Control Systems

Nyquist stability criterion

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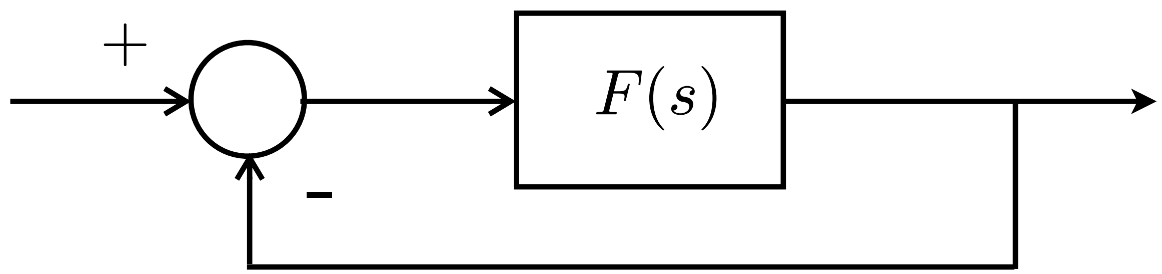
Outline

- polar plots when $F(j\omega)$ has no poles on the imaginary axis
- Nyquist stability criterion (first version)
- what happens when $F(j\omega)$ has poles on the imaginary axis
- Nyquist stability criterion (general case)
- general feedback system
- stability margins (gain and phase margin)
- Bode stability criterion
- effect of a delay in a feedback loop

Goal: establish a **necessary and sufficient stability criterion** for the **asymptotic stability** of the **closed-loop system** based on the information (Nyquist plot) of the open-loop system

Unit negative feedback



$\Rightarrow$ $W(s)$ closed-loop system transfer function

we have seen that

- in a unit feedback system, the **closed-loop** system has **hidden modes** if and only if the open loop has them
- the open-loop hidden modes are inherited **unchanged** by the closed loop

therefore we make the hypothesis that there exists

no open-loop hidden eigenvalue with non-negative real part

(since these would be inherited by the closed-loop system)

→ stability of the closed loop is only determined by the closed-loop poles

We are going to determine the

stability of the closed-loop system **from** the **open-loop system** features

(i.e. the graphical representation of the open-loop frequency response $F(j\omega)$)

Nyquist diagram: (closed) polar plot of $F(j\omega)$ with $\omega \in (-\infty, \infty)$

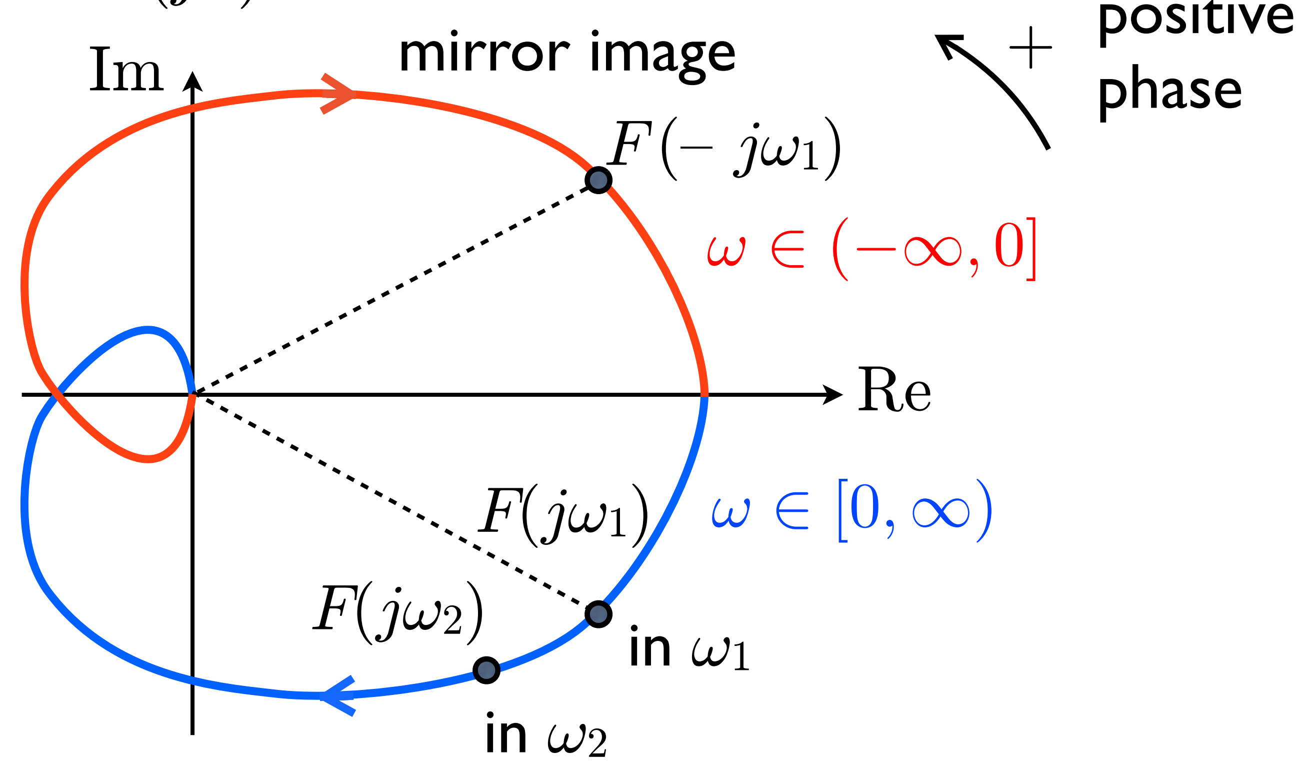
we plot the magnitude and phase on the same plot using the frequency as a parameter, that is we use the polar form for the complex number $F(j\omega)$

being $F(s)$ a rational function

(or rational function + delay)

$F(\text{-}j\omega) = F^*(j\omega)$

and therefore the plot for negative angular frequencies ω is the **symmetric** wrt the real axis of the one obtained for positive ω



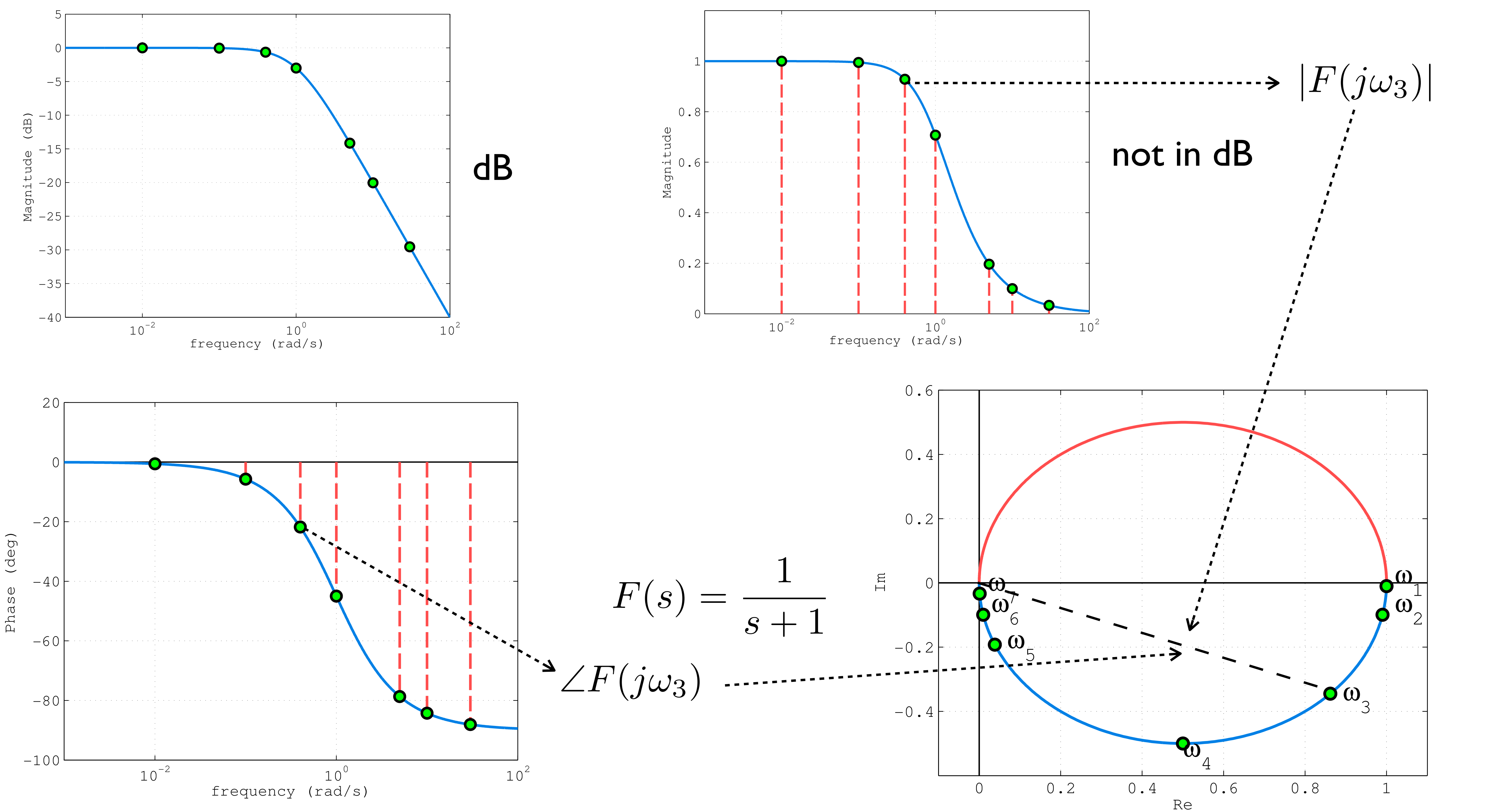
$F(-j\omega) = F^*(j\omega) \longrightarrow$

$$|F(j\omega)| = |F(-j\omega)|$$
$$\angle F(j\omega) = -\angle F(-j\omega)$$

Hyp. no open-loop poles on the imaginary axis (i.e. with $\mathrm{Re}[.] = 0$)

some **polar plots**

polar plot of $F(j\omega)$ can be obtained from the Bode diagrams (magnitude and phase information)



fact 1

The closed-loop system $W(s)$ has poles with $\mathrm{Re}[.] = 0$

if anf only if

the Nyquist plot of $F(j\omega)$ passes through the critical point (-1,0)

Proof.

Nyquist plot intersects the real axis in -1 therefore $\exists\,\bar{\omega}$ such that $F(j\bar{\omega}) = -1$

that is $F(j\bar{\omega}) + 1 = 0$ Being the closed-loop transfer function given by

$W(s) = \dfrac{F(s)}{1+F(s)}$ this shows that $s = j\bar{\omega}$ is a pole of $W(s)$

(and vice versa).

example: $F(s) = \dfrac{1}{s-1}$

fact II **Hyp**. no open-loop poles on the imaginary axis (i.e. with $\mathrm{Re}[.] = 0$)

let us define

- $n_F{}^+$ the number of **open-loop** poles with positive real part
- $n_W{}^+$ the number of **closed-loop** poles with positive real part
- N_{cc} the number of **encirclements** the Nyquist plot of $F(j\omega)$ makes around the point $(-1,\, 0)$ counted positive if counter-clockwise

a direct application of Cauchy's principle of argument gives

$$N_{cc} = n_F{}^+ - n_W{}^+$$

Obviously if the encirclements are defined positive clockwise, let them be N_c, the relationship changes sign and becomes $N_c = n_W{}^+$ - $n_F{}^+$

Hyp. no open-loop poles on the imaginary axis (i.e. with $\mathrm{Re}[.] = 0$)

(this hypothesis guarantees that, if $F(s)$ is strictly proper, the polar plot of $F(j\omega)$ is a closed contour and therefore we can determine the number of encirclements)

In order to guarantee closed-loop stability, we need $n_{W^+} = 0$ (no closed-loop poles with positive real part) and no poles with zero real part (which we saw being equivalent to asking that the Nyquist plot of $F(j\omega)$ does not pass through the point $(-1,\, 0)$)

Nyquist stability criterion (first version)

If the open-loop system has no poles on the imaginary axis,
the unit negative feedback system is **asymptotically stable**

if and only if

i) the Nyquist plot does not pass through the point $(-1,\, 0)$

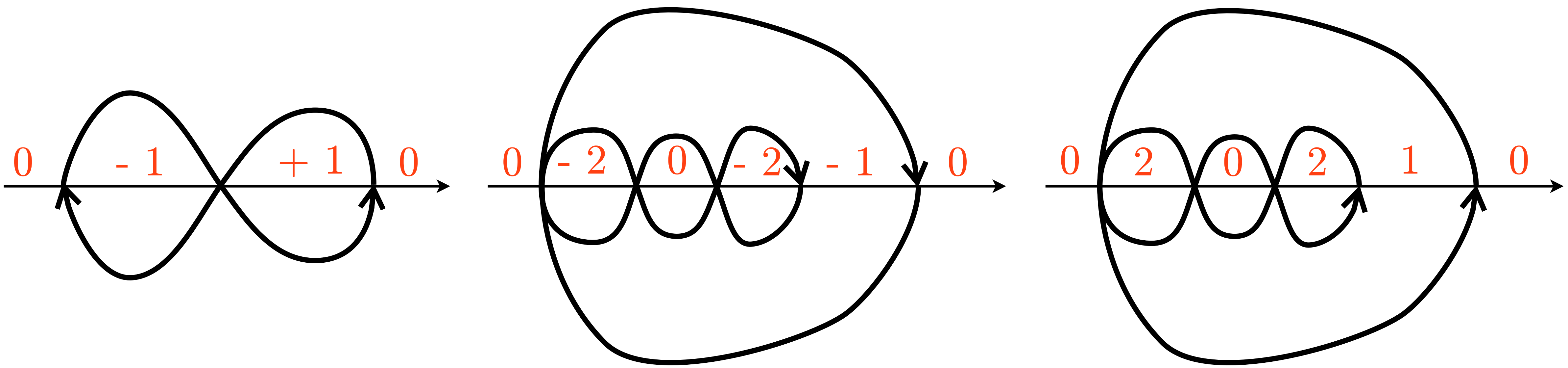
ii) the number of encirclements around the point $(-1,\, 0)$, counted positive if counter-clockwise, is equal to the number of open-loop poles with positive real part, i.e.

$$N_{cc} = n_{F^+}$$

Remarks

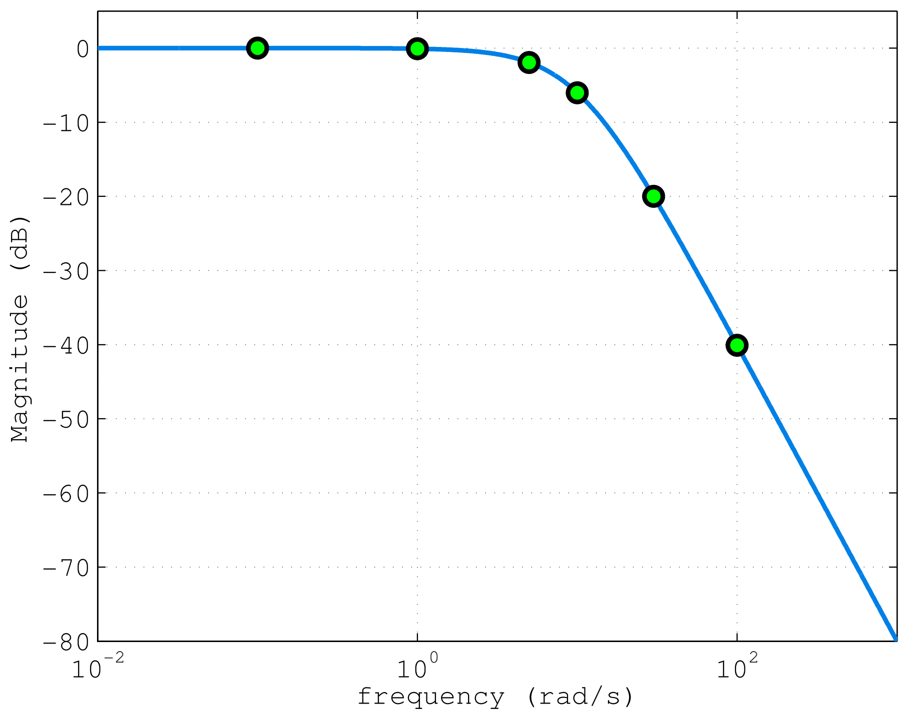
- if the open-loop system has no positive real part poles ($n_F{}^+ = 0$) then we obtain the simple N&S condition $N_{cc} = 0$ which requires the Nyquist plot not to encircle $(-1,\, 0)$
- if the stability condition is not satisfied (and N_{cc} exists) then we have an **unstable** closed-loop system with $n_W{}^+ = n_F{}^+ - N_{cc}$ positive real part poles
- condition i), which ensures that the closed-loop system does not have poles with zero real part, could be omitted by noting that if the Nyquist plot goes through the critical point $(-1,\, 0)$ then the number of encirclements is not well defined

examples on the number of encirclements depending on where is the critical point

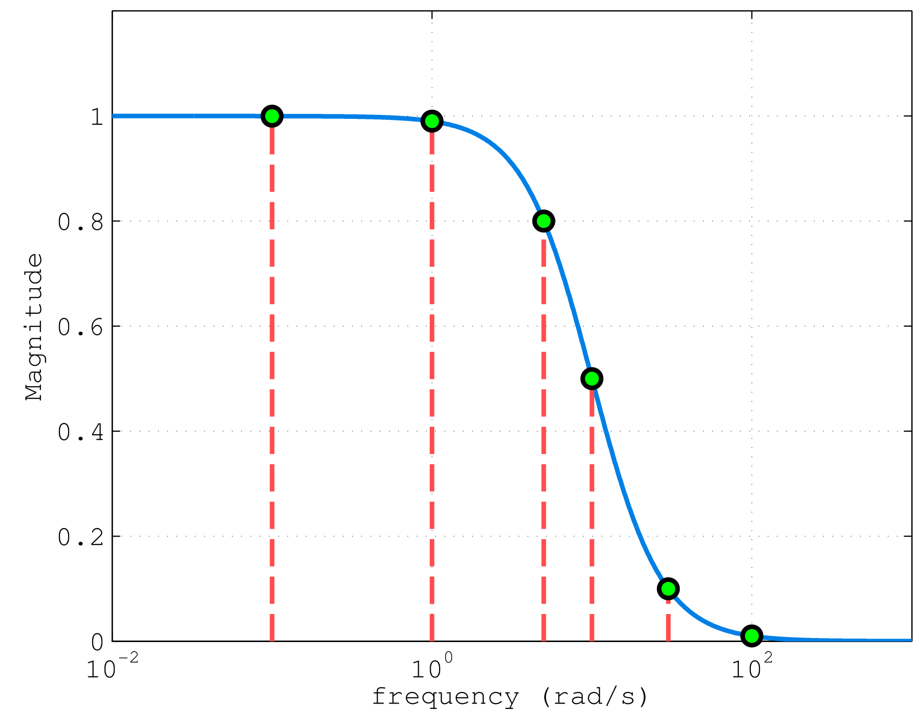


example I

in dB



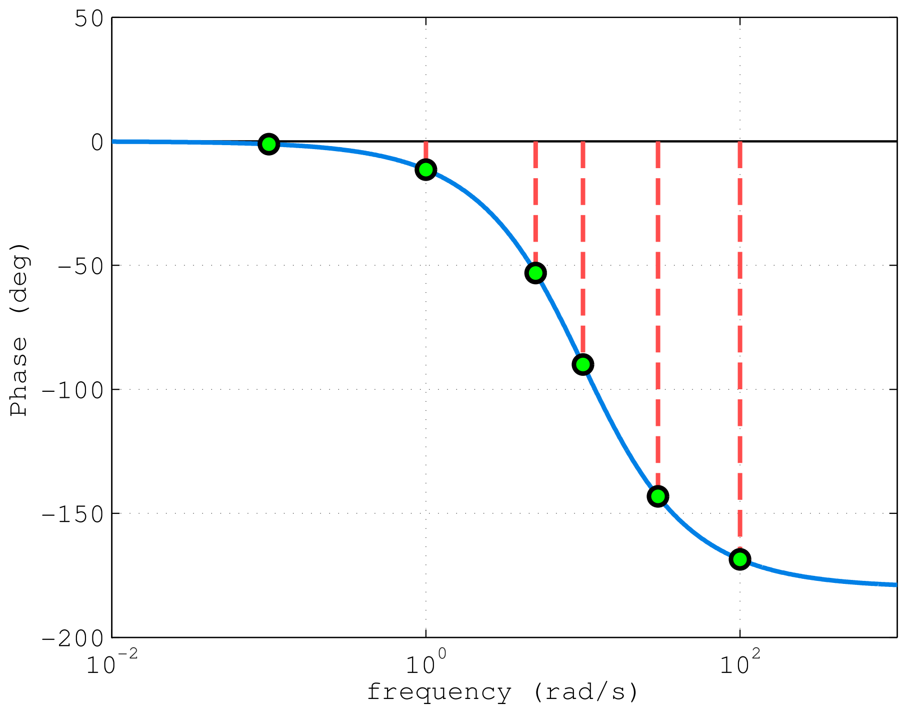
not in dB



open loop system

$$F(s) = \frac{100}{(s+10)^2}$$

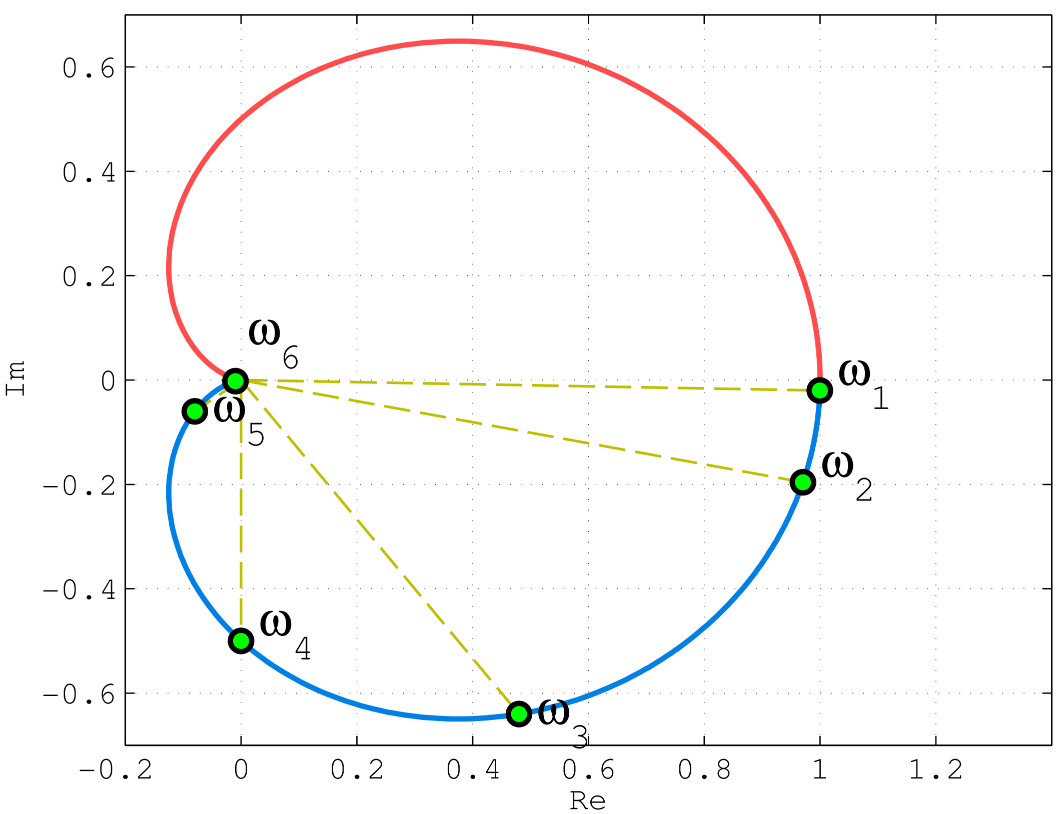
phase



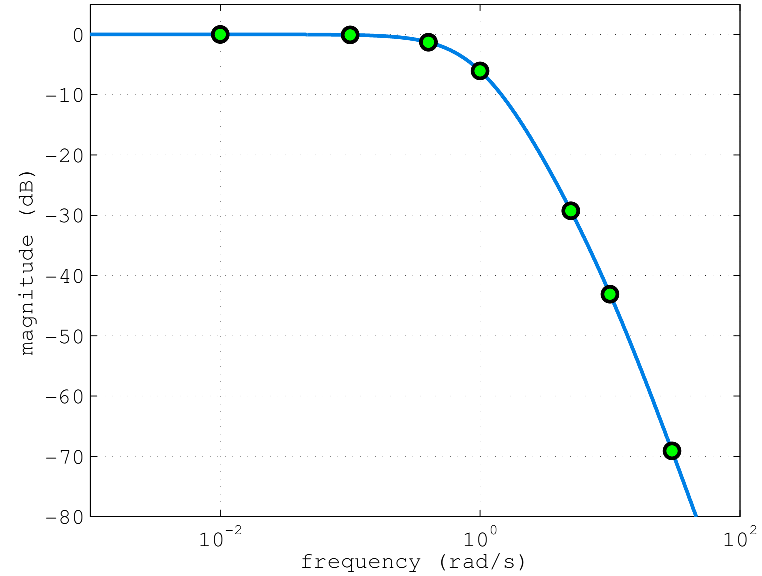
$$N_{cc} = n_F{}^+ = 0$$

Nyquist criterion is verified thus the closed-loop system is asymptotically stable

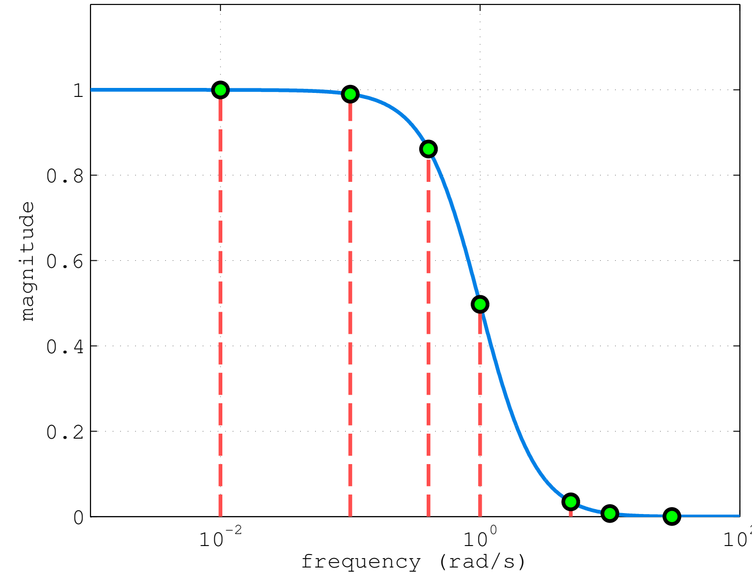
Nyquist plot



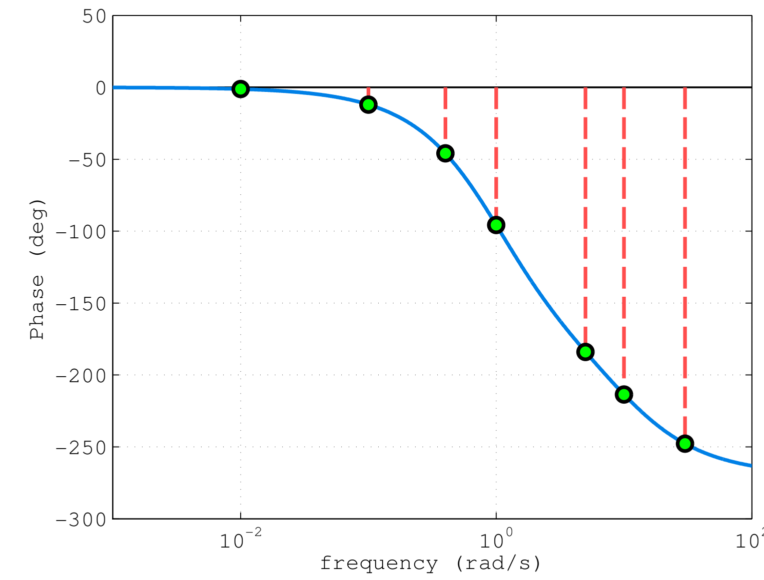
magnitude (dB)



magnitude



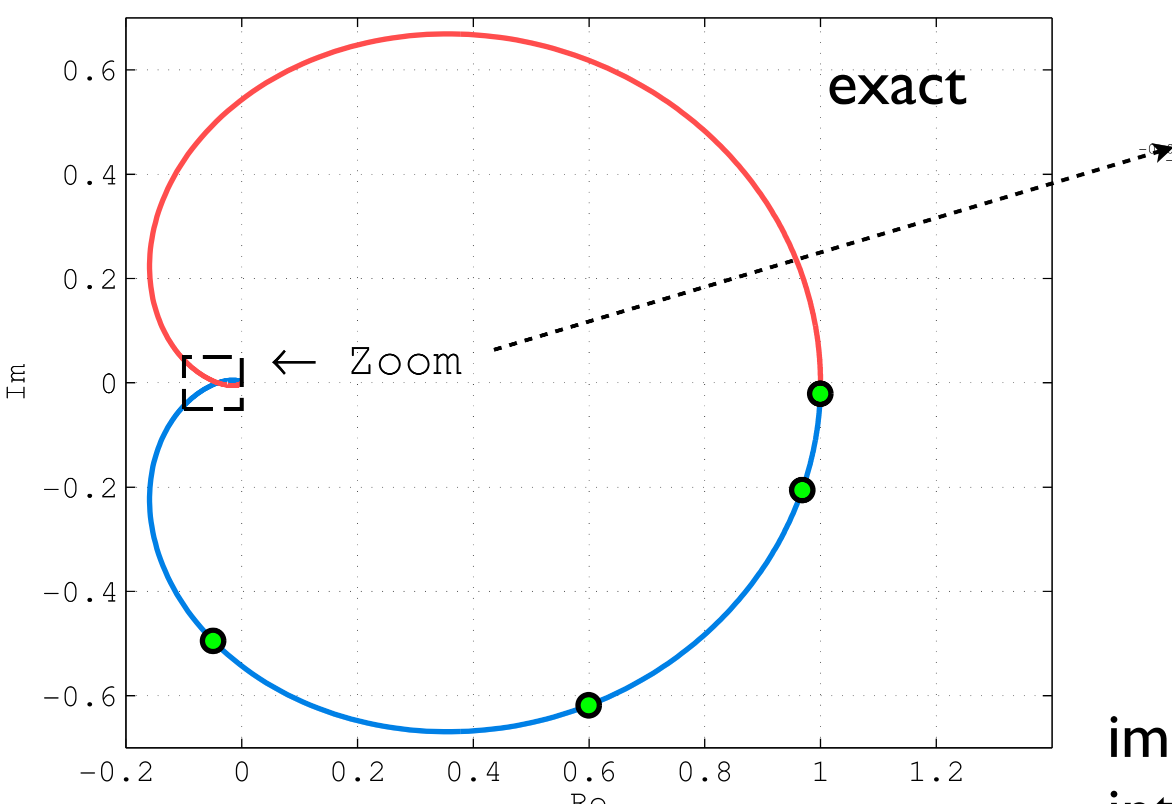
phase

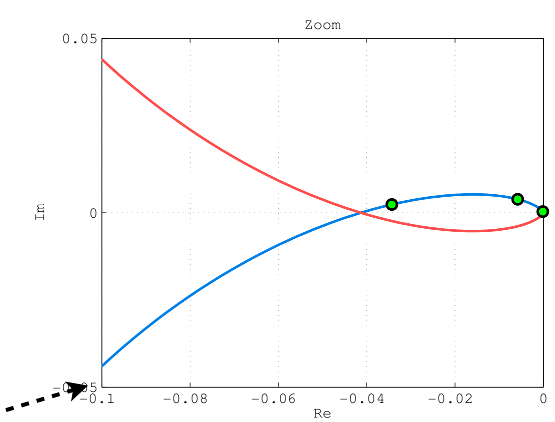


open loop system

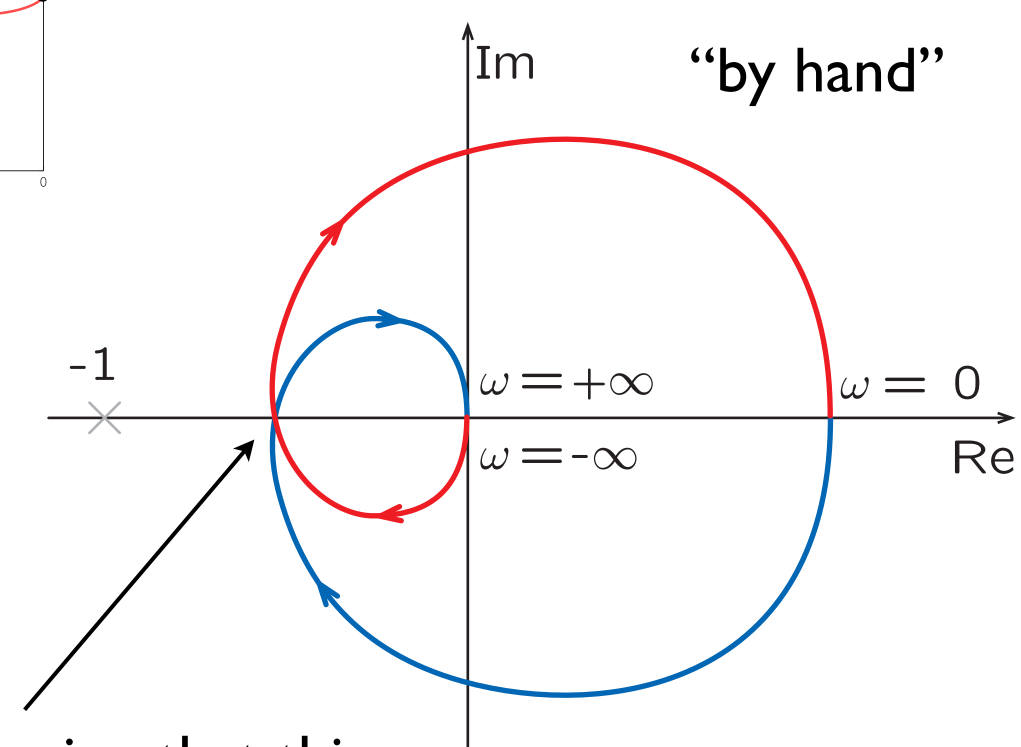
$$F(s) = \frac{10}{(s+1)^2(s+10)}$$

example II



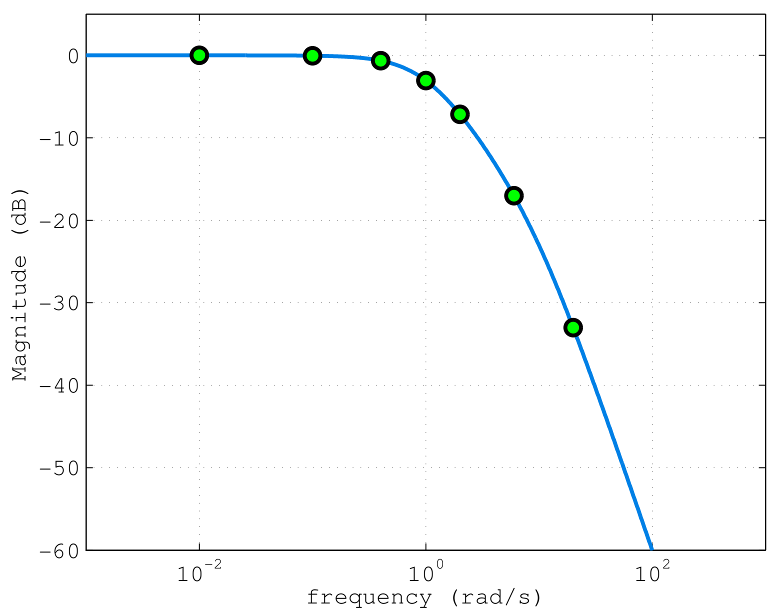


Nyquist plot

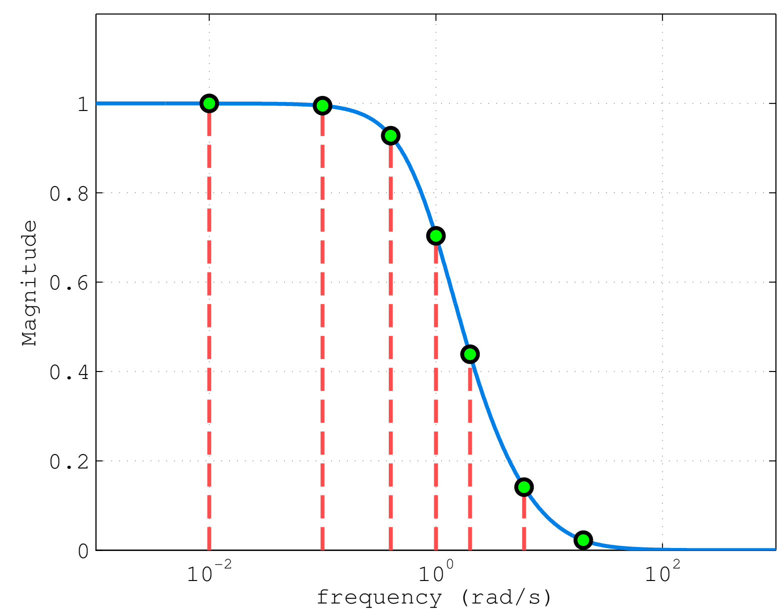


important to determine that this intersection is on the right of -1

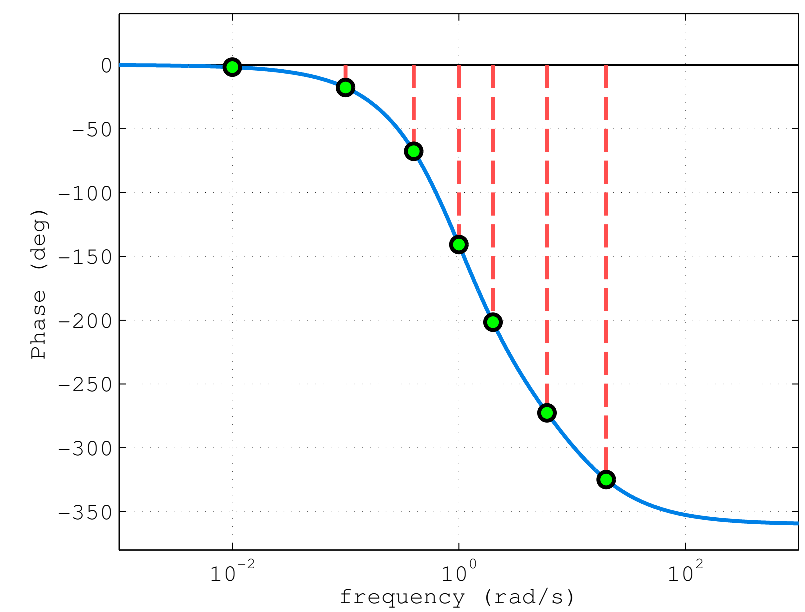
magnitude (dB)



magnitude



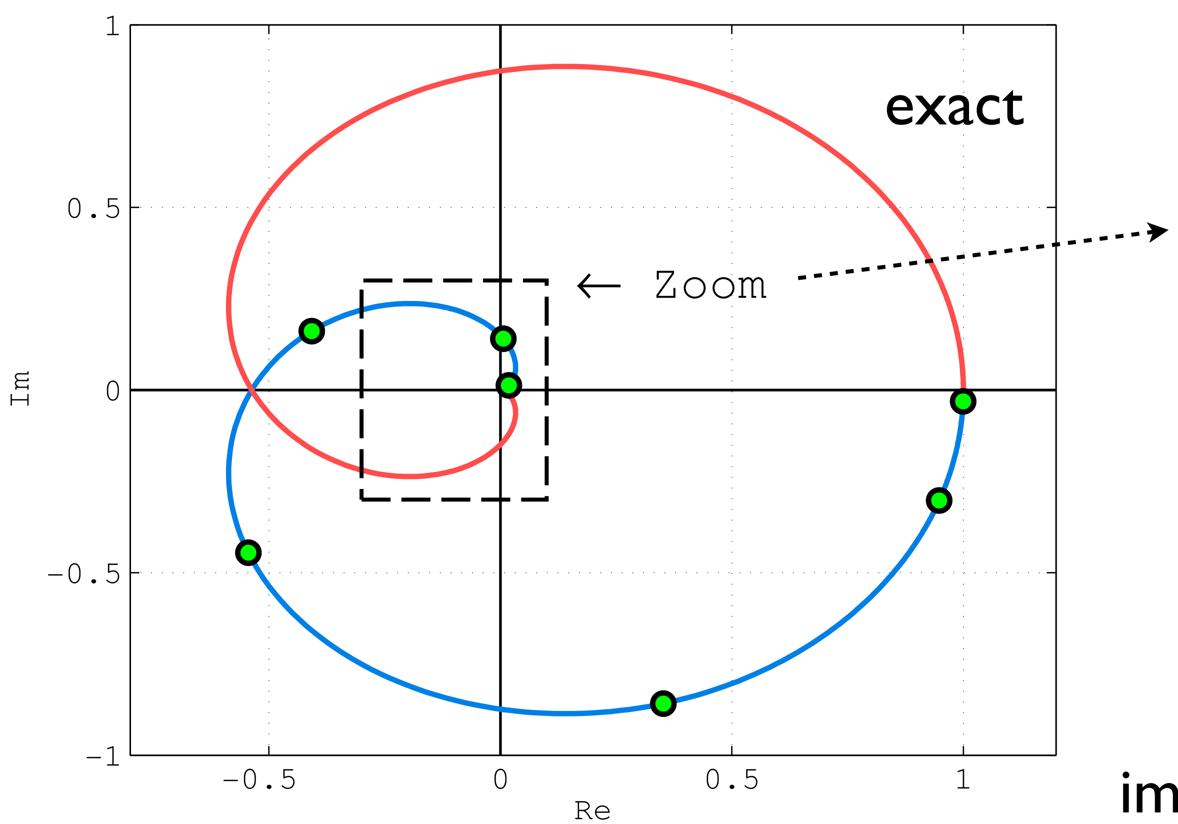
phase

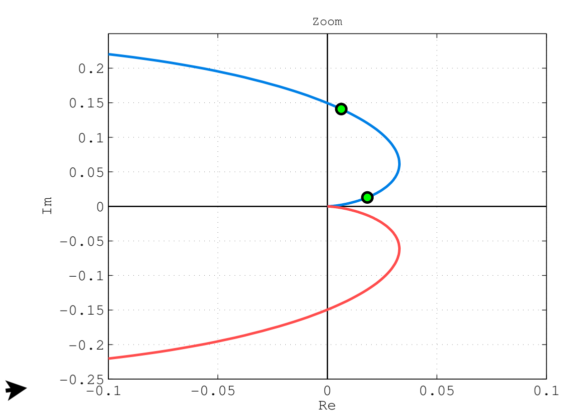


open loop system

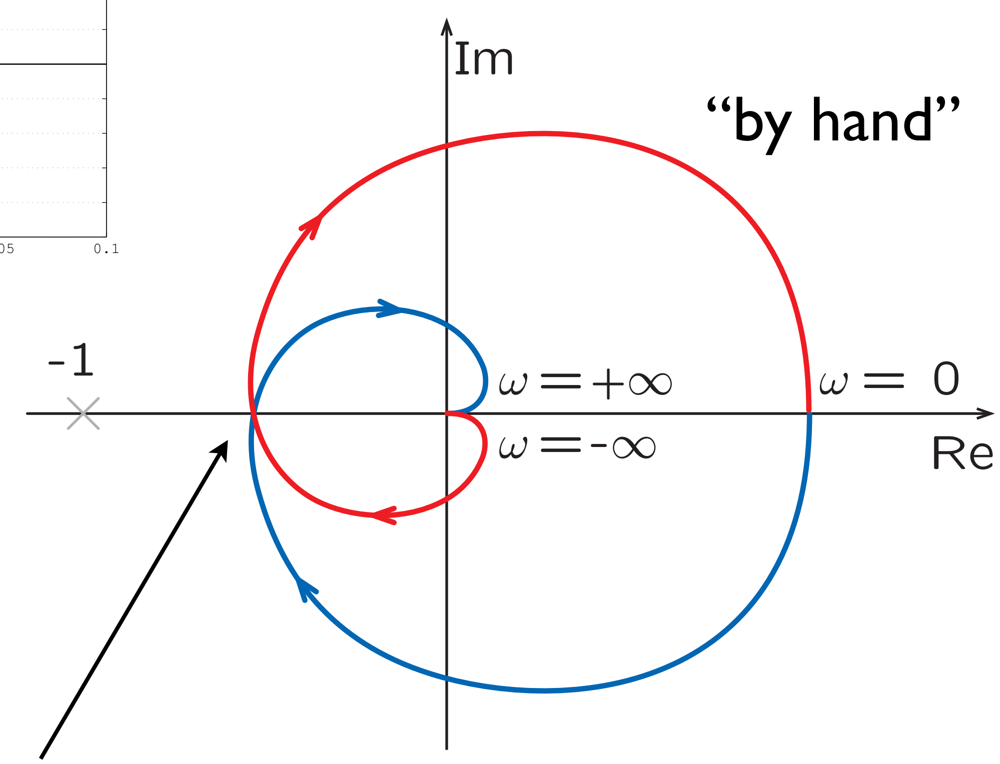
$$F(s) = \frac{-10(s-1)}{(s+1)^2(s+10)}$$

example III



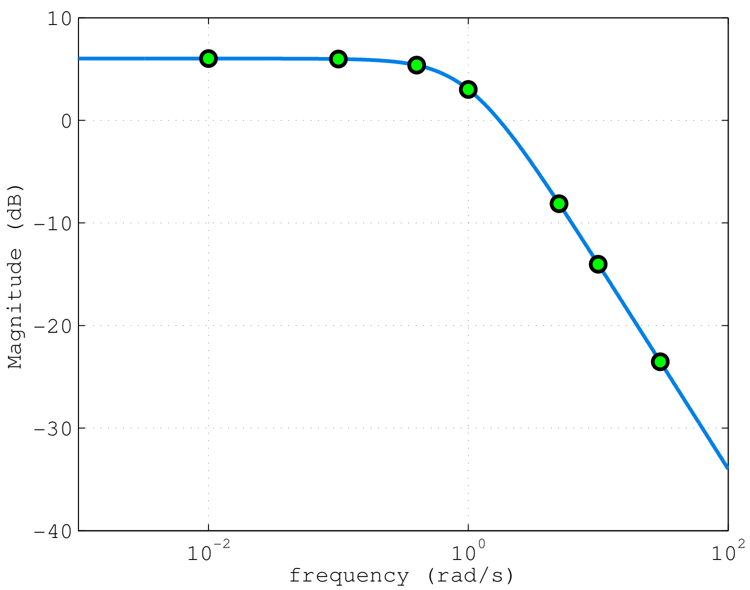


Nyquist plot

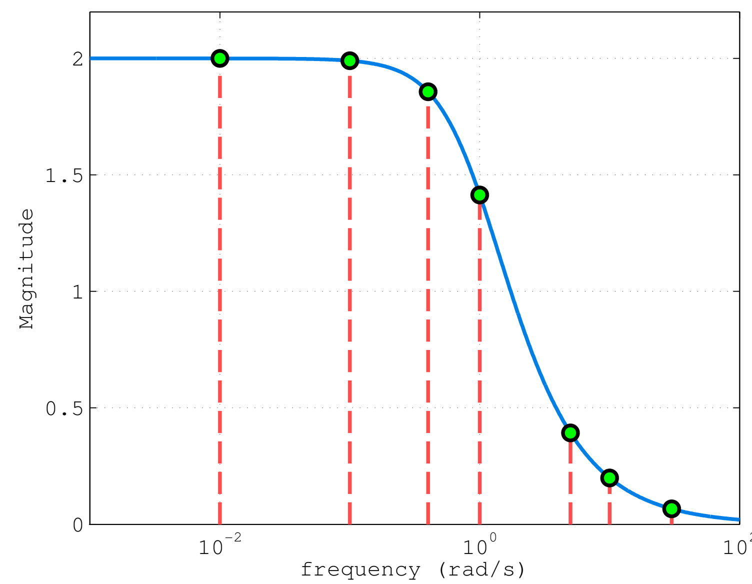


important to determine that this intersection is on the right of -1

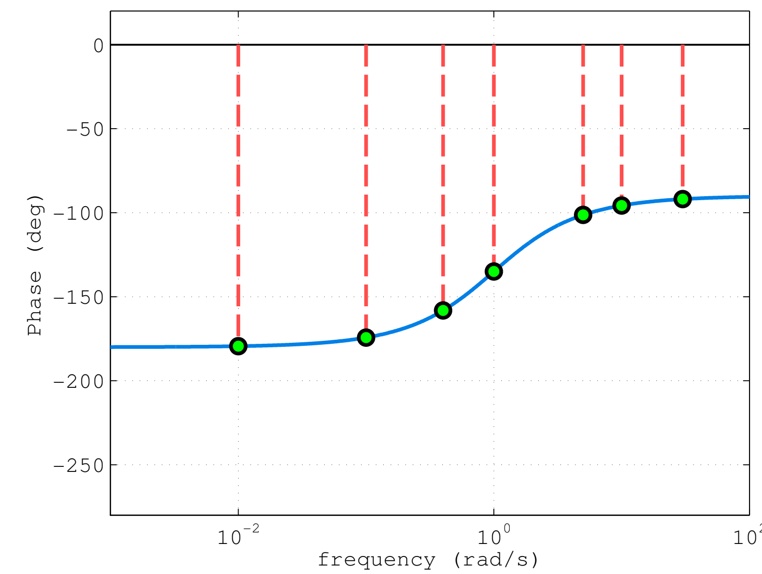
magnitude (dB)



magnitude



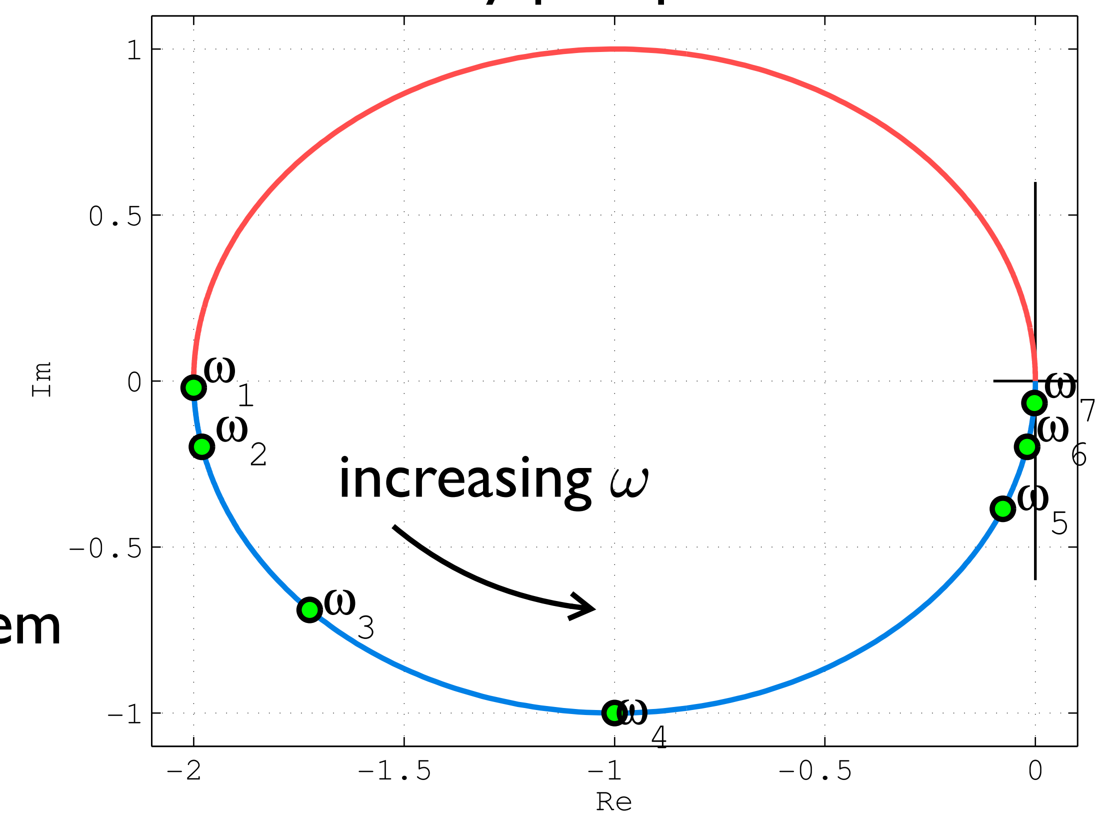
phase



$$F(s) = \frac{2}{s-1}$$

unstable open-loop system

exact Nyquist plot



$$N_{cc} = n_F^+ = 1$$

example IV

Nyquist criterion is verified: closed-loop system is asymptotically stable

Let's remove the hypothesis of "no open-loop poles on the imaginary axis" (i.e. with $\mathrm{Re}[.] = 0$)

Open-loop poles on the imaginary axis (i.e. with $\mathrm{Re}[.] = 0$) come from:

- one or more integrators (pole in $s = 0$)
- resonance (imaginary poles in $s = +/-\ j\omega_n$)

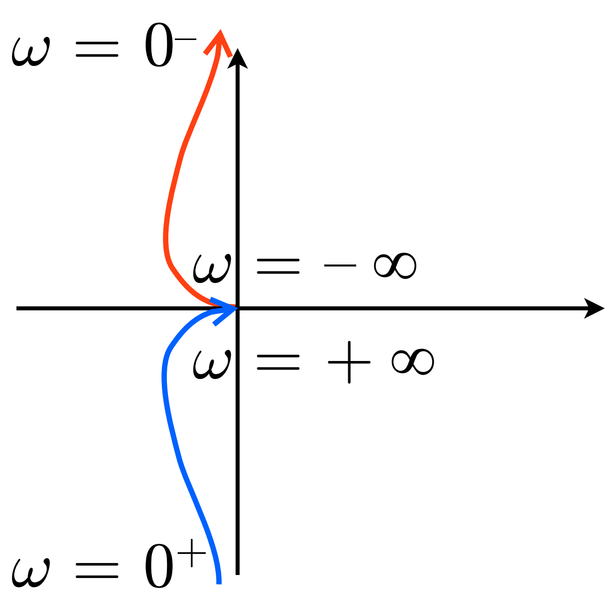
and give a discontinuity of $-\pi$ in the phase

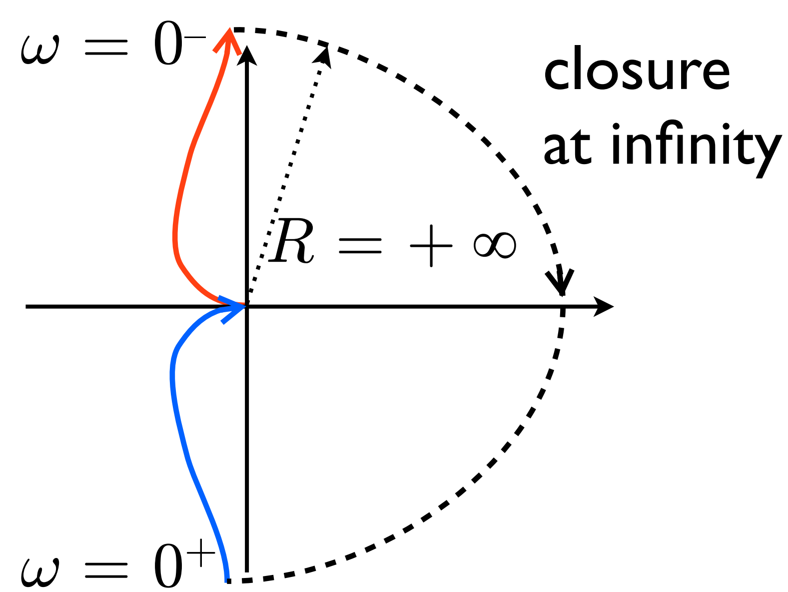
- passing from $\pi/2$ to $-\pi/2$ when ω switches from 0^- to 0^+
- or from 0 to $-\pi$ when ω switches from $\omega_n{}^-$ to $\omega_n{}^+$

while the magnitude is at infinity

In order to obtain a closed polar plot, we introduce **closures at infinity** which consists in rotating of π **clockwise** (corresponding to a variation of $-\pi$) with an infinite radius (for **every pole** with $\mathrm{Re}[.] = 0$) for increasing frequencies, at those values of the frequency corresponding to singularities of the transfer function $F(s)$ lying on the imaginary axis (poles of the open-loop system with $\mathrm{Re}[.] = 0$)

$$F(s) = \frac{1}{s(s+1)}$$





closures at infinity examples

$F(s) = \dfrac{K}{s(1+\tau_1 s)}$	π clockwise ($= -\pi$) at infinity from $\omega = 0^-$ to $\omega = 0^+$
$F(s) = \dfrac{K}{s^2(1+\tau_1 s)}$	2π clockwise ($= -2\pi$) at infinity from $\omega = 0^-$ to $\omega = 0^+$
$F(s) = \dfrac{K(1+\tau_2 s)}{s^3(1+\tau_1 s)}$	3π clockwise at infinity from $\omega = 0^-$ to $\omega = 0^+$
$F(s) = \dfrac{K}{(s^2+\omega_1^2)(1+\tau_1 s)}$	π clockwise at infinity from $\omega = -\omega_1{}^-$ to $\omega = -\omega_1{}^+$ π clockwise at infinity from $\omega = \omega_1{}^-$ to $\omega = \omega_1{}^+$
$F(s) = \dfrac{K}{(s^2+\omega_1^2)^2(1+\tau_1 s)}$	2π clockwise at infinity from $\omega = -\omega_1{}^-$ to $\omega = -\omega_1{}^+$ 2π clockwise at infinity from $\omega = \omega_1{}^-$ to $\omega = \omega_1{}^+$
$F(s) = \dfrac{K(1+\tau_2 s)}{s^2(s^2+\omega_1^2)(1+\tau_1 s)}$	π clockwise at infinity from $\omega = -\omega_1{}^-$ to $\omega = -\omega_1{}^+$ 2π clockwise at infinity from $\omega = 0^-$ to $\omega = 0^+$ π clockwise at infinity from $\omega = \omega_1{}^-$ to $\omega = \omega_1{}^+$

Nyquist stability criterion (no restriction on open-loop poles)

Let the open-loop system have ${n_F}^+$ poles with positive real part.

The unit negative feedback system is **asymptotically stable**

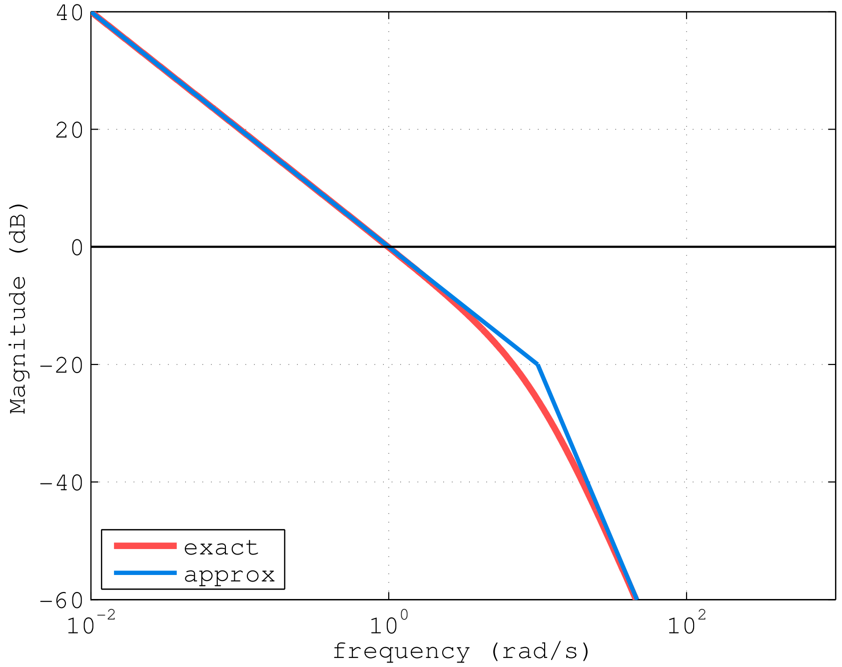
if and only if

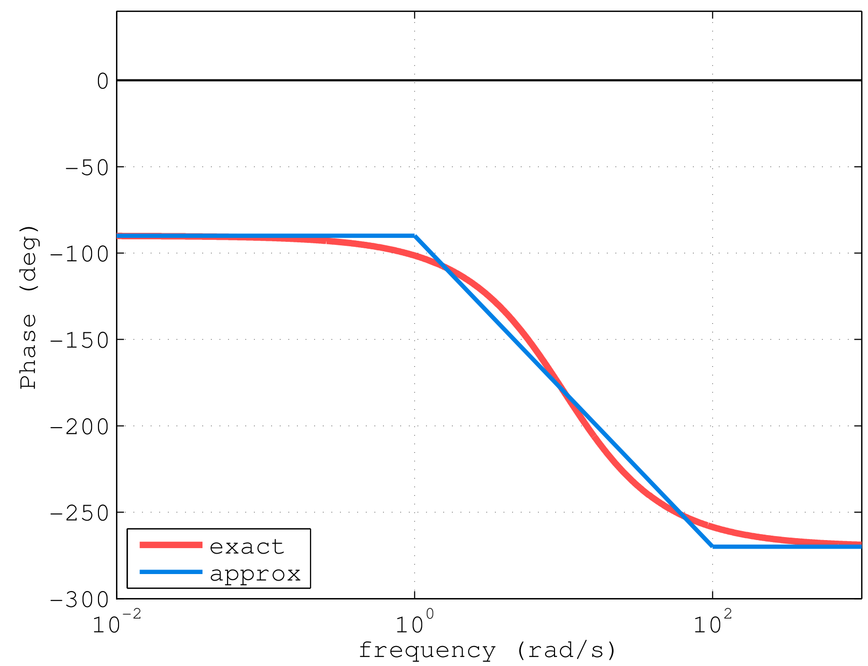
i) the Nyquist plot does not pass through the point $(-1,\,0)$

ii) the number of encirclements around the point $(-1,\,0)$ counted positive if counter-clockwise is equal to the number of open-loop poles with positive real part, i.e.

$$N_{cc} = {n_F}^+$$

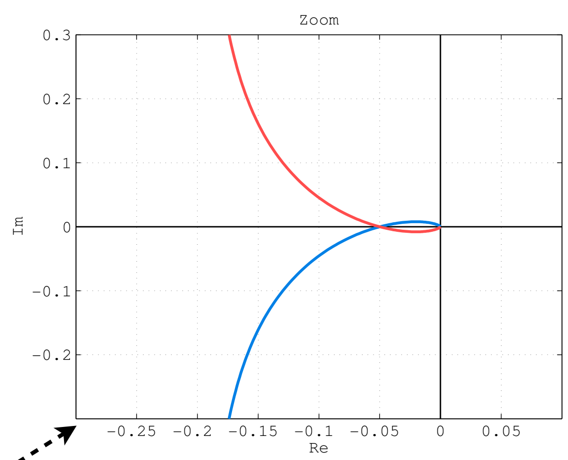
That is the same result shown before, valid under the hypothesis of no open-loop poles on the imaginary axis (i.e. with $\mathrm{Re}[.] = 0$), still holds provided we define how to obtain the closures at infinity.



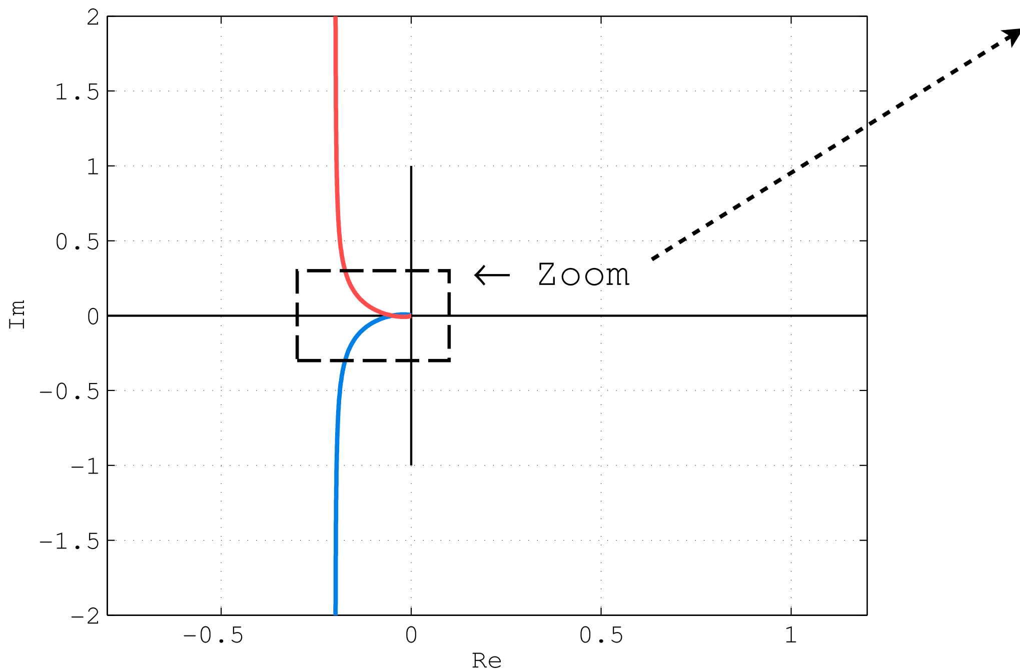


example V

$$F(s) = \frac{100}{s(s+10)^2}$$

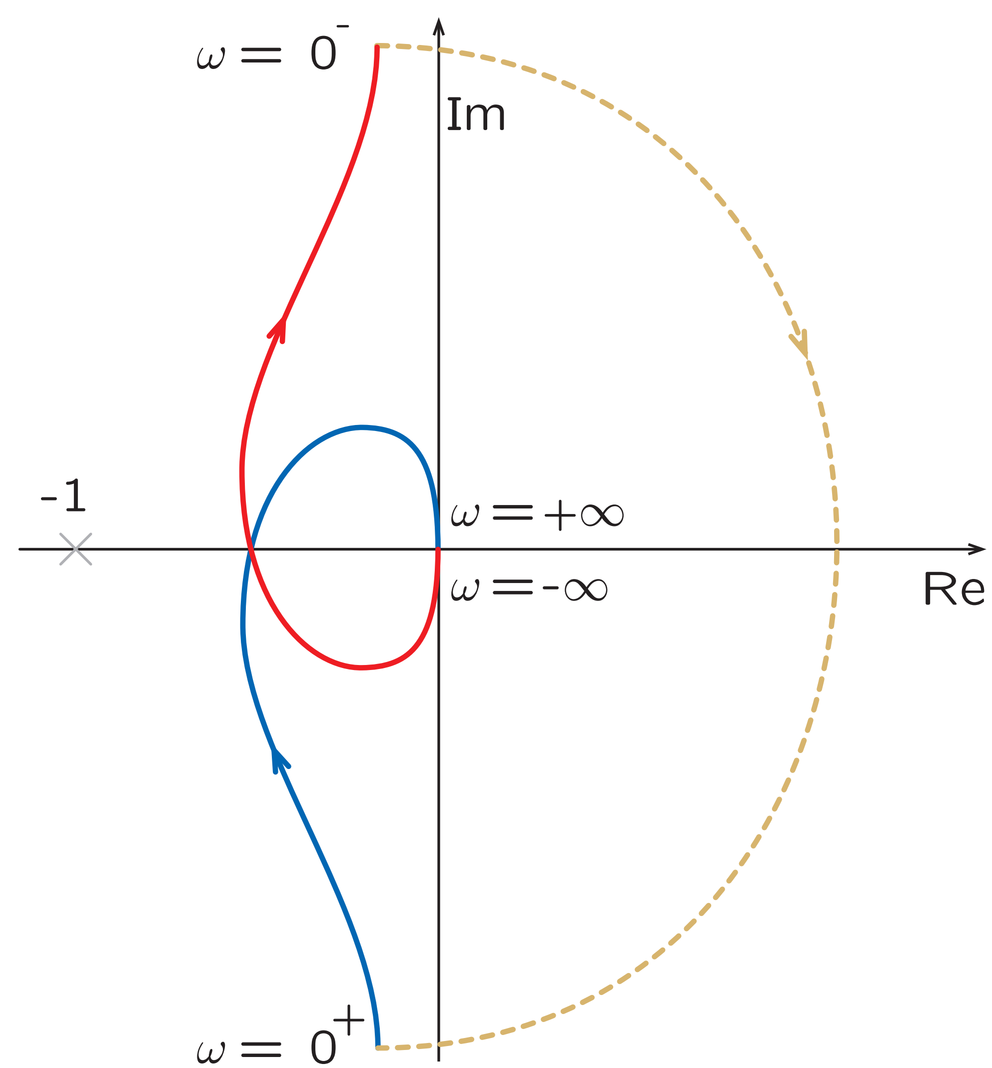


the zoom is just to show that the phase goes to $-3\pi/2$



$$N_{cc} = n_{F^+} = 0$$

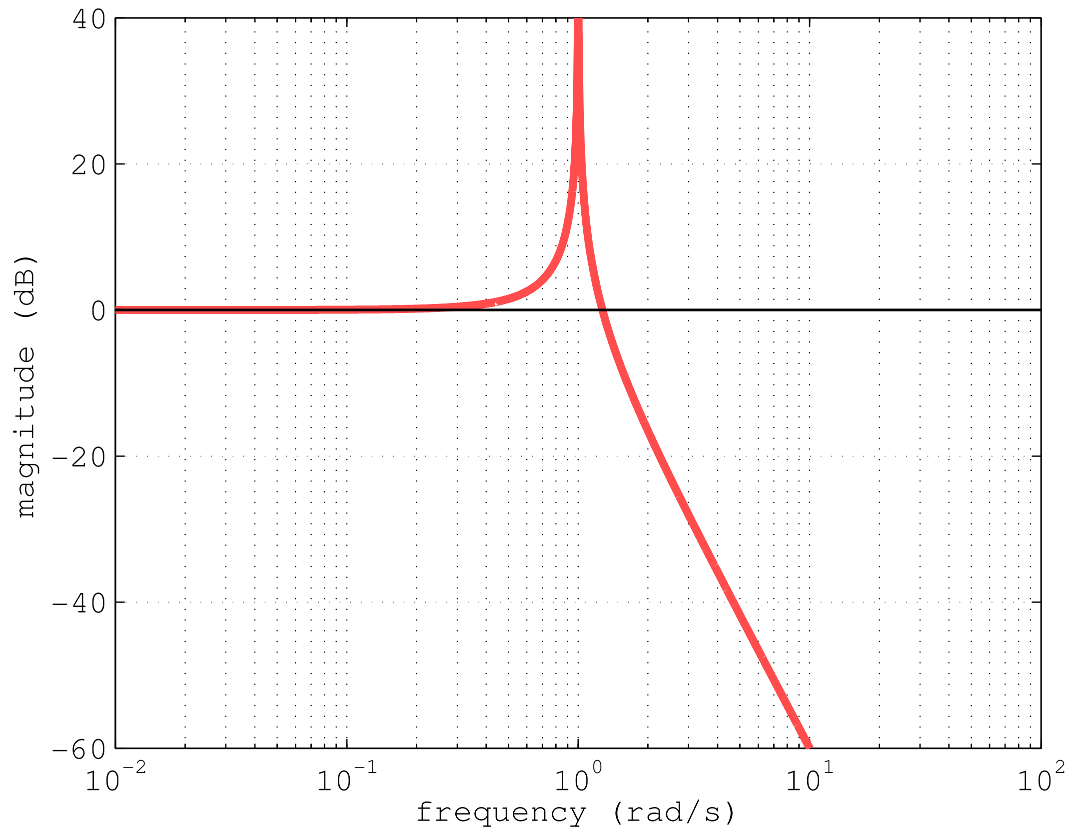
π clockwise with infinite radius from 0^- to 0^+

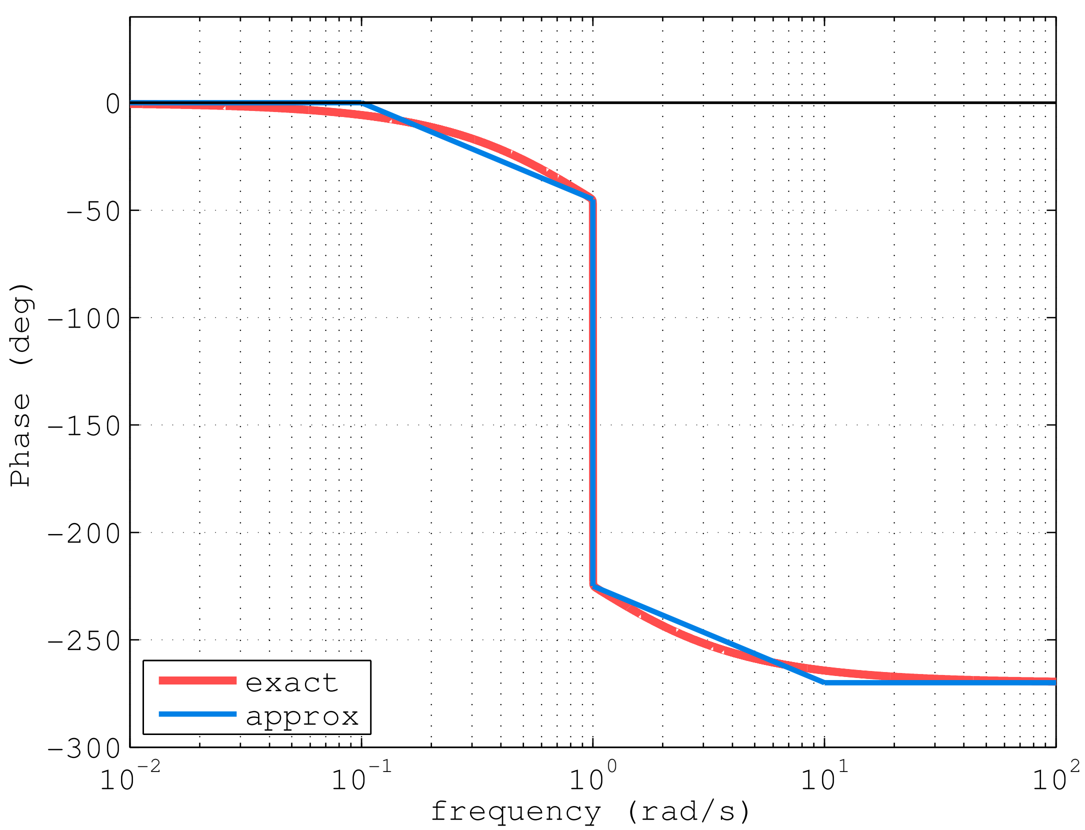


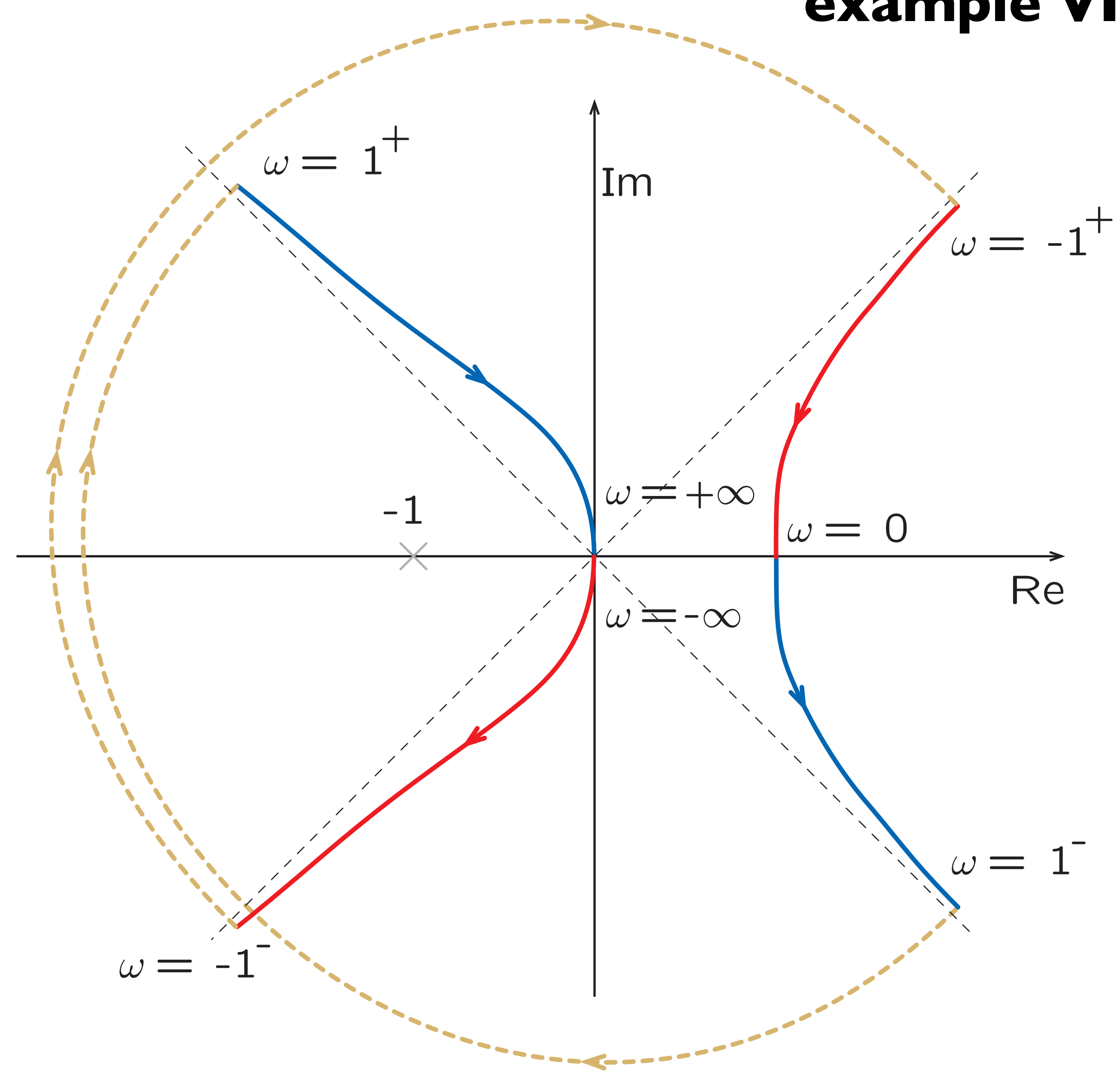
$$F(s) = \frac{1}{(s+1)(s^2+1)}$$

one pole in $+j$
one pole in $-j$

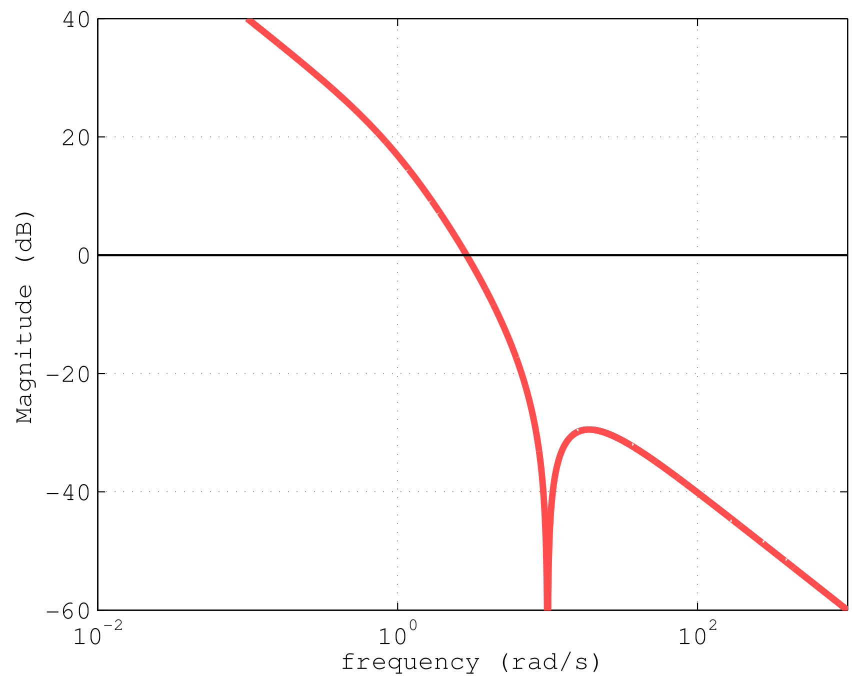
example VI

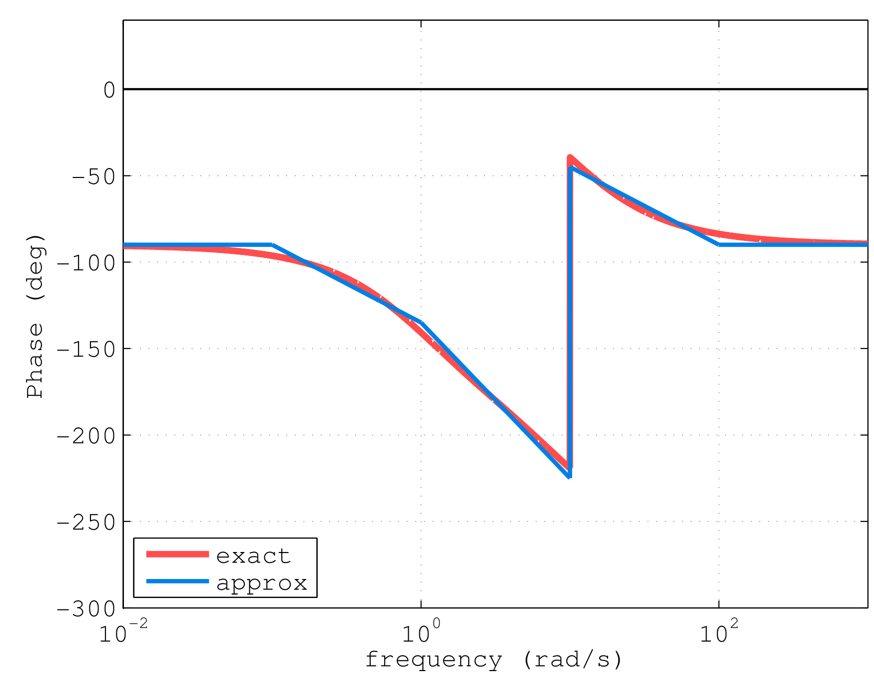






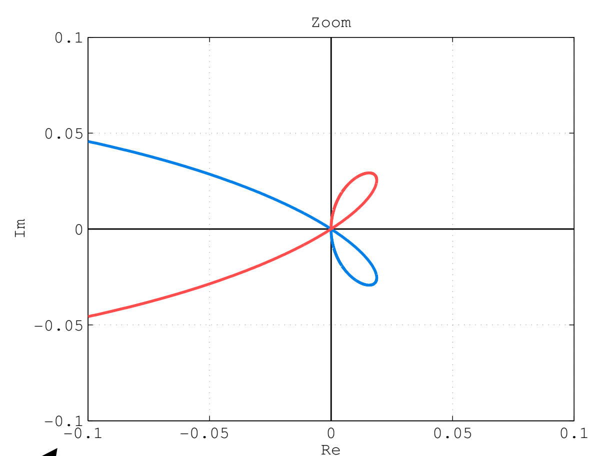
π clockwise with infinite radius from -1^- to -1^+
π clockwise with infinite radius from 1^- to 1^+

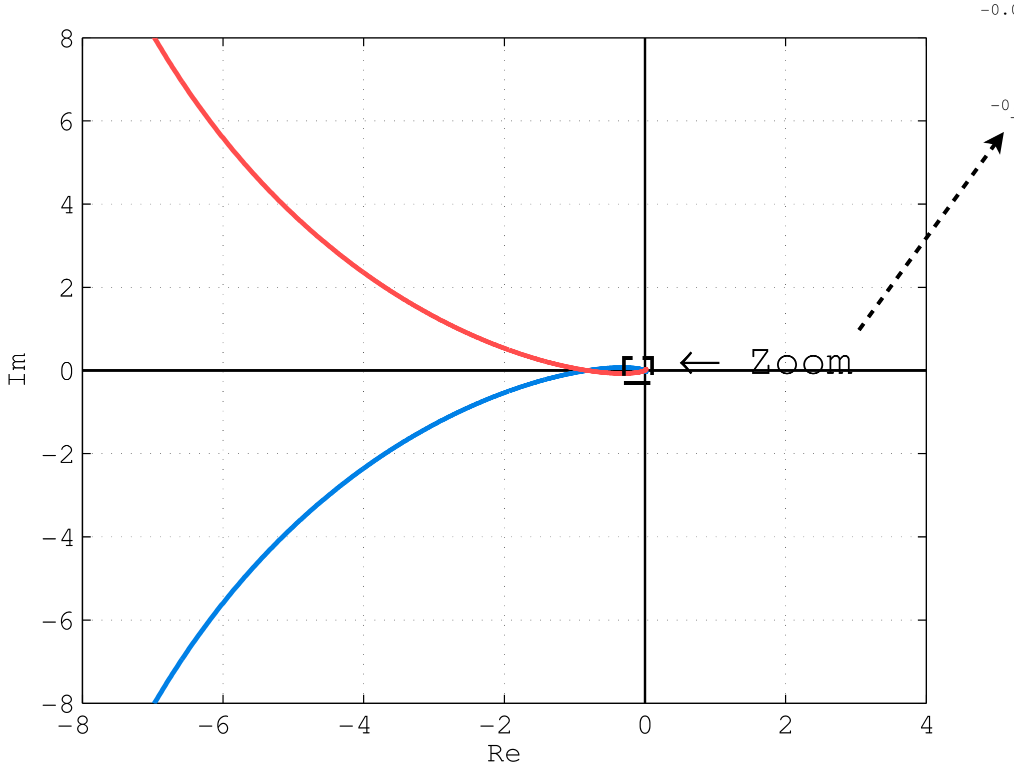




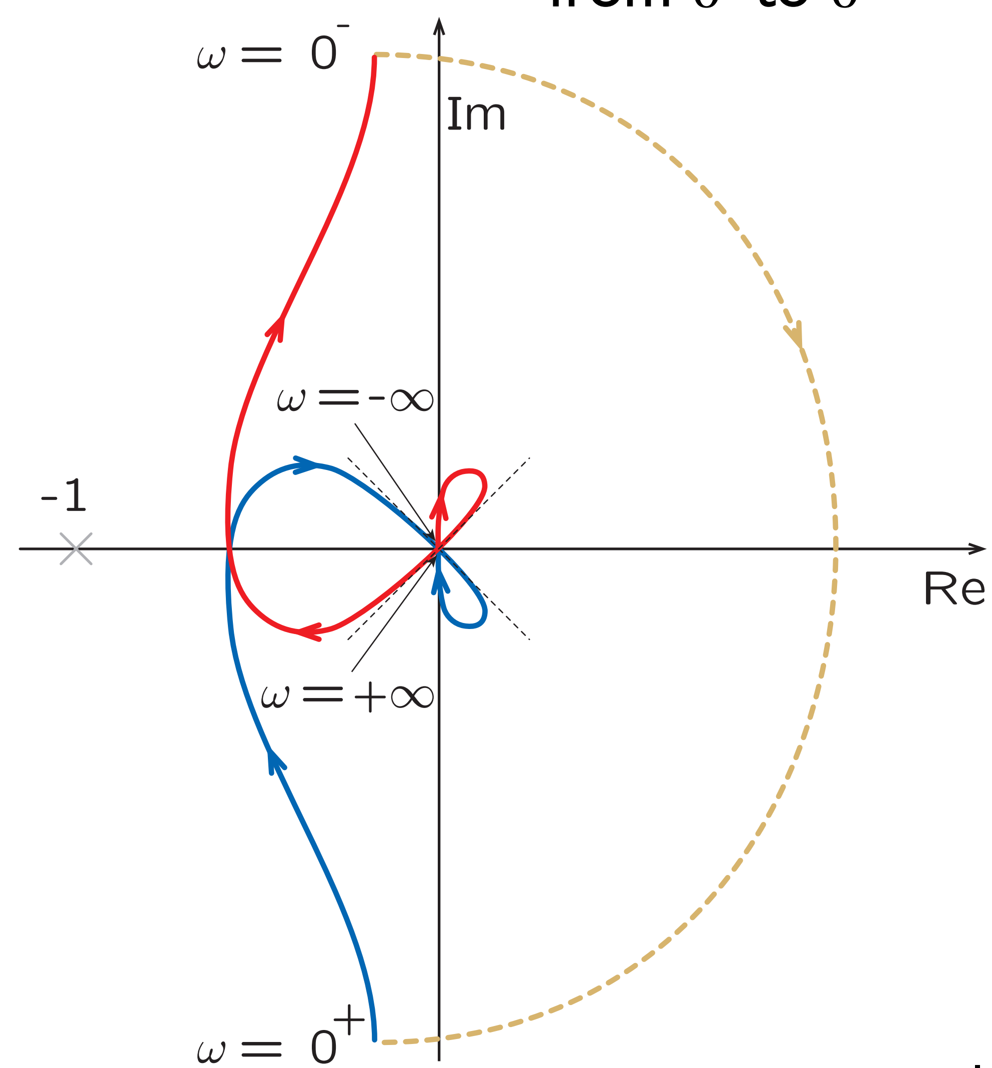
example VII

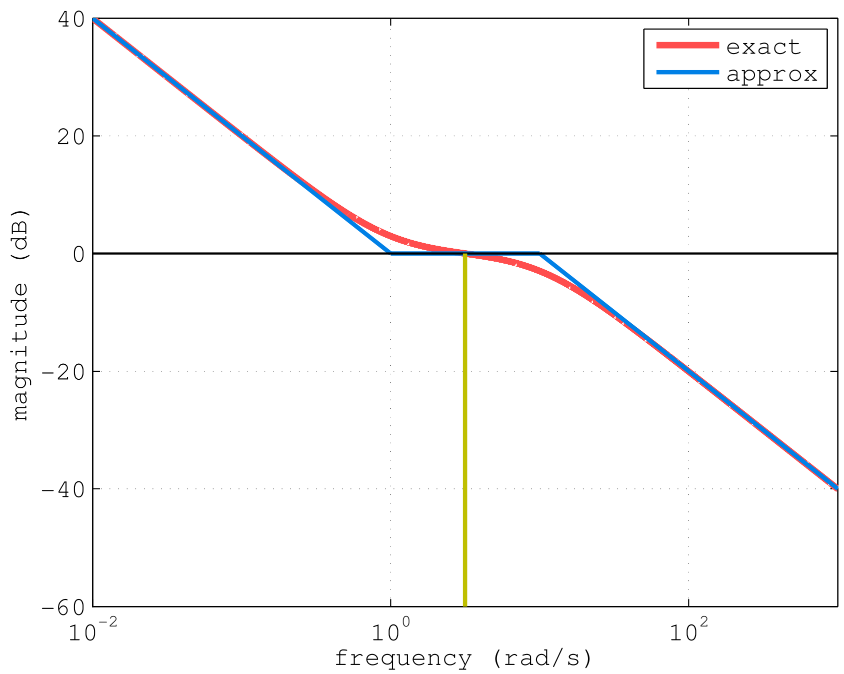
$$F(s) = \frac{s^2 + 100}{s(s+1)(s+10)}$$





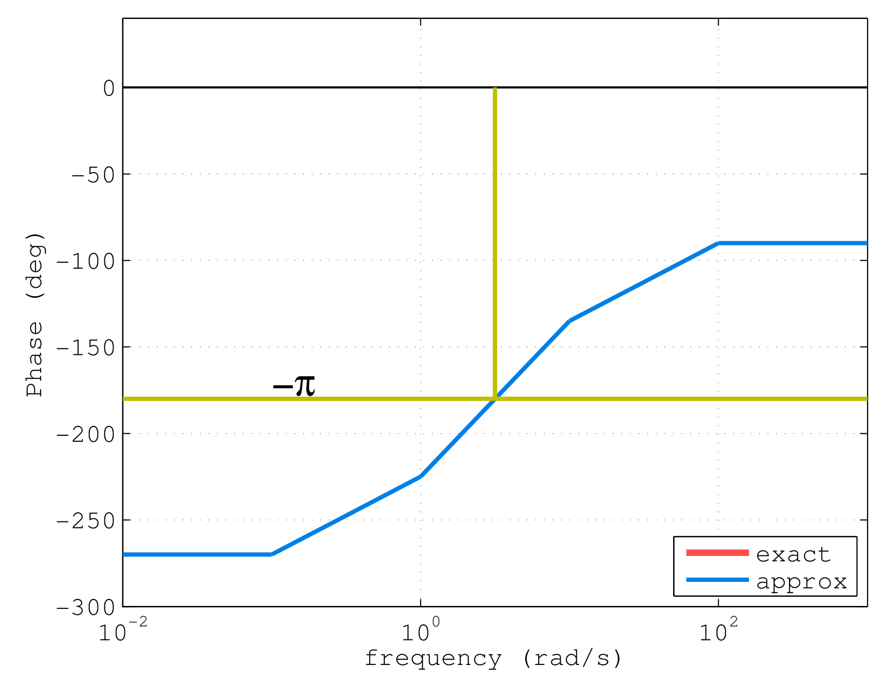
π clockwise with infinite radius from 0$^-$ to 0$^+$

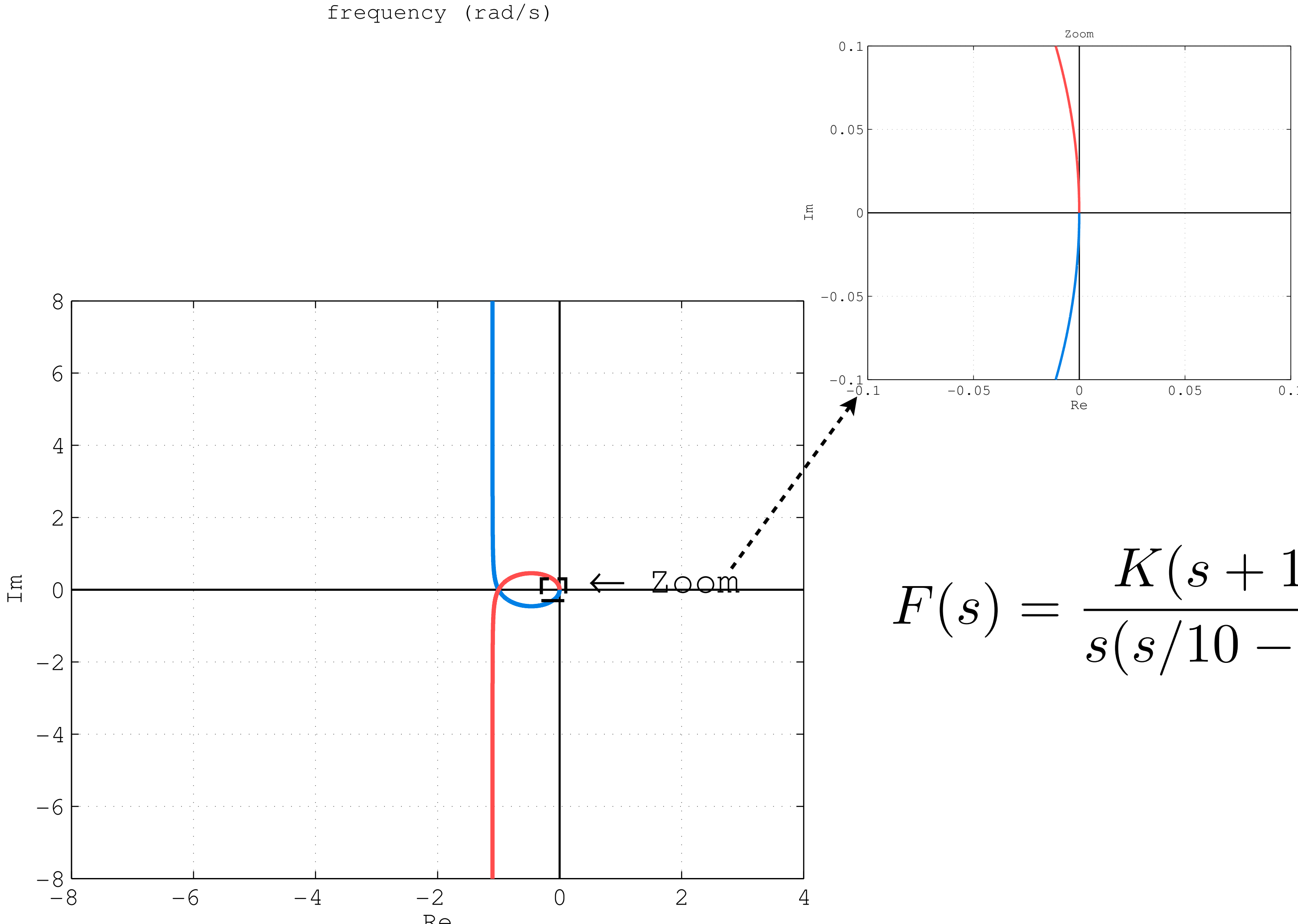




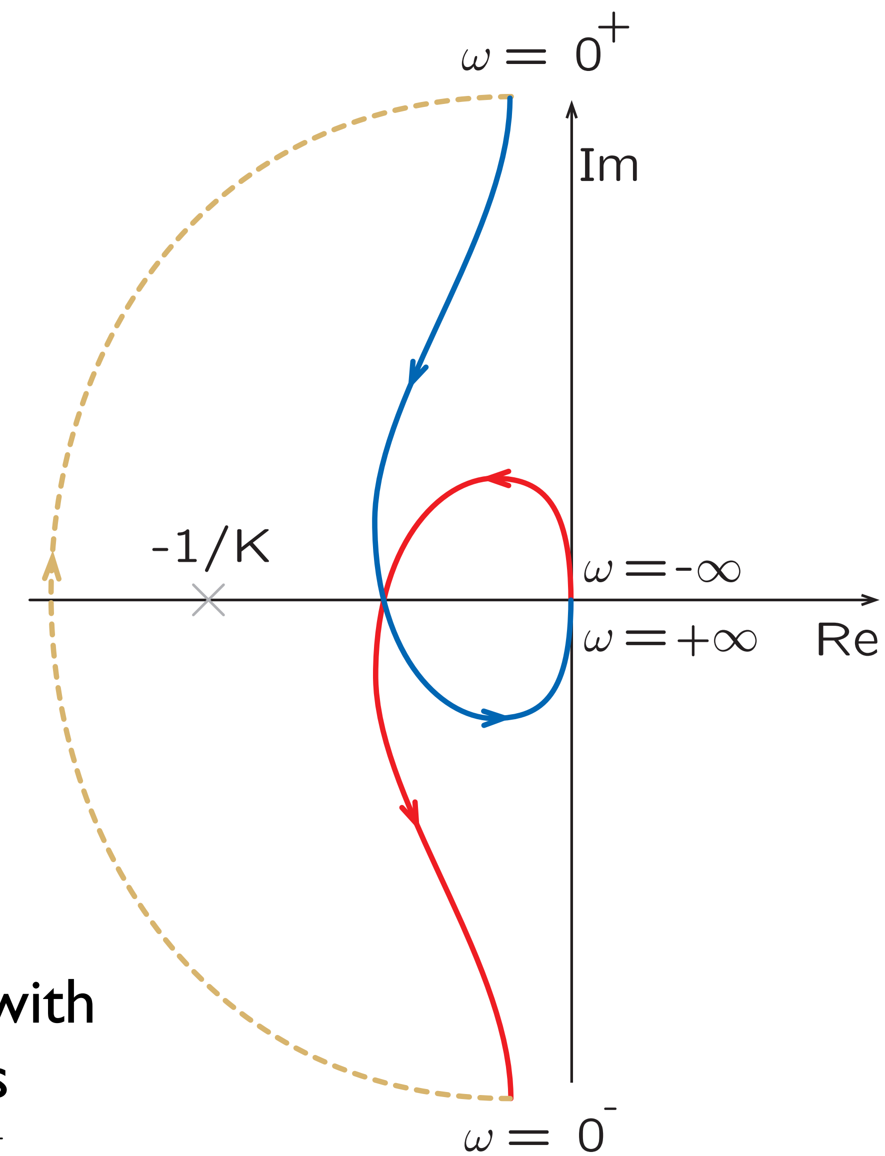
example VIII

for $K = 1$ plot crosses $(-1, 0)$



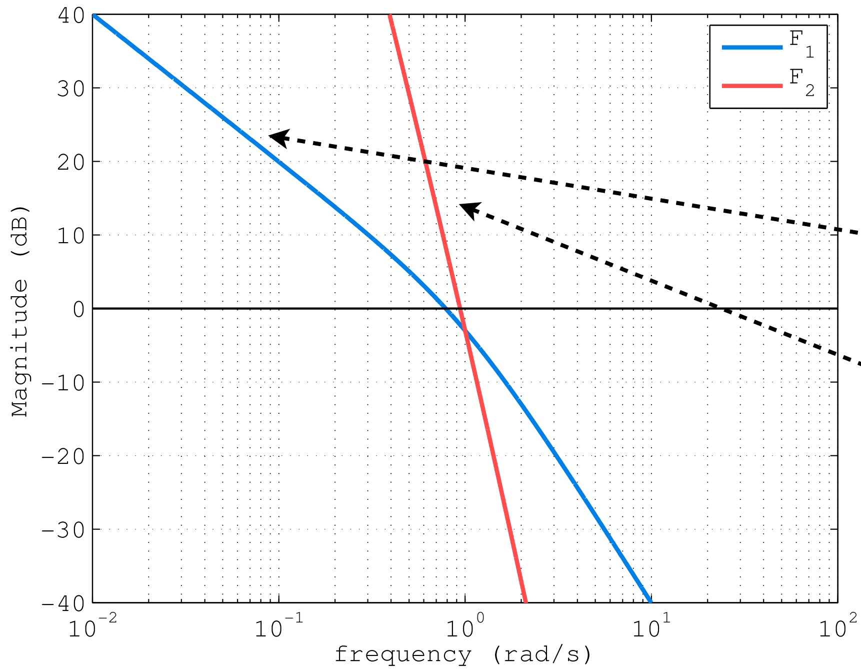


$$F(s) = \frac{K(s+1)}{s(s/10-1)}$$



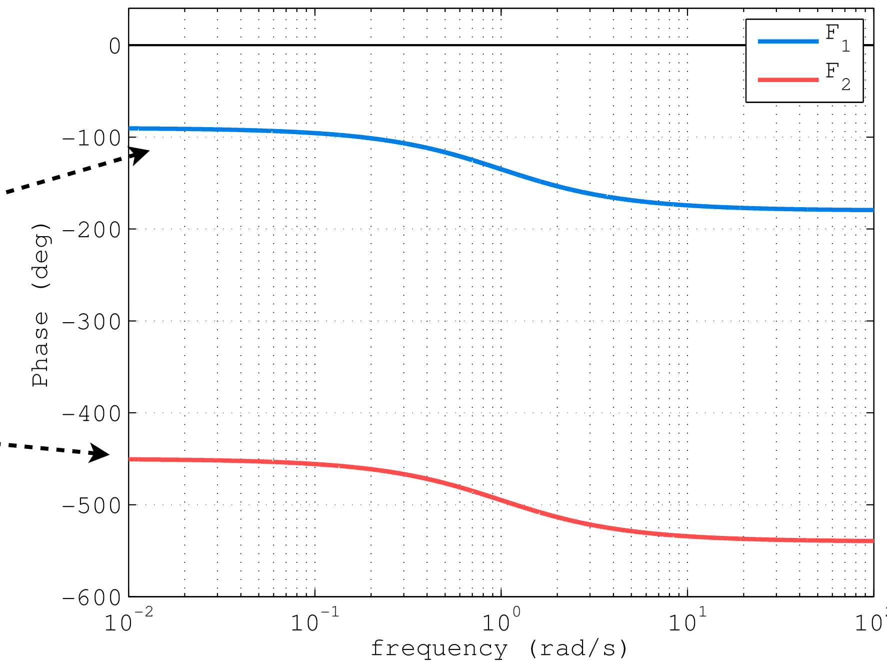
π clockwise with infinite radius from 0^- to 0^+

example IX

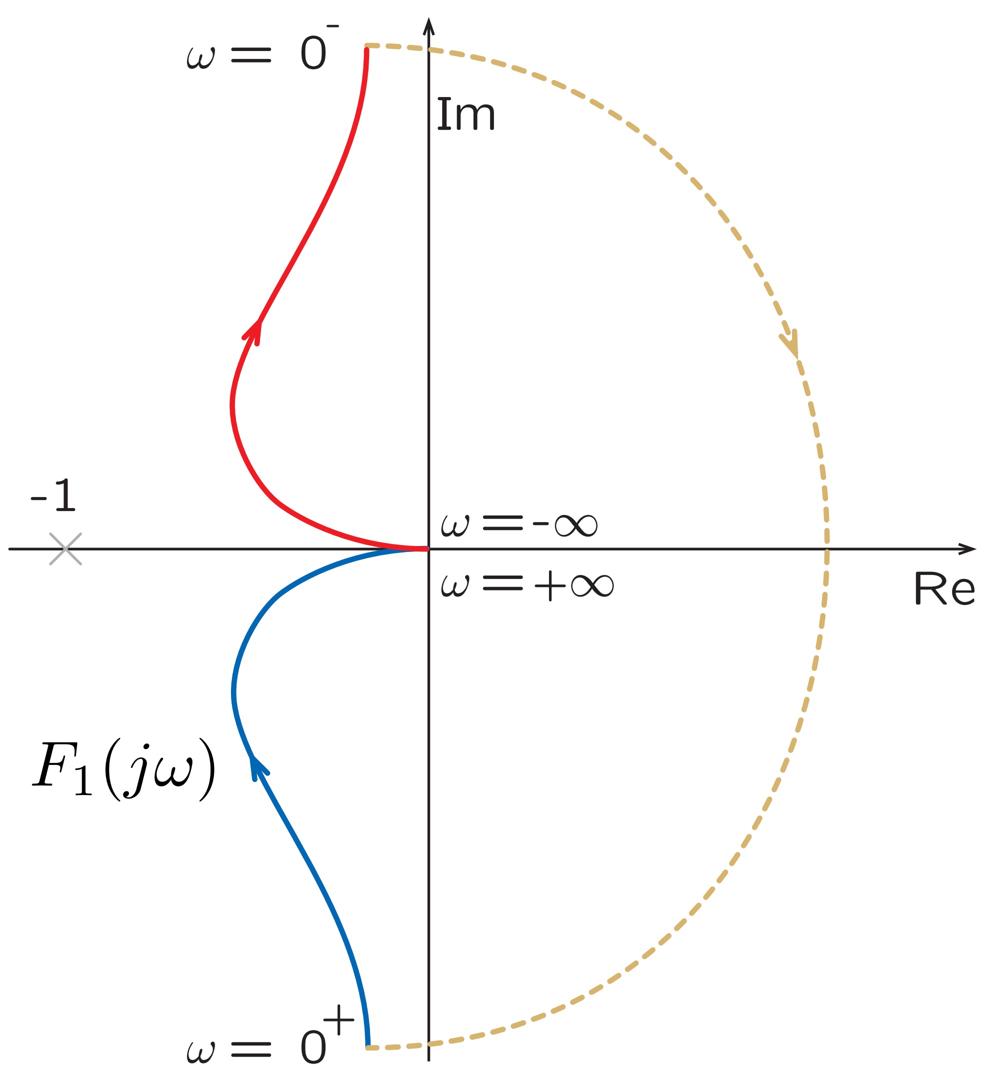


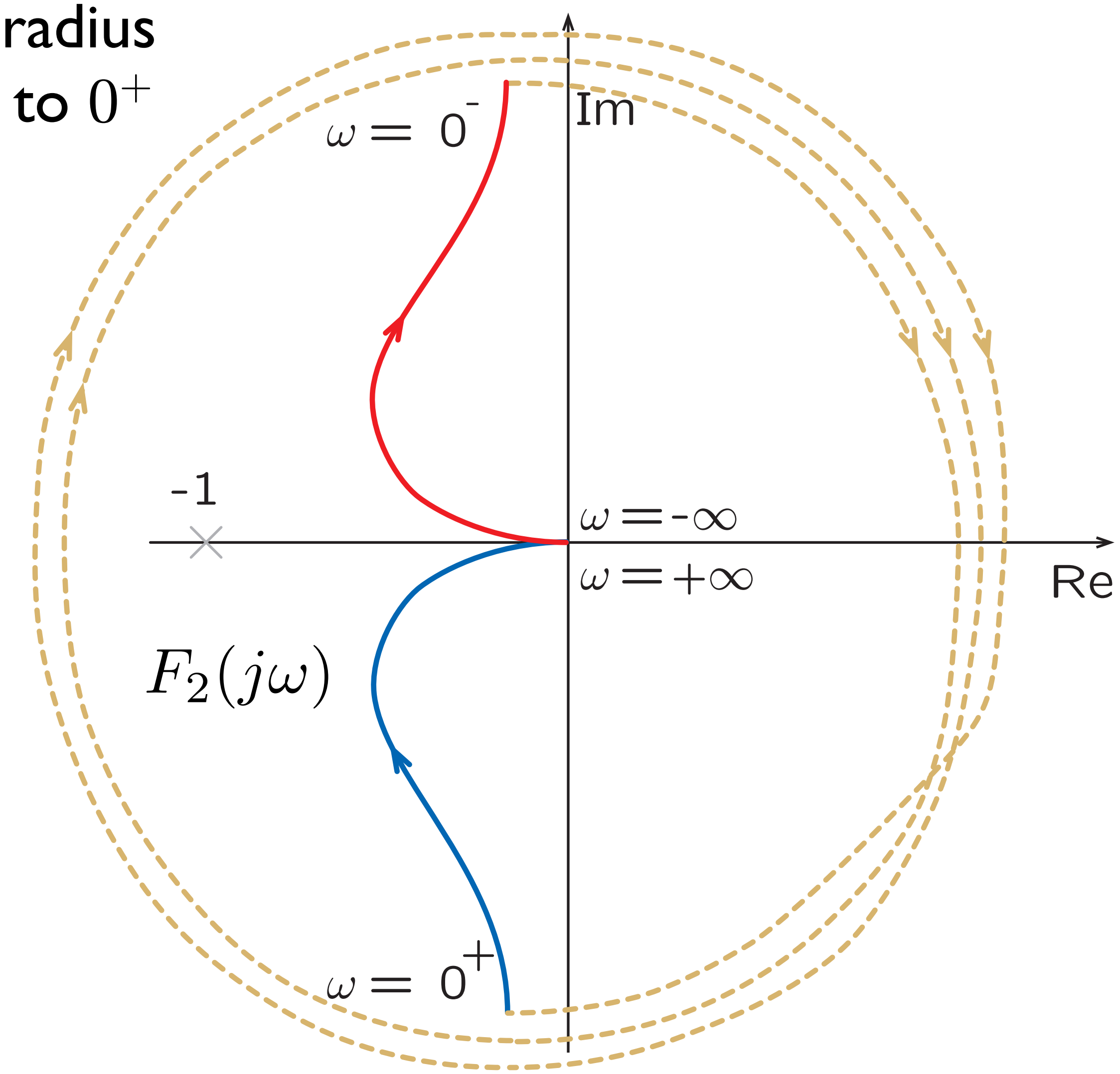
$$F_1(s) = \frac{1}{s(s+1)}$$

$$F_2(s) = \frac{1}{s^5(s+1)}$$

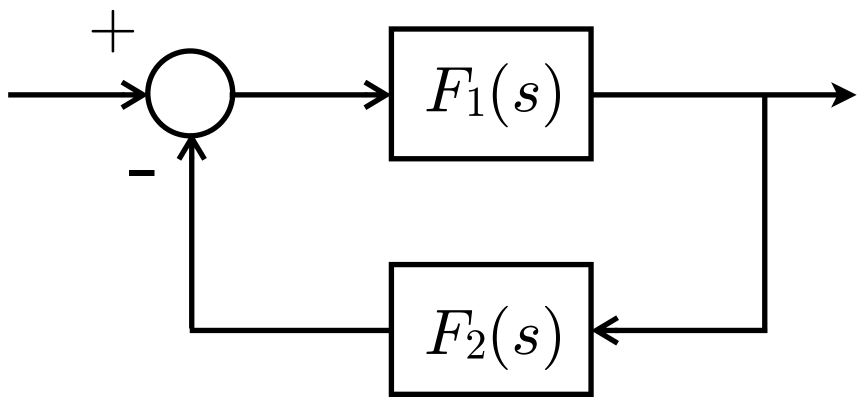


5 times π clockwise with infinite radius from 0^- to 0^+





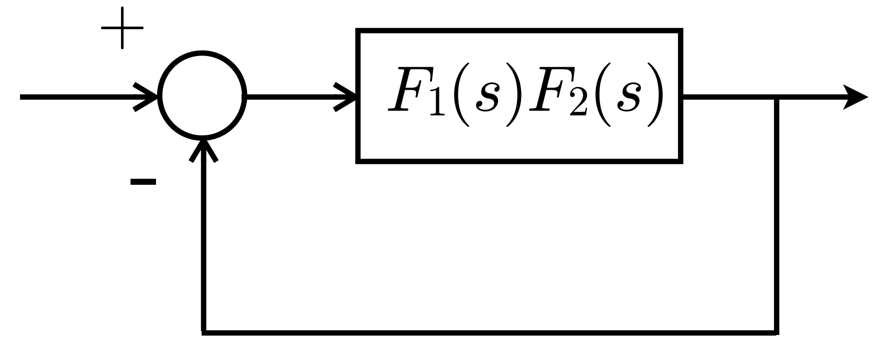
general negative feedback



for **stability** these two schemes are equivalent

$$F_1(s) = N_1(s)/D_1(s)$$

$$F_2(s) = N_2(s)/D_2(s)$$



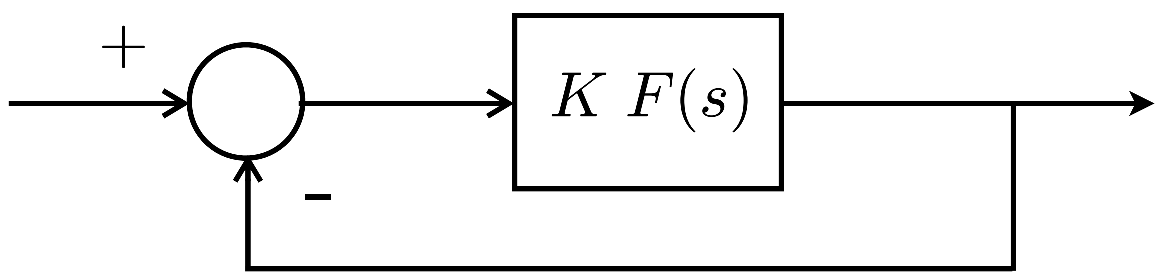
$$\begin{aligned} W_1(s) &= \frac{F_1(s)}{1 + F_1(s)F_2(s)} \\ &= \frac{N_1(s)D_2(s)}{D_2(s)D_1(s) + N_1(s)N_2(s)} \end{aligned}$$

$$\begin{aligned} W_2(s) &= \frac{F_1(s)F_2(s)}{1 + F_1(s)F_2(s)} \\ &= \frac{N_1(s)N_2(s)}{D_2(s)D_1(s) + N_1(s)N_2(s)} \end{aligned}$$

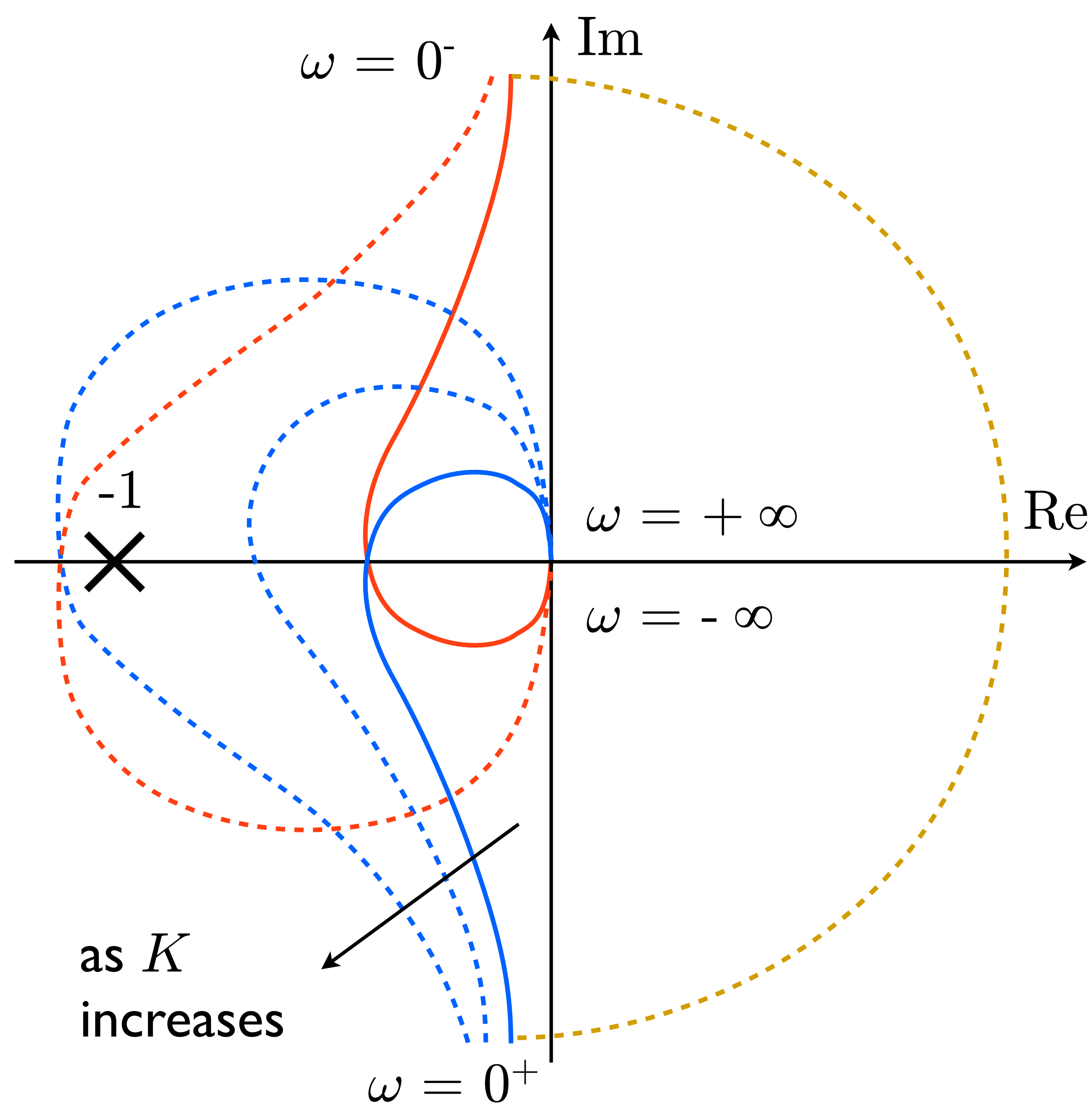
same denominator
same poles
same stability properties

Typical pattern for a control system:

open-loop system with no positive real part poles $n_{F^+} = 0$, therefore the closed-loop system will be asymptotically stable if and only if the Nyquist plot makes **no encirclements** around the point $(-1,\,0)$. We want to explore how the closed-loop stability varies as a gain K in the open-loop system increases.



As K increases over a critical value the closed-loop system goes from asymptotically stable to unstable



In this context, the proximity to the critical point $(-1,\ 0)$ is an indicator of the proximity of the closed-loop system to instability.

We can define two quantities:

gain margin k_{GM}

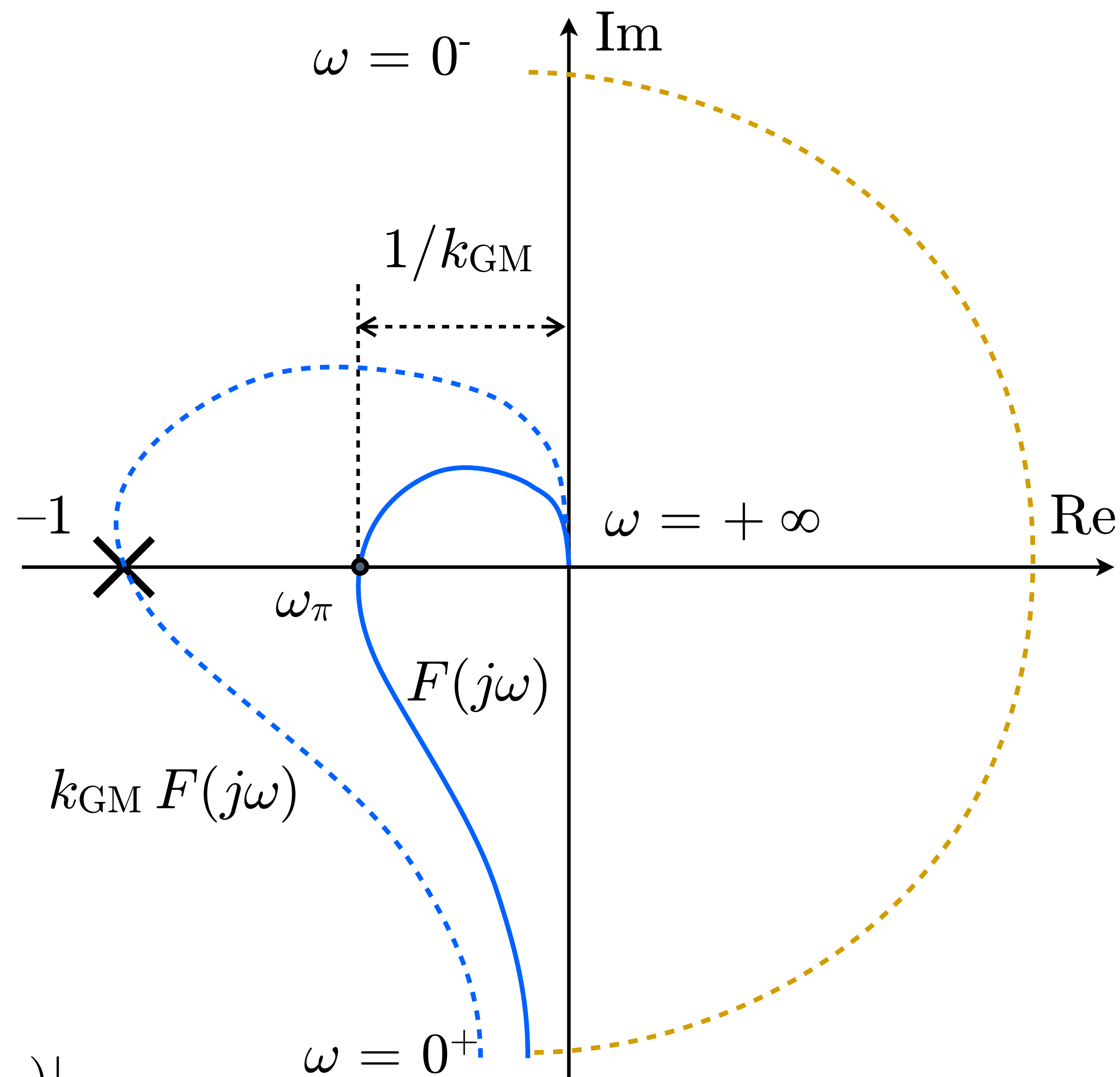
If we multiply $F(j\omega)$ by the quantity k_{GM} the Nyquist diagram will pass through the critical point

the gain margin k_{GM} is the largest gain factor that the closed-loop system can tolerate (strictly) before it becomes unstable

$$\omega_\pi \ : \ \angle F(j\omega_\pi) = -\pi$$

$$k_{GM} = \frac{1}{|F(j\omega_\pi)|}$$

$$k_{GM}|_{dB} = -|F(j\omega_\pi)|_{dB}$$



only positive angular frequencies are shown for ease of exposition

phase margin PM

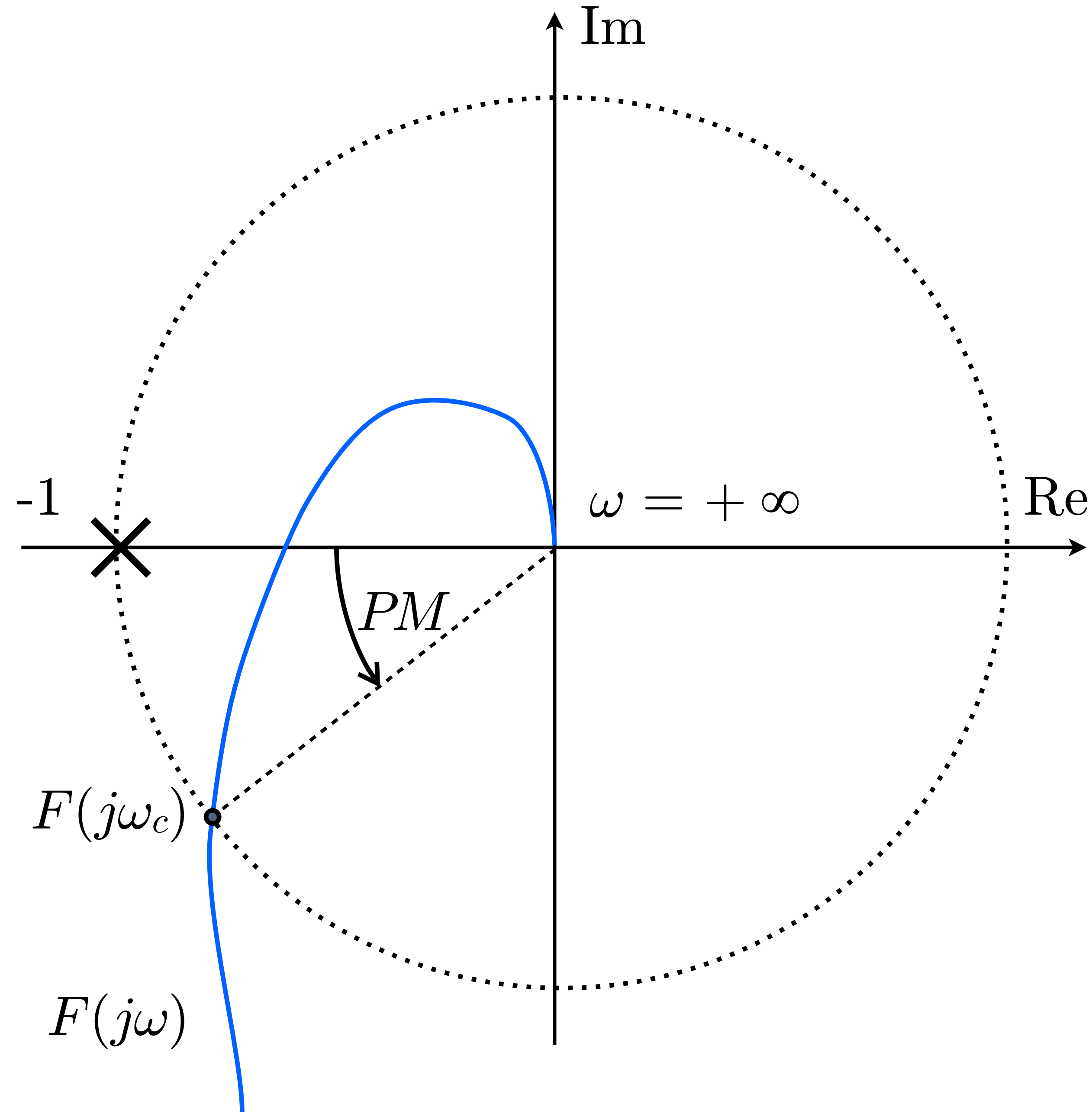
the phase margin PM is the largest amount of lag the closed-loop system can tolerate (strictly) before it becomes unstable

ω_c angular frequency at which the gain is unity is defined as **crossover frequency** (or gain crossover frequency)

$$\omega_c \; : \quad |F(j\omega_c)| = 1$$

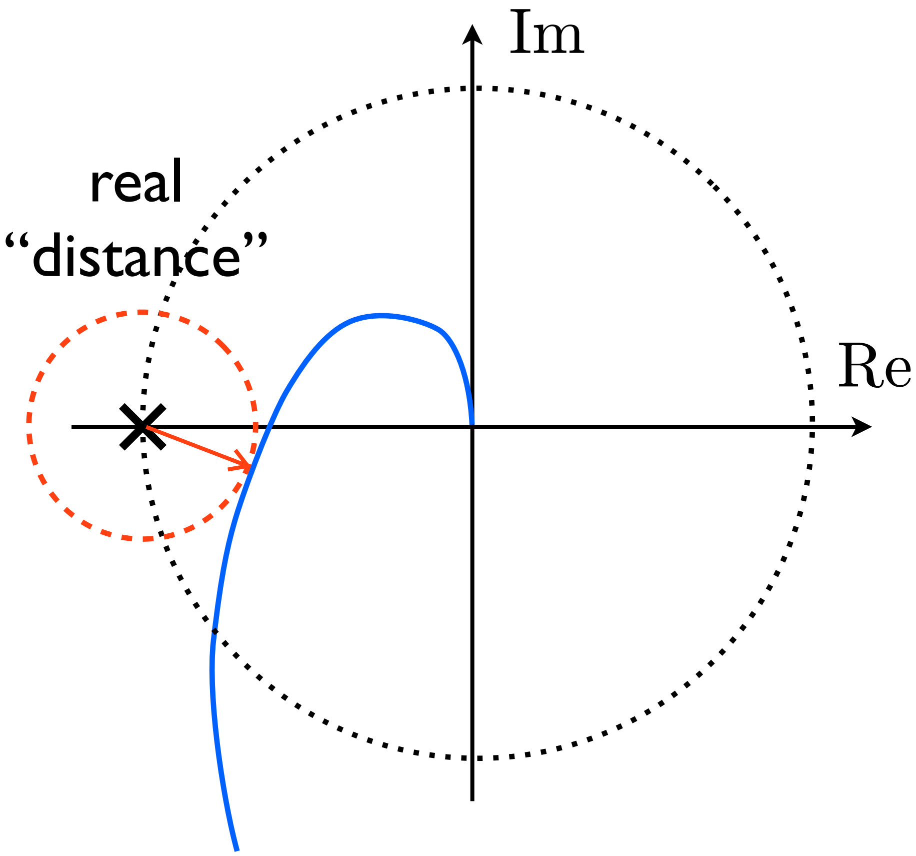
$$\omega_c \; : \quad |F(j\omega_c)|_{dB} = 0\,dB$$

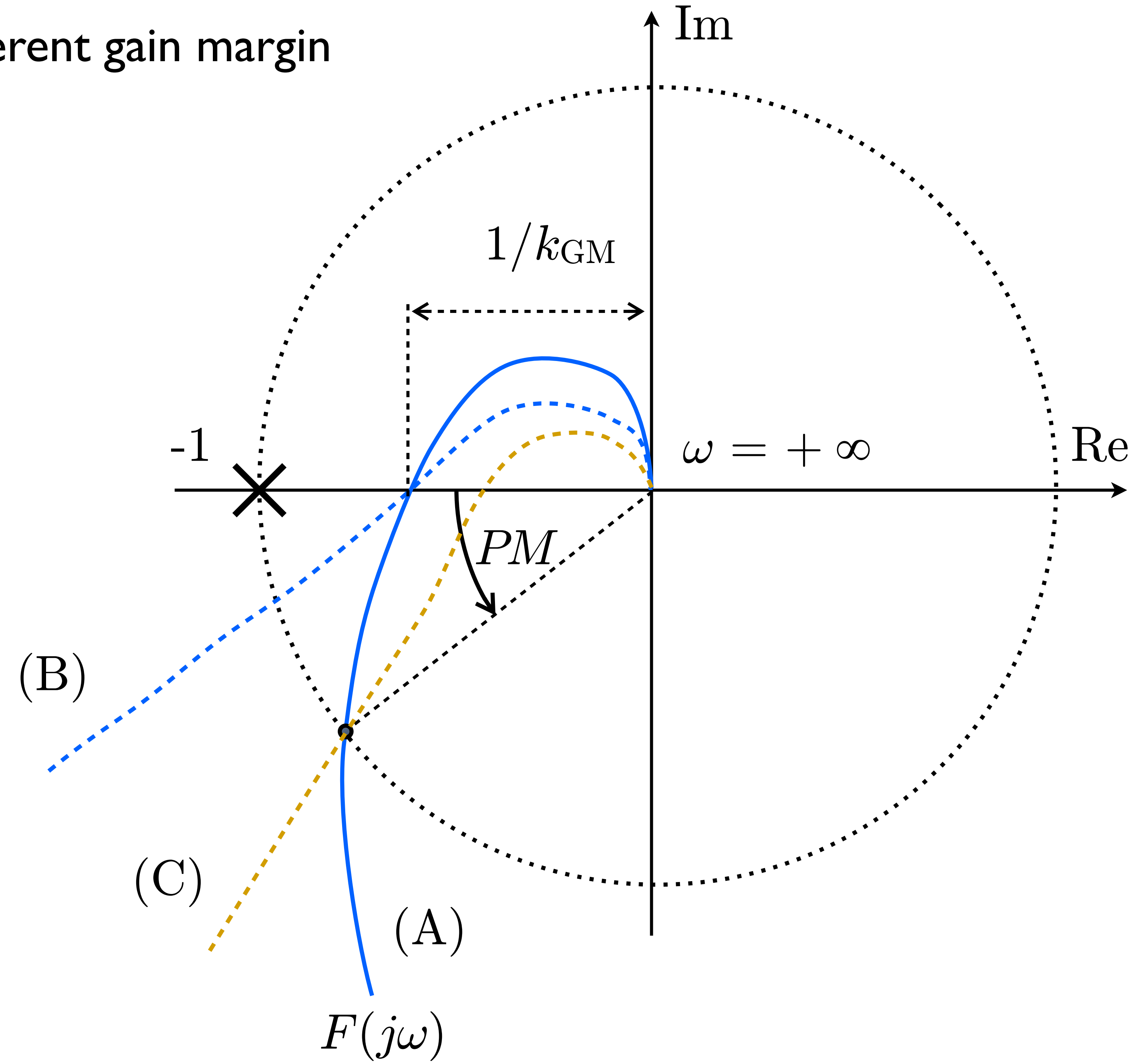
$$PM = \pi + \angle F(j\omega_c)$$



(B) same gain margin as (A) but different phase margin

(C) same phase margin as (A) but different gain margin

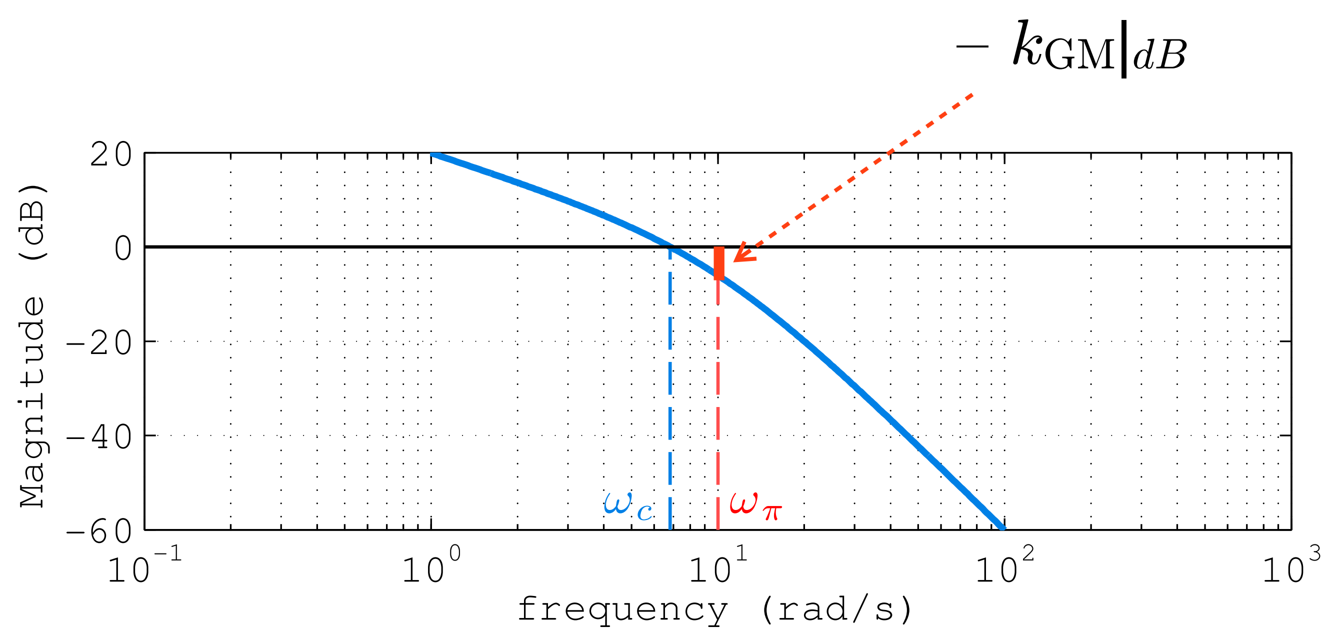




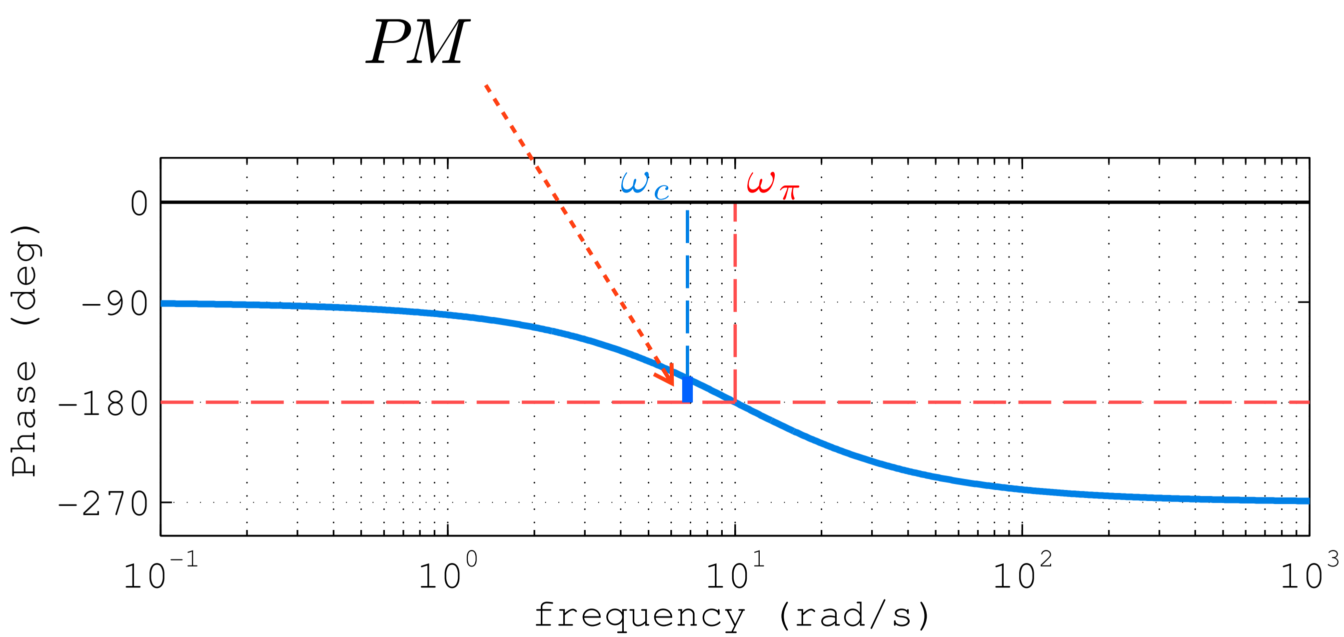
stability margins on Bode

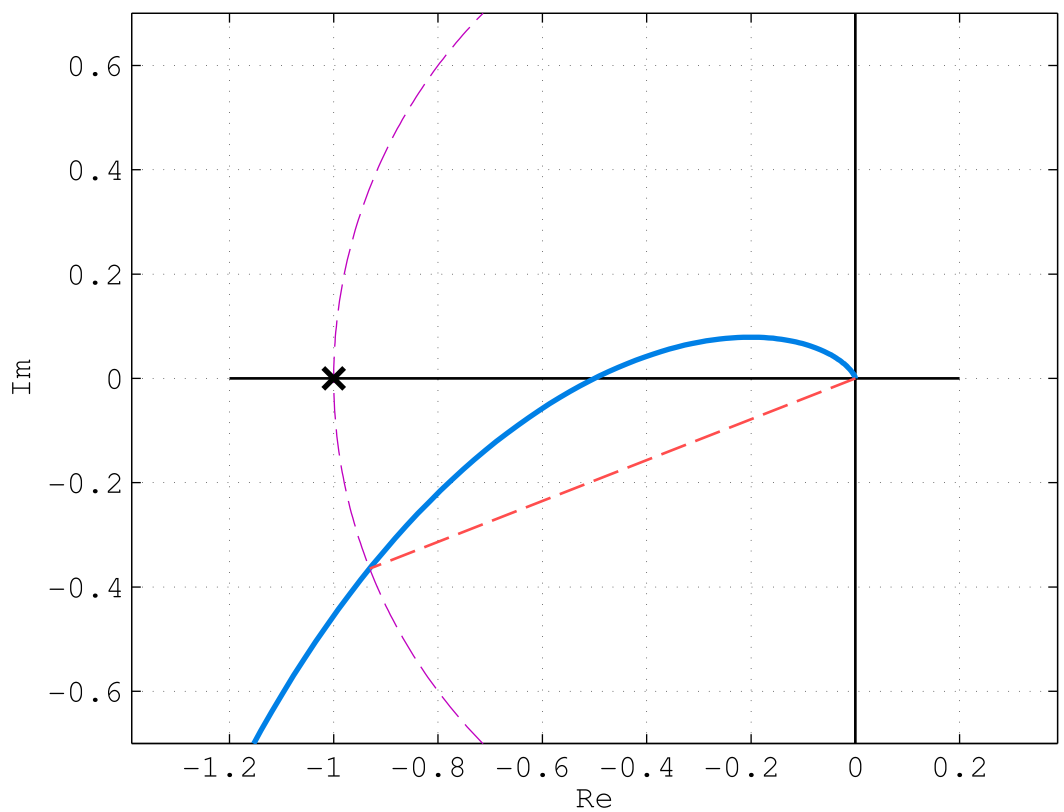
$$k_{\mathrm{GM}}|_{dB} = -|F(j\omega_\pi)|_{dB}$$

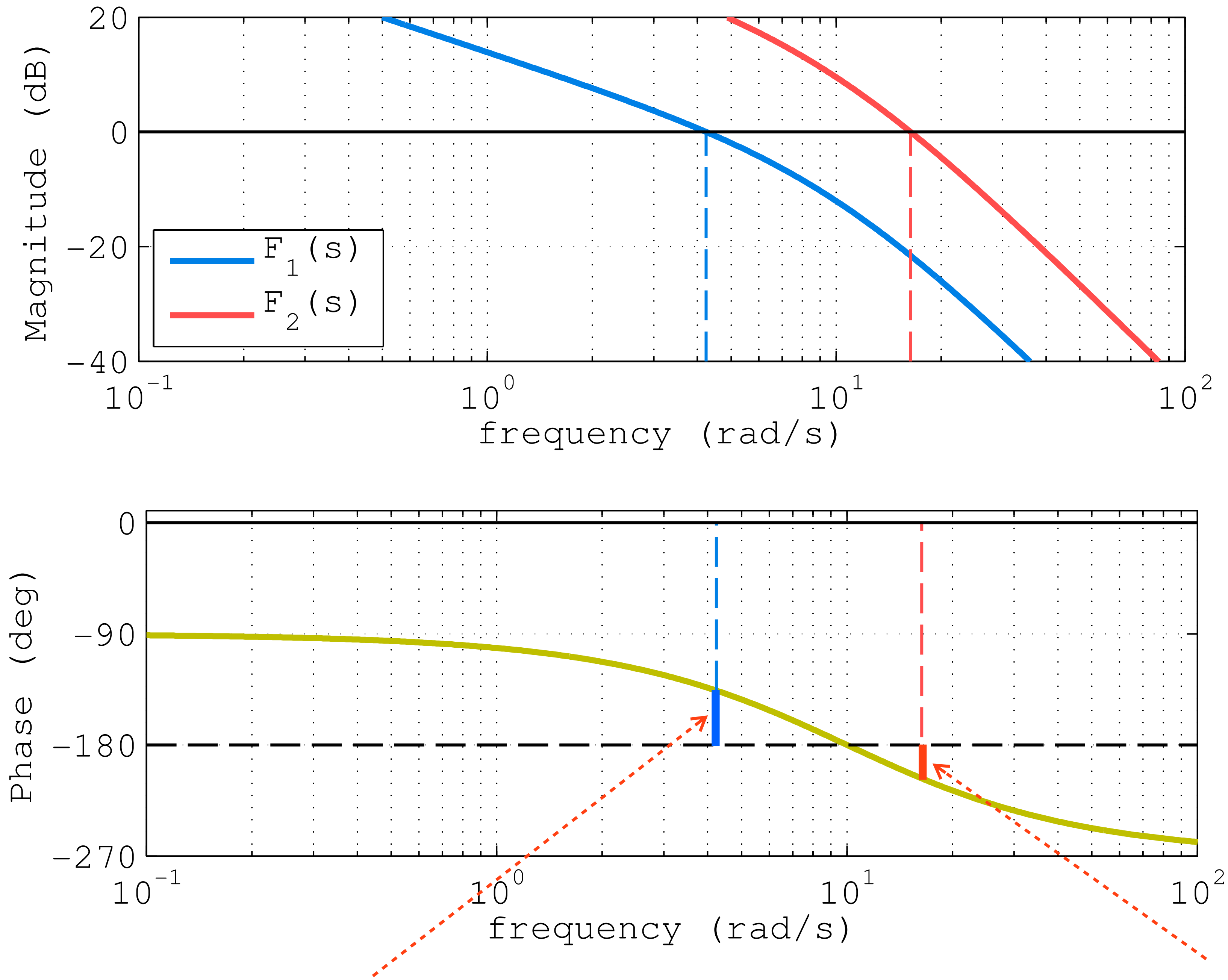
$$PM = \pi + \angle F(j\omega_c)$$



$$F(s) = \frac{1000}{s(s+10)^2}$$







$$F_1(s) = \frac{500}{s(s+10)^2}$$

$$F_1(s) = \frac{6000}{s(s+10)^2}$$

these are scaled (by 0.5 and 6) wrt the previous system

both have same phase but different crossover frequencies and therefore different phase margins

positive phase margin (and thus for this example $N_{cc} = 0$)

negative phase margin (and thus for this example a non-zero N_{cc})

asymptotically stable closed-loop system

both systems with $n_F^+ = 0$

unstable closed-loop system

(suggested exercise: check with Routh)

Bode stability theorem

Let the open-loop system $F(s)$ be with **no positive real part poles** (i.e., $n_{F^+} = 0$) and such that there exists a **unique crossover frequency** ω_c (i.e., such that $|F(j\omega_c)| = 1$) then the **closed-loop** system is **asymptotically stable**

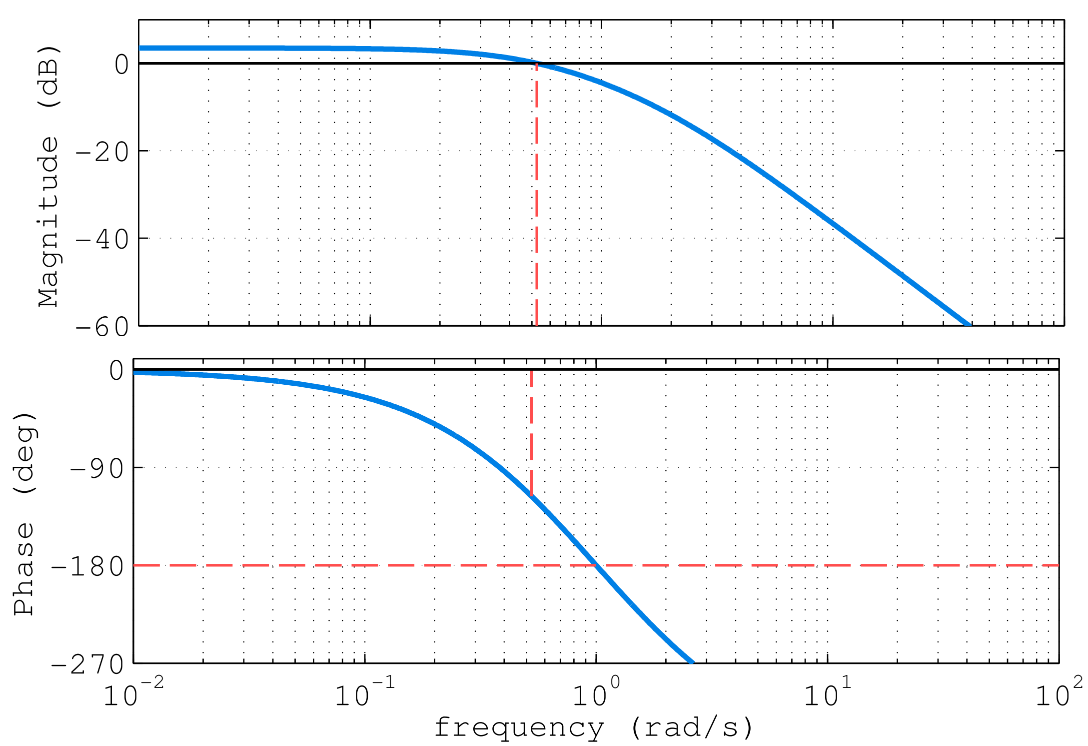
if and only if

the open-loop system's **generalized** gain is **positive**

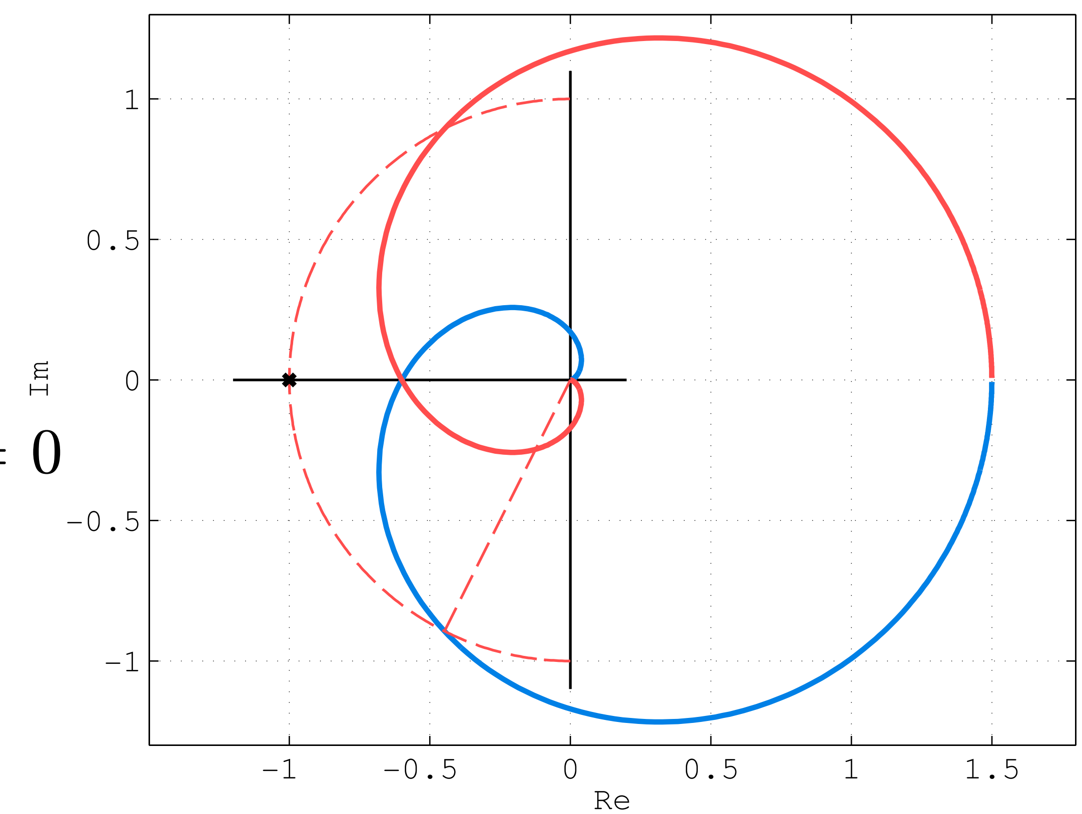
&

the **phase margin** (PM) is **positive**

$$F(s) = \frac{1.5(1-s)}{(s+1)(2s+1)(0.5s+1)}$$



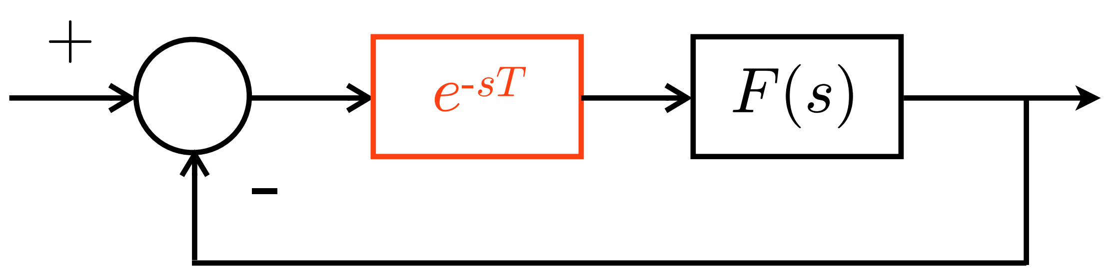
$$N_{cc} = n_{F^+} = 0$$



Bode stability theorem

- stability margins are useful to evaluate stability **robustness** wrt parameters variations (for example the gain margin directly states how much gain variation we can tolerate)

- phase margin is also useful to evaluate stability **robustness** wrt delays in the feedback loop. Recall that, from the time shifting property of the Laplace transform, a delay is modeled by e^{-sT} and that



$$\angle e^{-j\omega T} = -\omega T$$
$$|e^{-j\omega T}| = 1$$

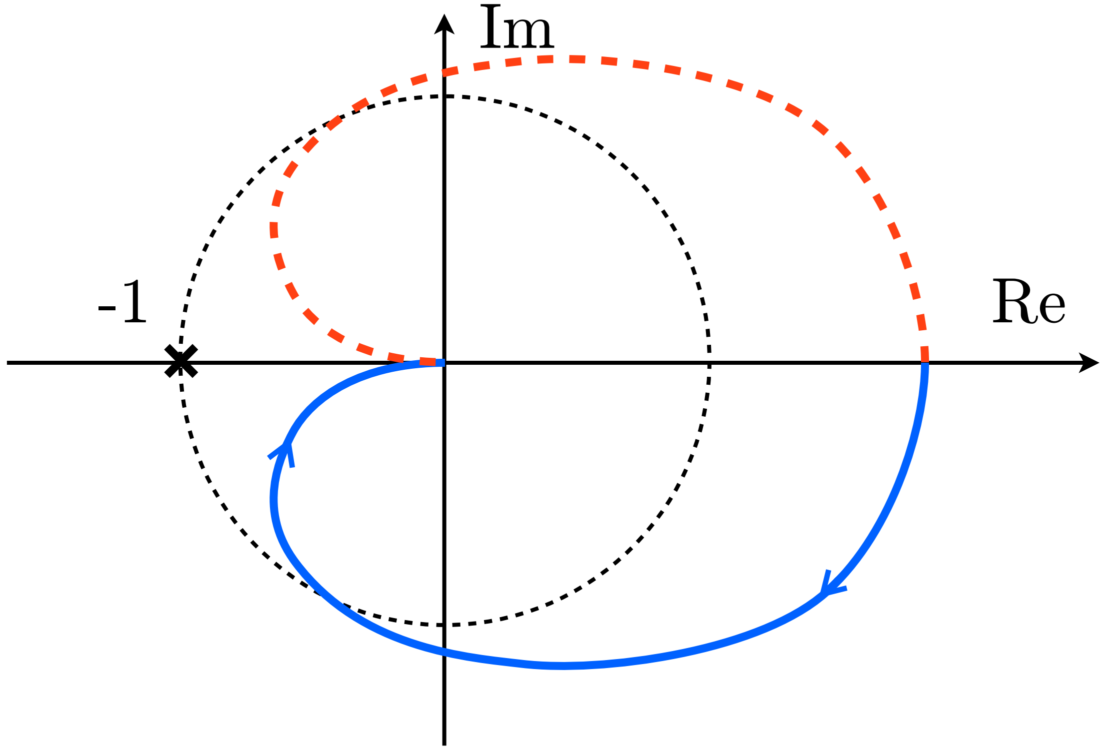
delay
of T sec

$\angle e^{-j\omega T} = -\omega T$ ⟶ a delay introduces a phase lag and therefore it can easily "destabilize" a system (note that the abscissa in the Bode diagrams is in $\log_{10}$ scale so the phase decreases very fast)

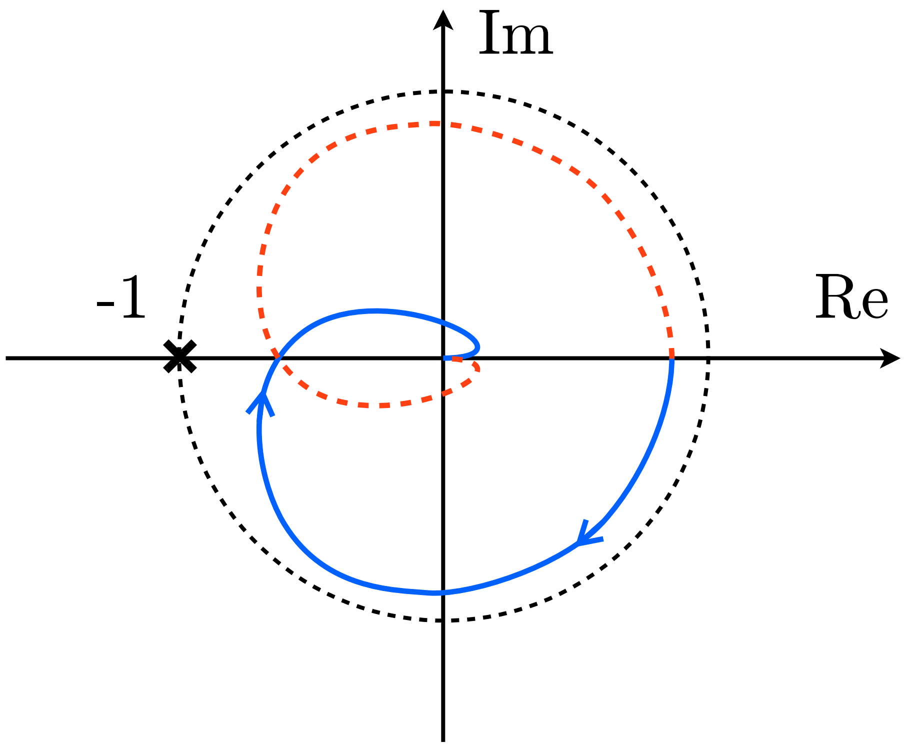
$|e^{-j\omega T}| = 1$ ⟶ a delay in the loop does not alter the magnitude (0 dB contribution)

Special cases

- infinite gain margin



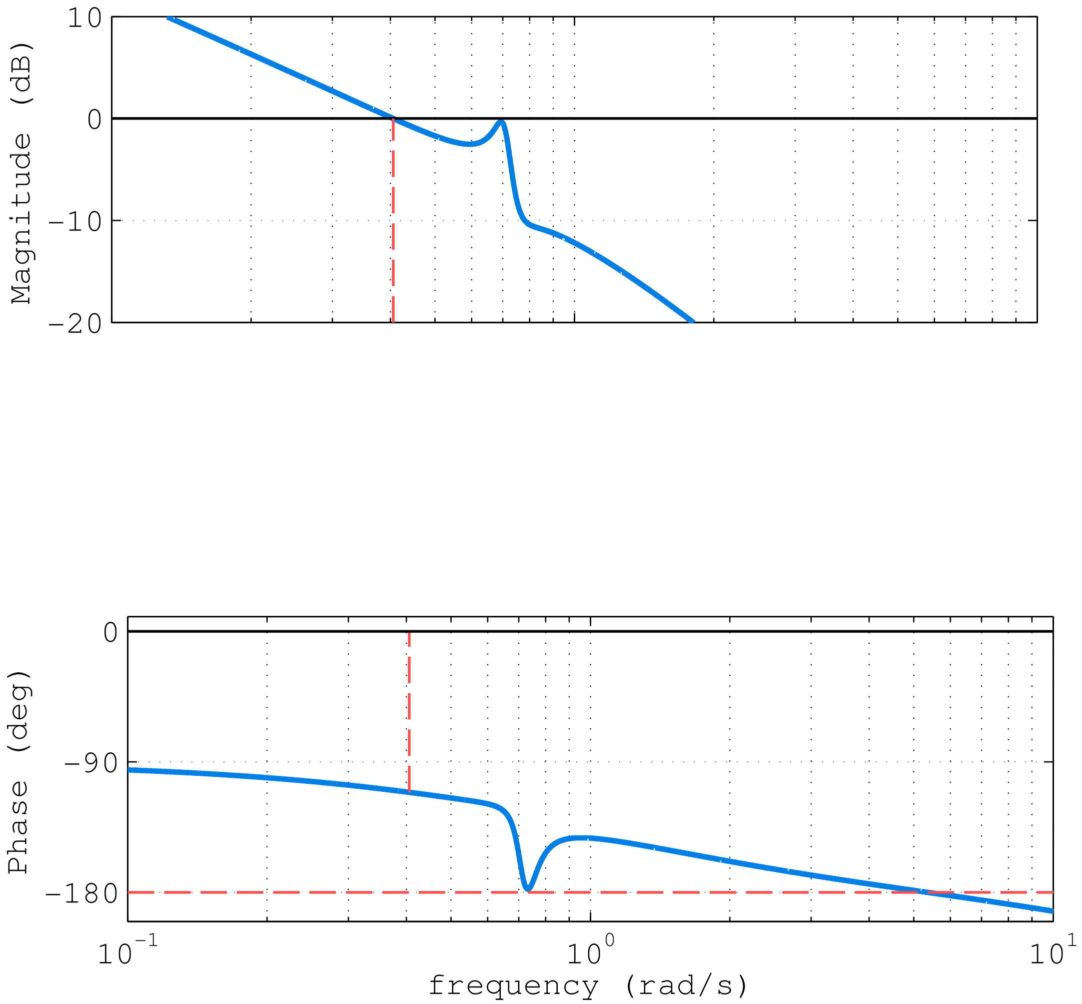
- infinite phase margin

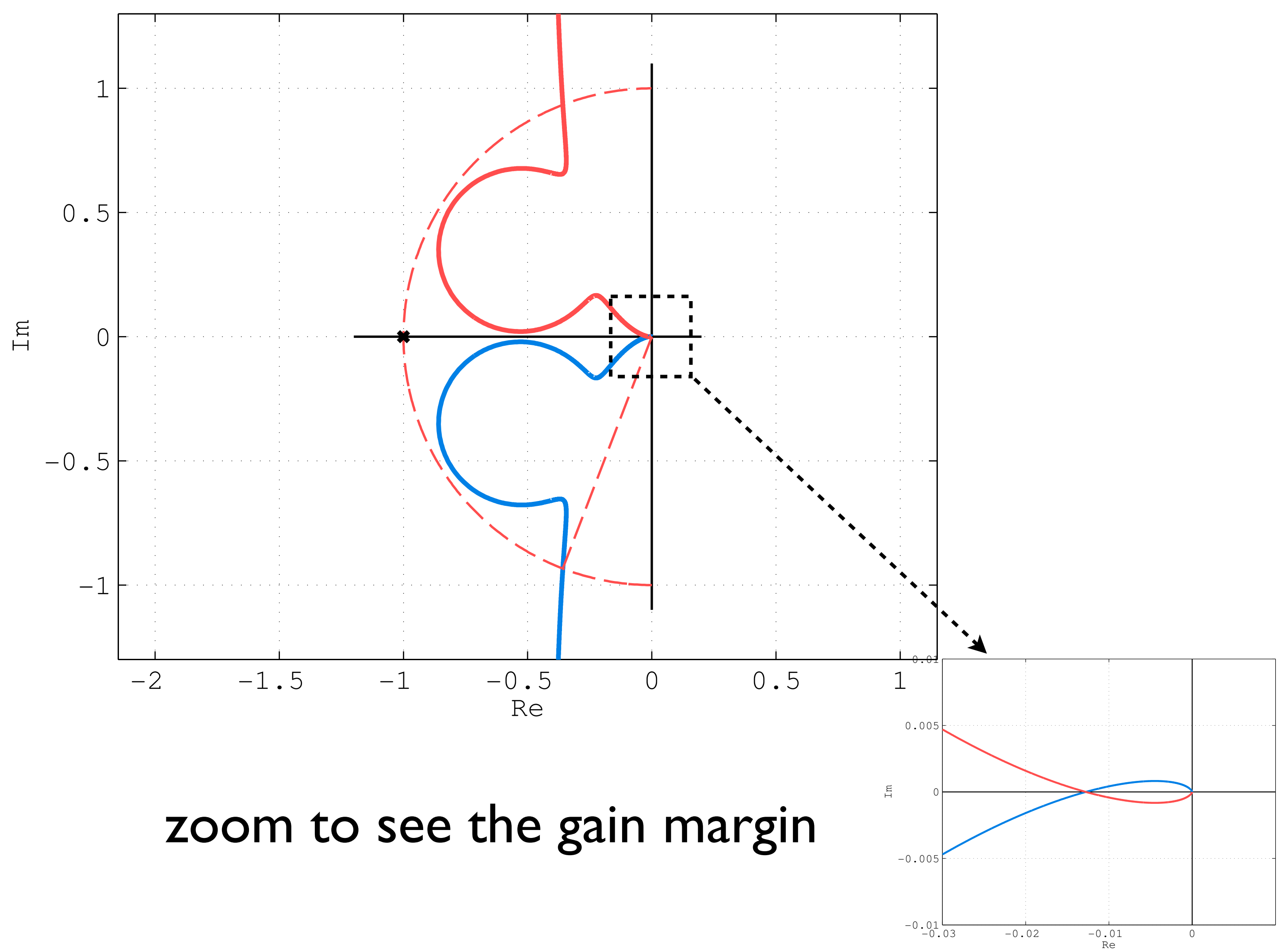


Particular example

good gain and phase margins but close to critical point

$$F(s) = \frac{0.38(s^2 + 0.1s + 0.55)}{s(s+1)(s/30+1)(s^2+0.06s+0.5)}$$





zoom to see the gain margin