) = K_P e(t) + K_I \int_0^t e(\tau)d\tau + K_D \dot{e}(t)$$



$$C_{PID}(s) = \frac{m(s)}{e(s)} = K_P + \frac{K_I}{s} + K_D s$$

ideal PID controller transfer function

$$= K_P \left(1 + \frac{1}{T_I s} + T_D s\right)$$

with

$$T_I = \frac{K_P}{K_I}$$

$$T_D = \frac{K_D}{K_P}$$

- widely spread
- fixed structure with 3 tunable parameters $K_P K_D K_I$ (or $K_P T_D T_I$)
- can be tuned automatically even with scarce knowledge of the (simple) plant
- basic tuning refers only to static specifications

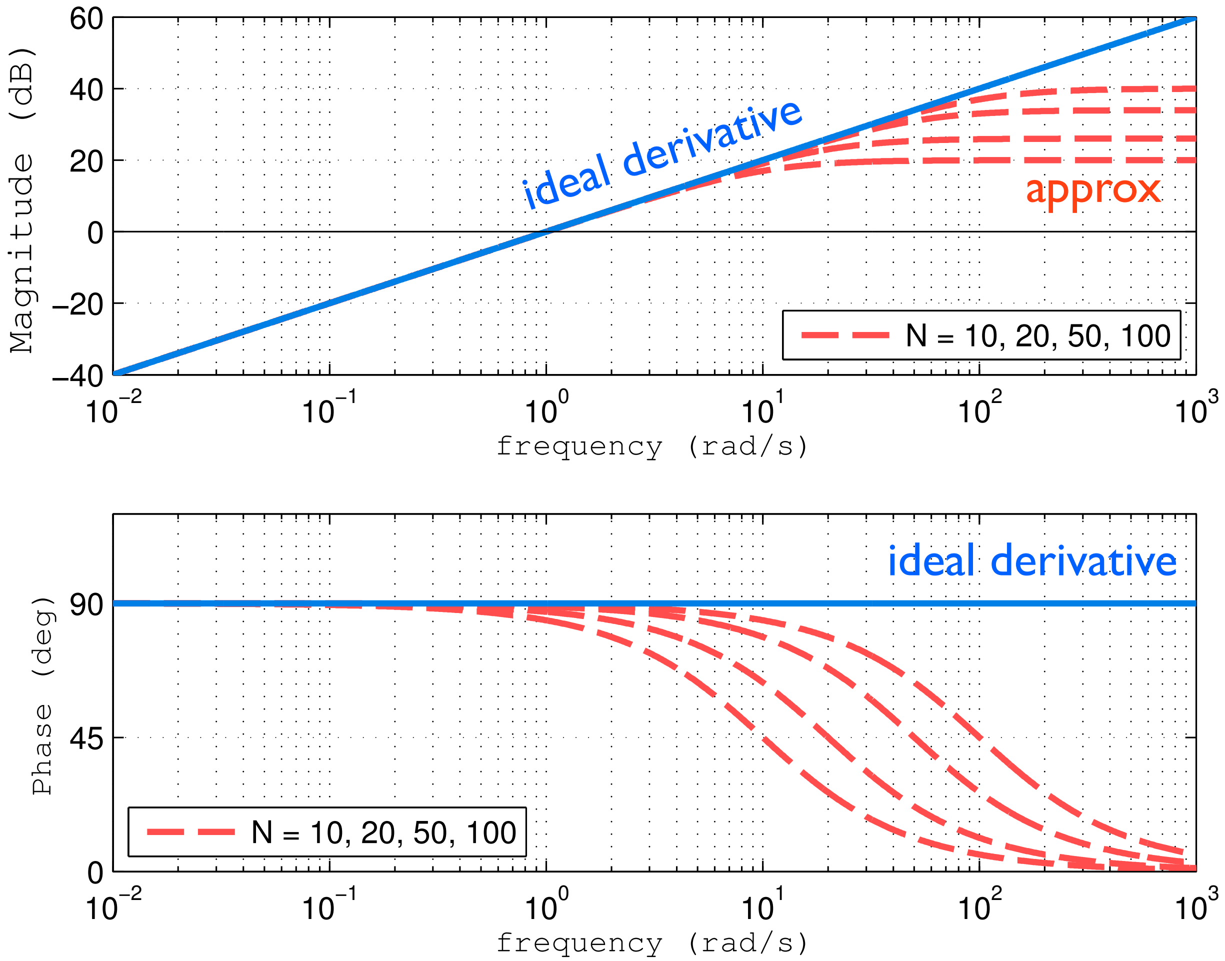
Derivative action is physically not realizable (improper transfer function) we need to approximate

→ add a high-frequency pole in

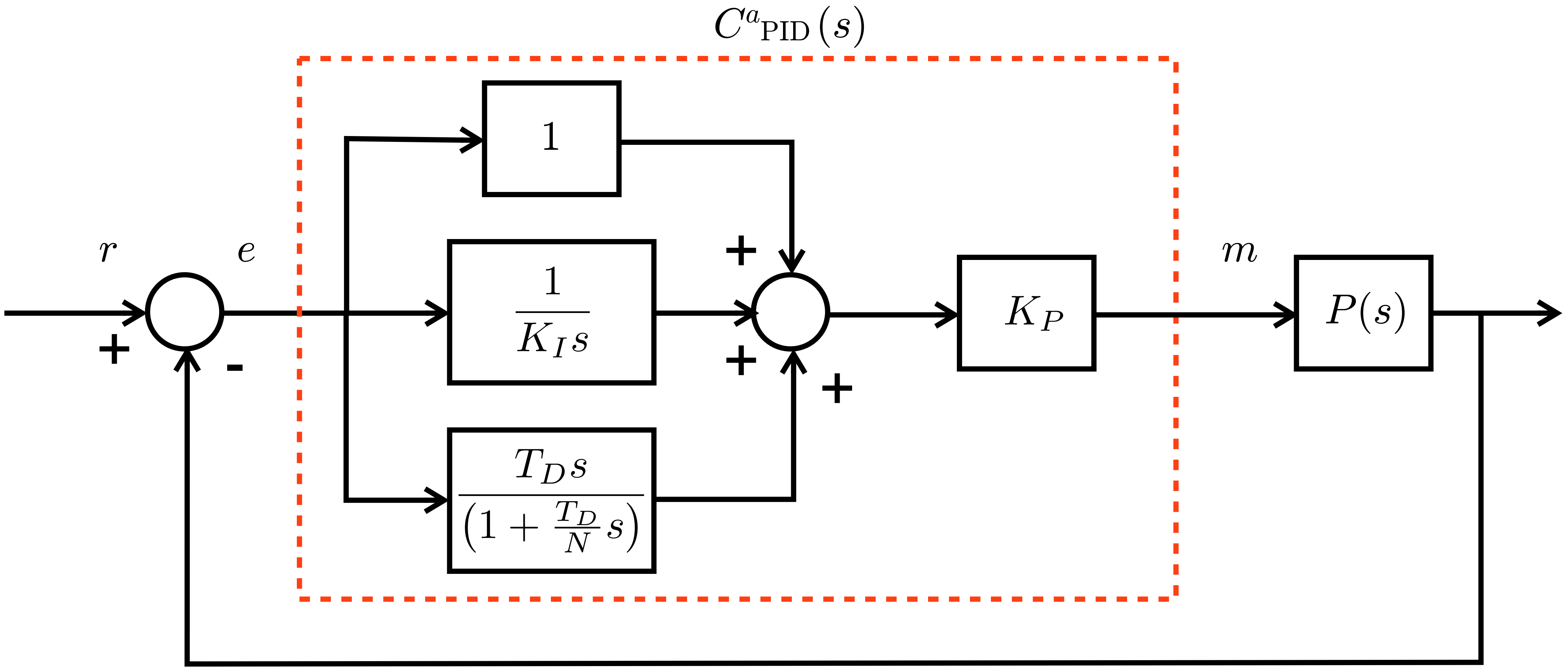
$$s = -\frac{K_P}{K_D} N = -\frac{N}{T_D}$$

$$C_D^a(s) = \frac{T_D s}{\left(1 + \frac{T_D}{N} s\right)}$$

← high frequency pole



approximated derivative action for various values of N



Basic PID feedback control scheme

typical configurations

- proportional
- proportional + derivative (approximate)
- proportional + integrative

PD configuration

equivalent to a Lead compensator

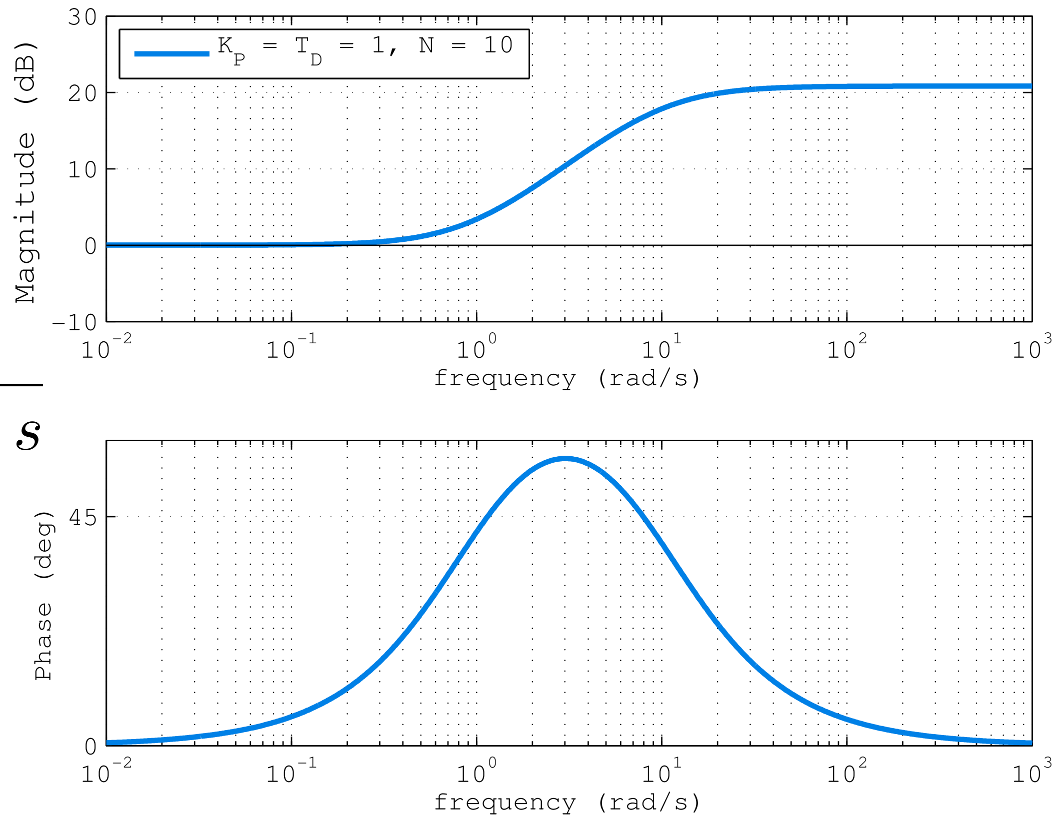
τ_a

$$C^a_{PD}(s) = K_P\left(1+\frac{s\,T_D}{1+s\frac{T_D}{N}}\right) = K_P\,\frac{1+\left(T_D\,\frac{N+1}{N}\right)s}{1+\frac{1}{N+1}\left(T_D\,\frac{N+1}{N}\right)s}$$

$1/m_a$

$$= K_P\,\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}$$

$m_a > 1$

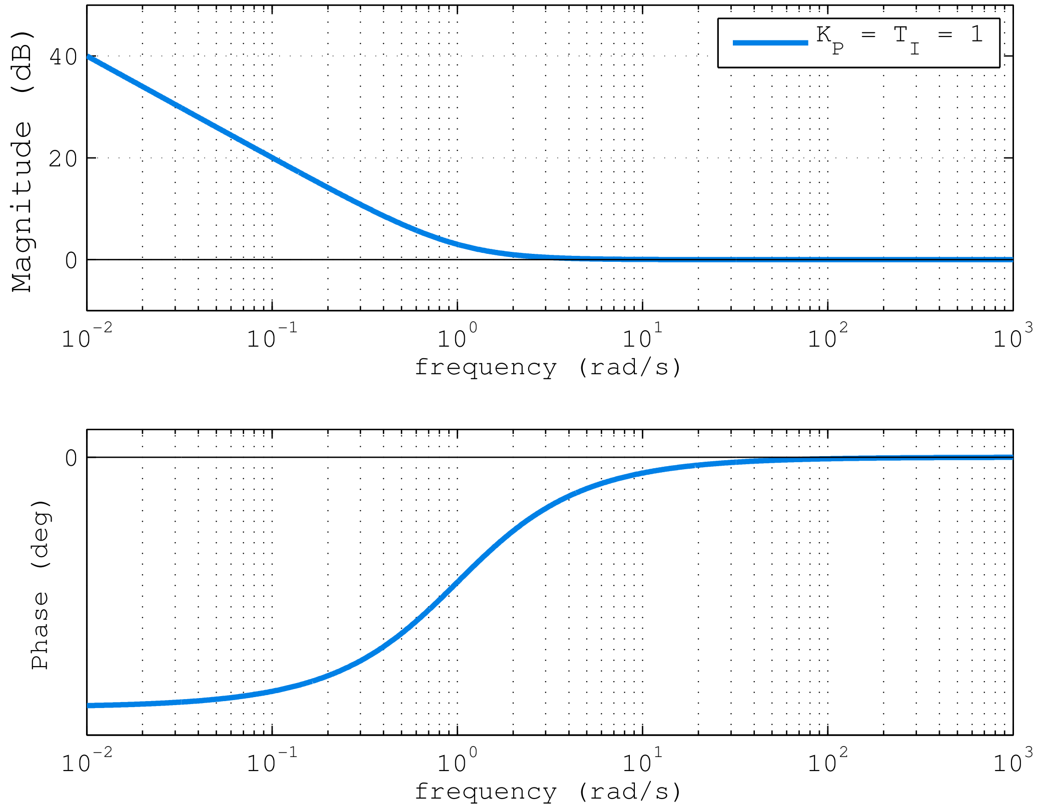


PI configuration

$$C_{PI}(s) = K_P\left(1+\frac{1}{sT_I}\right) = \frac{K_P}{T_I}\,\frac{(1+T_I s)}{s}$$

zero in s = -$1/T_I$ + pole in $s = 0$

compensates the phase lag introduced by the pole in $s = 0$



Vocabulary

English	Italiano
lead compensator	funzione anticipatrice
lag compensator	funzione attenuatrice
phase lead	anticipo di fase
phase lag	ritardo di fase
attenuation	attenuazione
amplification	amplificazione
open-loop shaping	sintesi per tentativi