Control Systems

System as a filter

L. Lanari

Dipartimento di Ingegneria Informatica
Automatica e Gestionale Antonio Ruberti

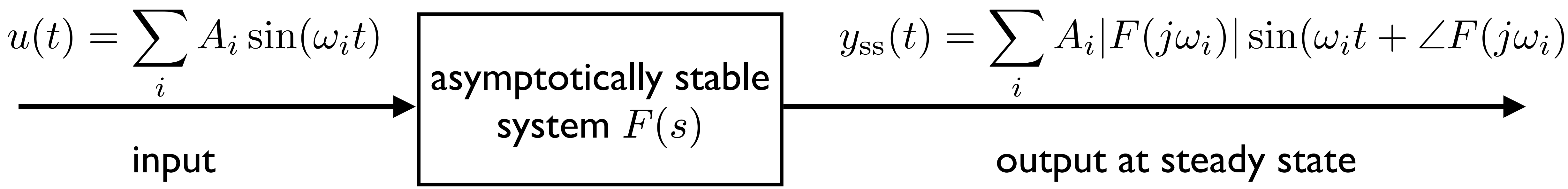
Sapienza
Università di Roma

Outline

- preliminaries
- steady state example for a first order system
- importance of the phase
- steady state example for a second order system
- transient: bandwidth
- transient: resonance peak
- transient: frequency vs time domain characterization
- Mass-Spring-Damper system
- other examples
- quarter-car system

preliminaries

- system linearity guarantees that



that is the steady state output of an asymptotically stable system having as input a linear combination of sinusoids coincides with the same linear combination of the steady state responses of the system to each individual sinusoid

- moreover recall that a periodic signal can be expanded in a Fourier series which is an infinite sum of weighted sines and cosines

$\Longrightarrow$ we can compute the steady state response to more complex signals

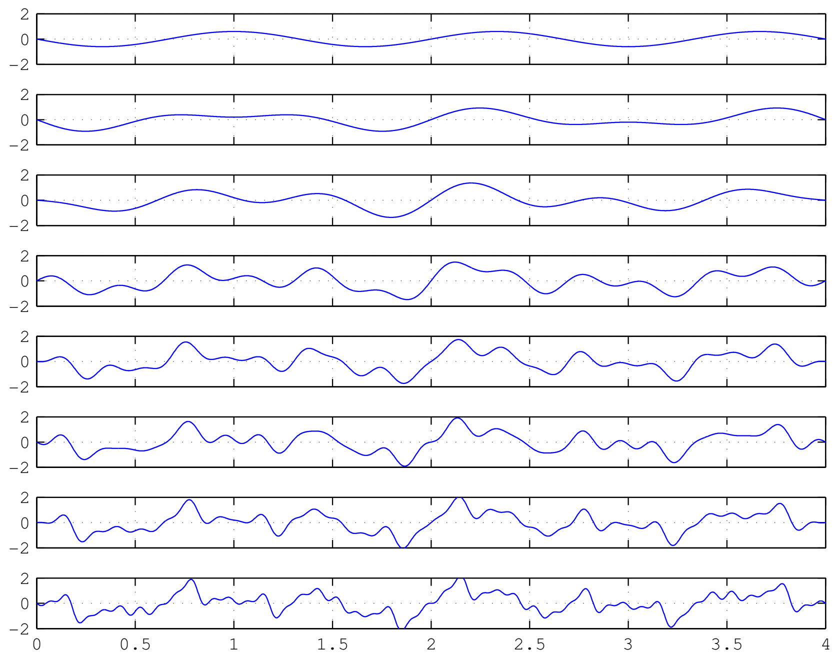
In the following the system will be considered as a signal transformer: from the input to the output (ex. a time profile for a force as input and the position evolution as output in the Mass-Spring_Damper system)

example: a periodic input signal

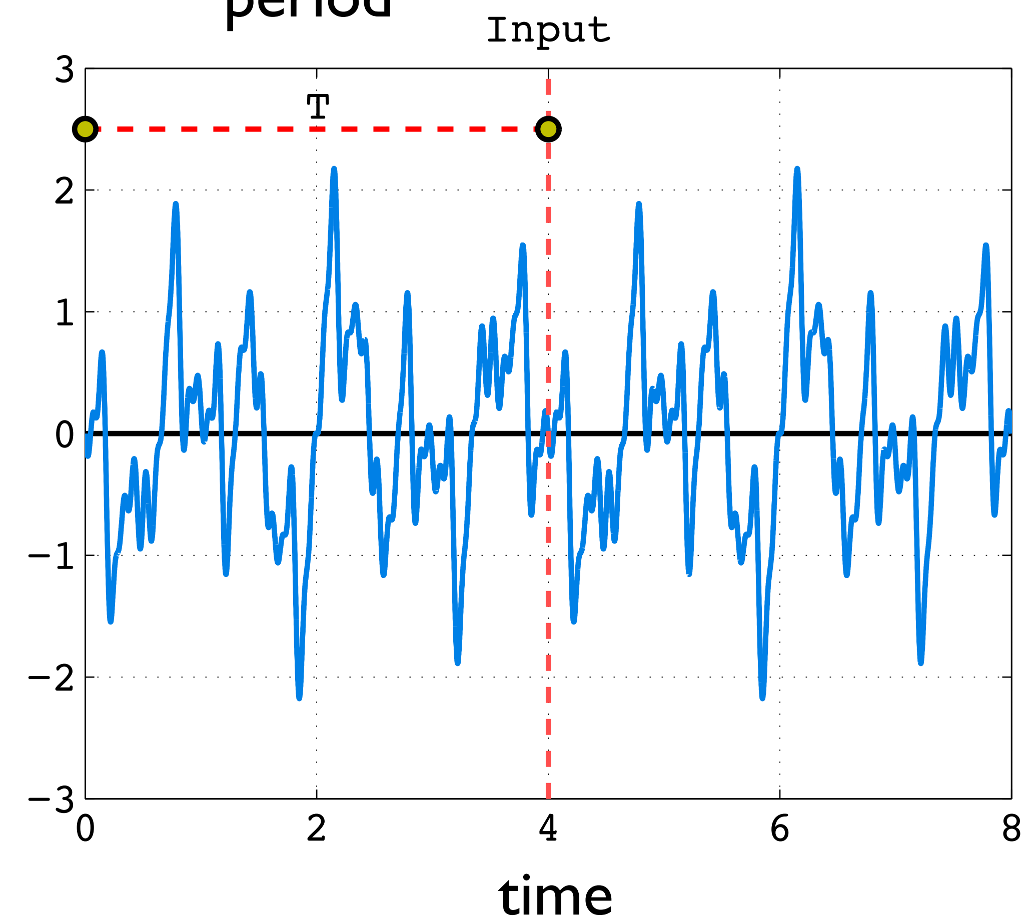
$$u(t) = -0.6\sin(f_1 t) - 0.4\sin(f_2 t) + 0.5\sin(f_3 t) + 0.5\sin(f_4 t) - 0.3\sin(f_5 t) - 0.2\sin(f_6 t) + 0.2\sin(f_7 t) - 0.2\sin(f_8 t)$$

$$f_1 = 2\pi 0.75,\ f_2 = 2\pi 1.25,\ f_3 = 2\pi 1.5,\ f_4 = 2\pi 3,\ f_5 = 2\pi 5,\ f_6 = 2\pi 6,\ f_7 = 2\pi 8,\ f_8 = 2\pi 11$$

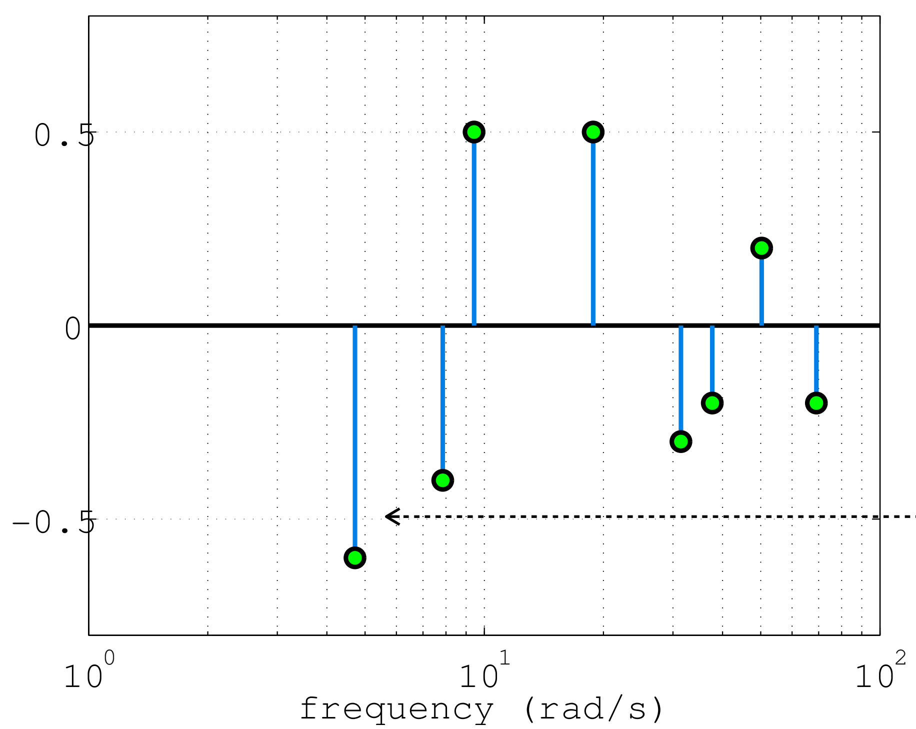
time



adding one contribution at a time



Frequency spectrum of the input



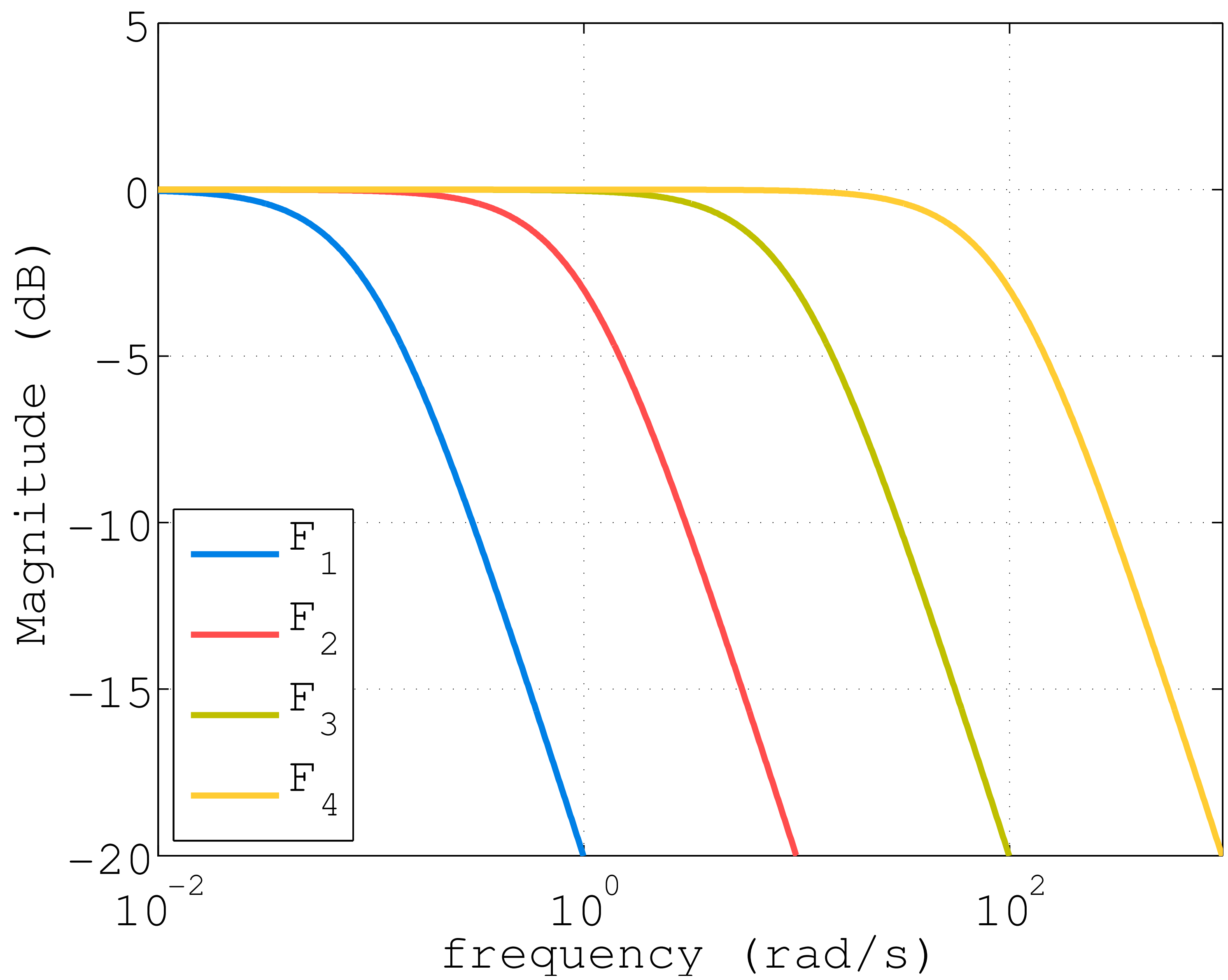
same information in the frequency domain

magnitude of the sine function at that frequency

behavior at steady state: example 1

4 different systems
(all first order and with unit gain)

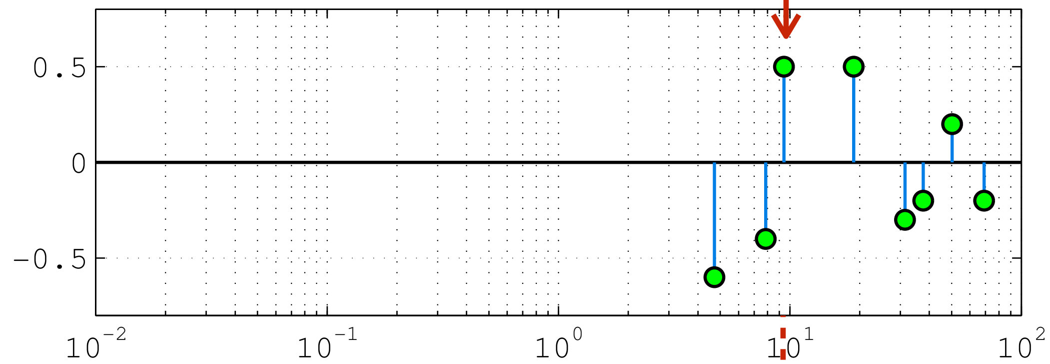
$$F_1(s) = \frac{1}{1+10s}, \quad F_2(s) = \frac{1}{1+s}, \quad F_3(s) = \frac{1}{1+0.1s}, \quad F_4(s) = \frac{1}{1+0.01s}$$



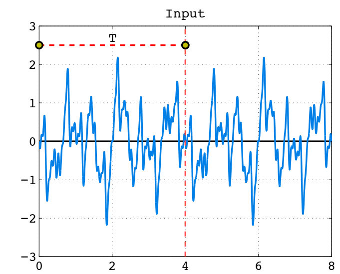
frequency

$0.5 \sin(1.5\ \pi\ t)$

time



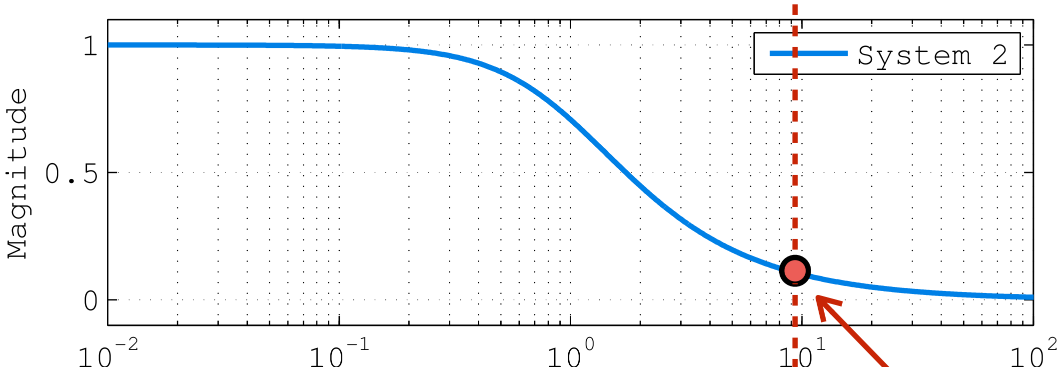
input



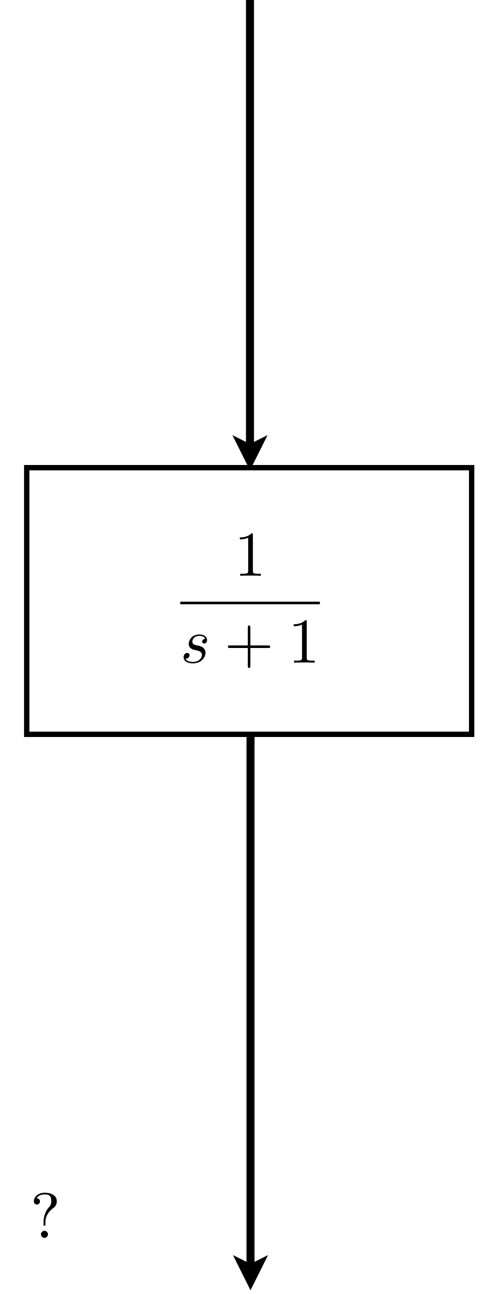
each input component with frequency ω_i is amplified/attenuated by $|P(j\omega_i)|$

we need to multiply

magnitude not in dB

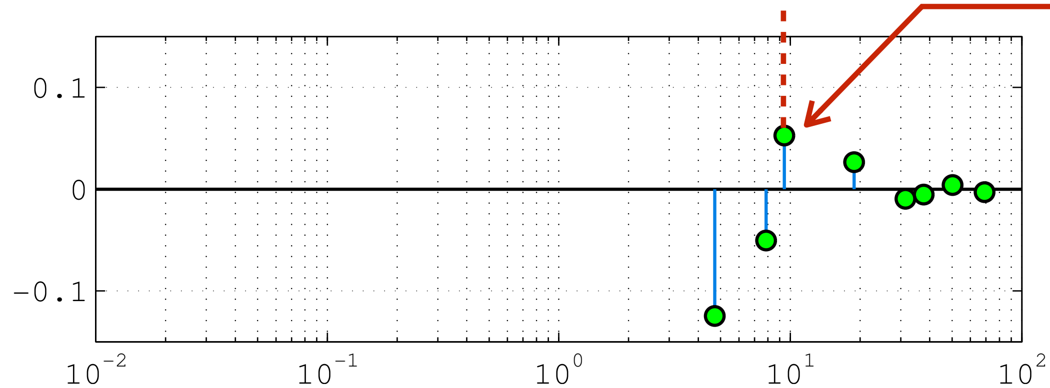


system



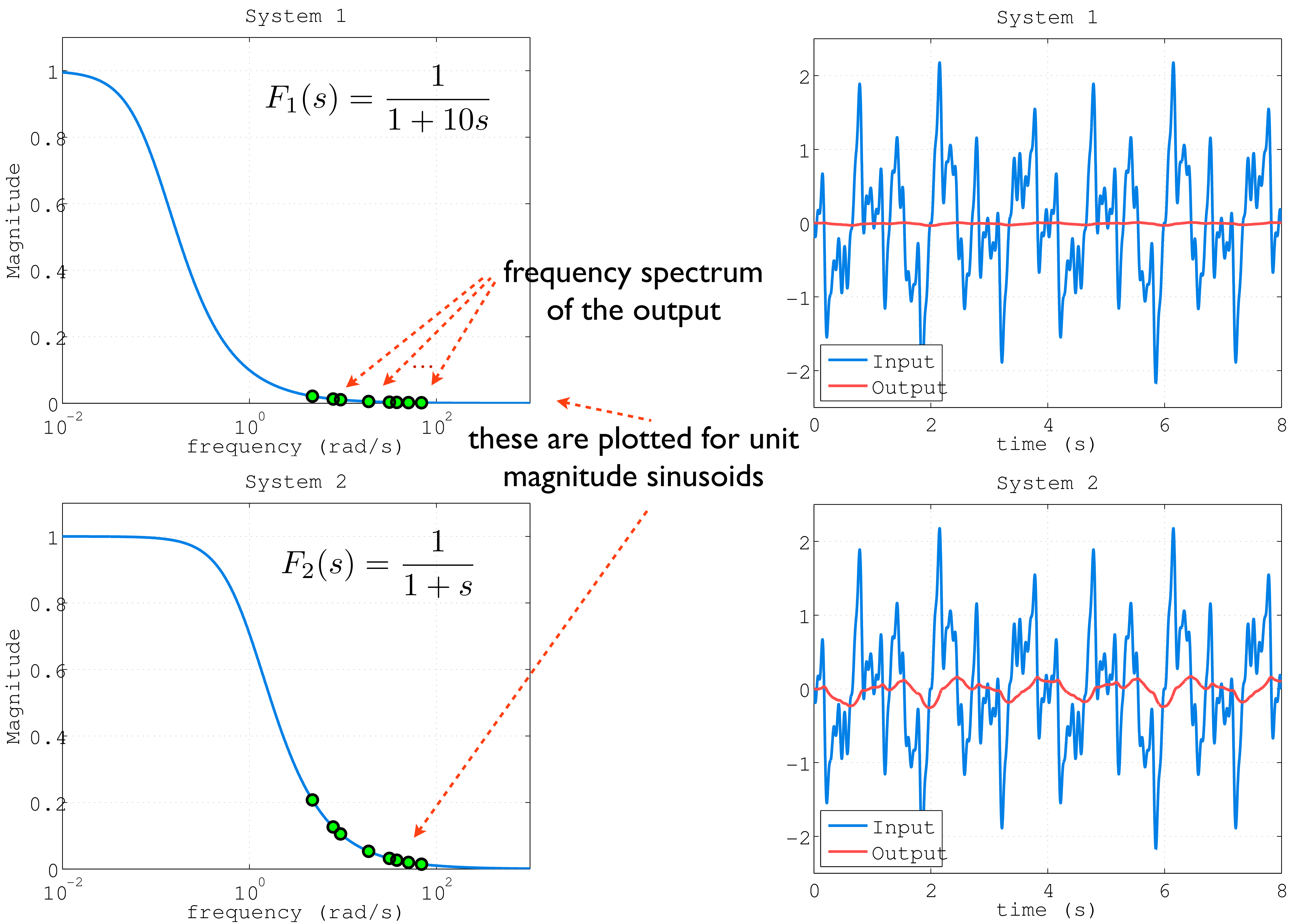
$|\ P\ (j\ 1.5\ \pi)\ | = 0.1055$

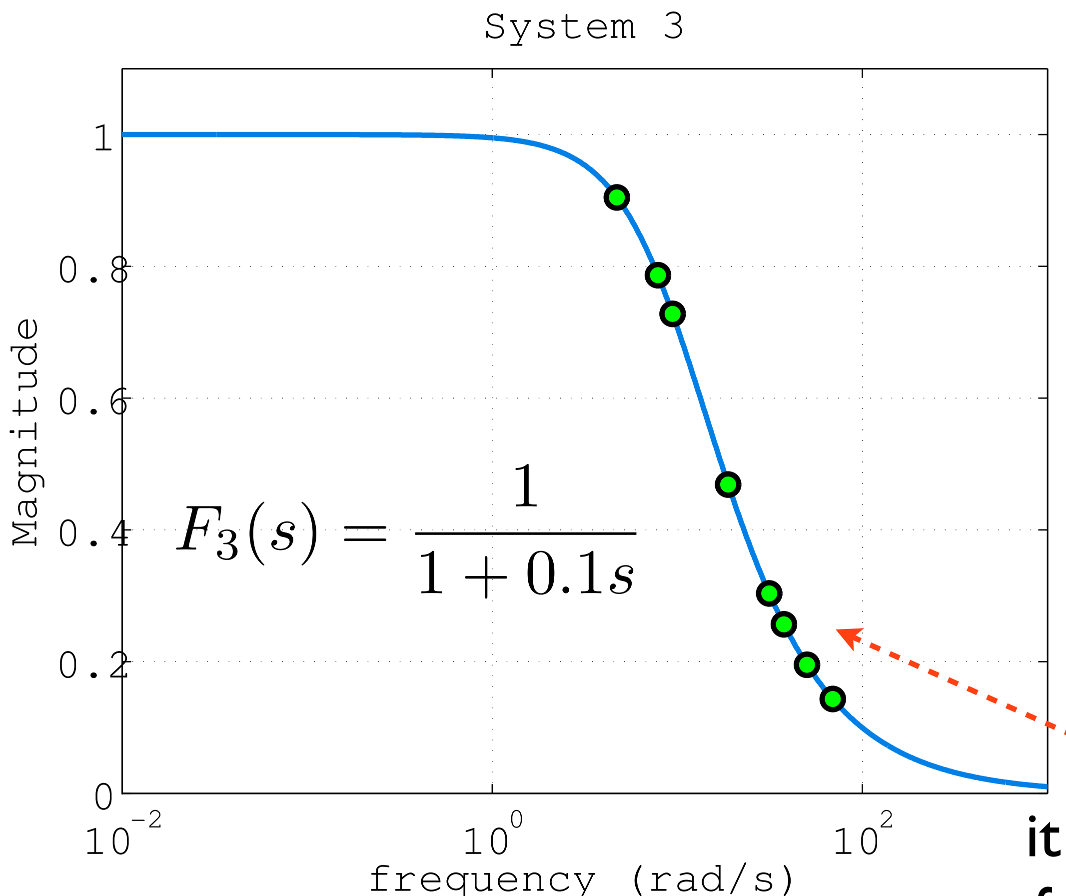
$0.5 \times 0.1055 \sin(1.5\ \pi\ t)$

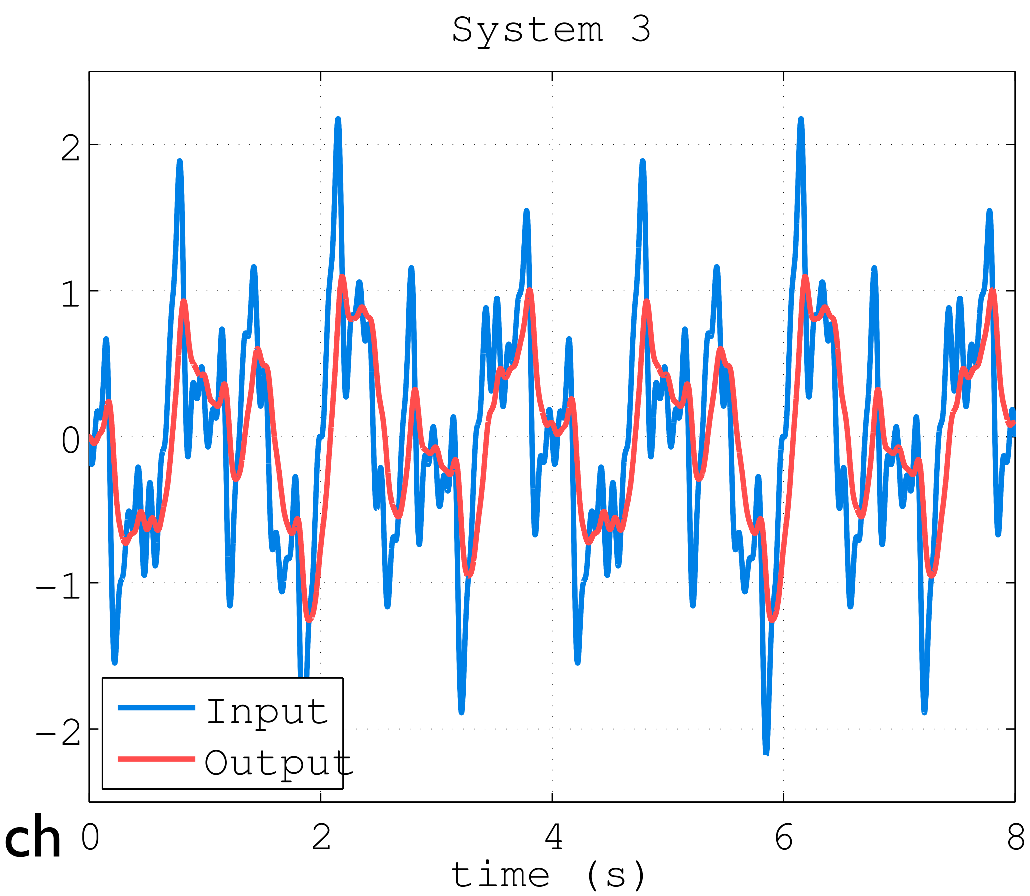


output

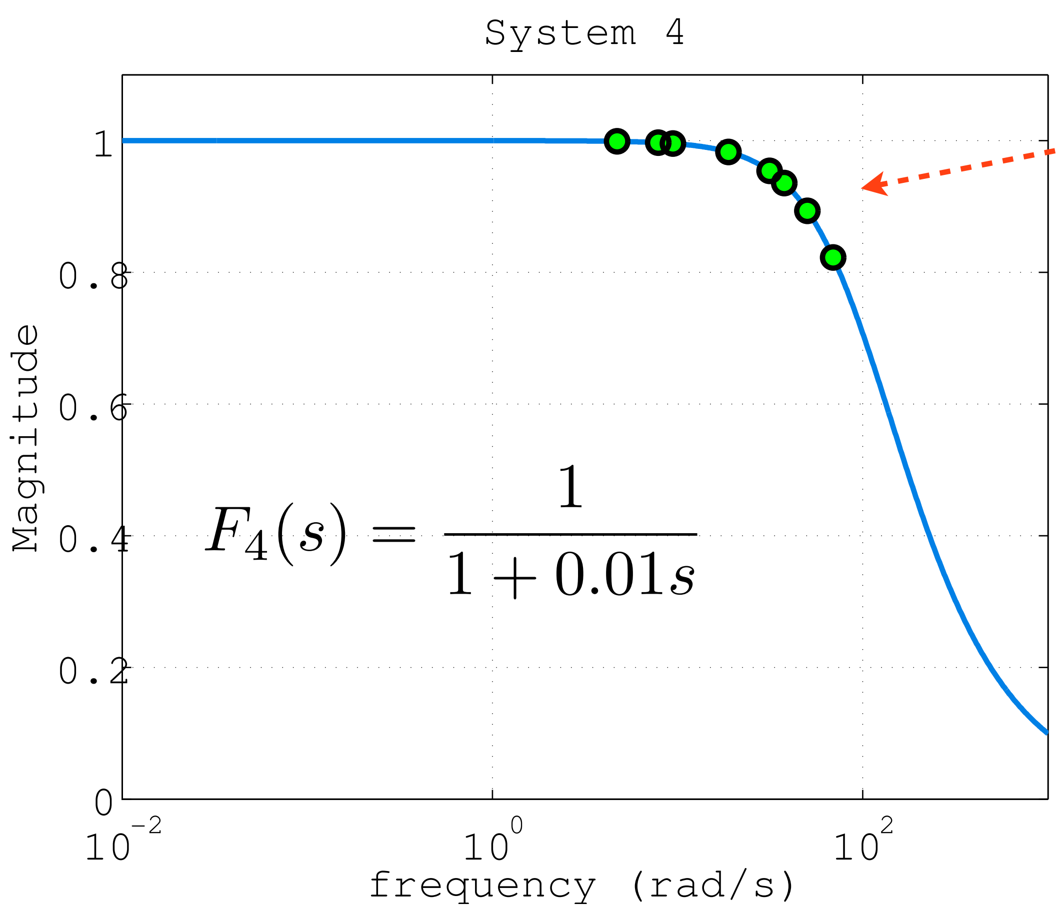
?

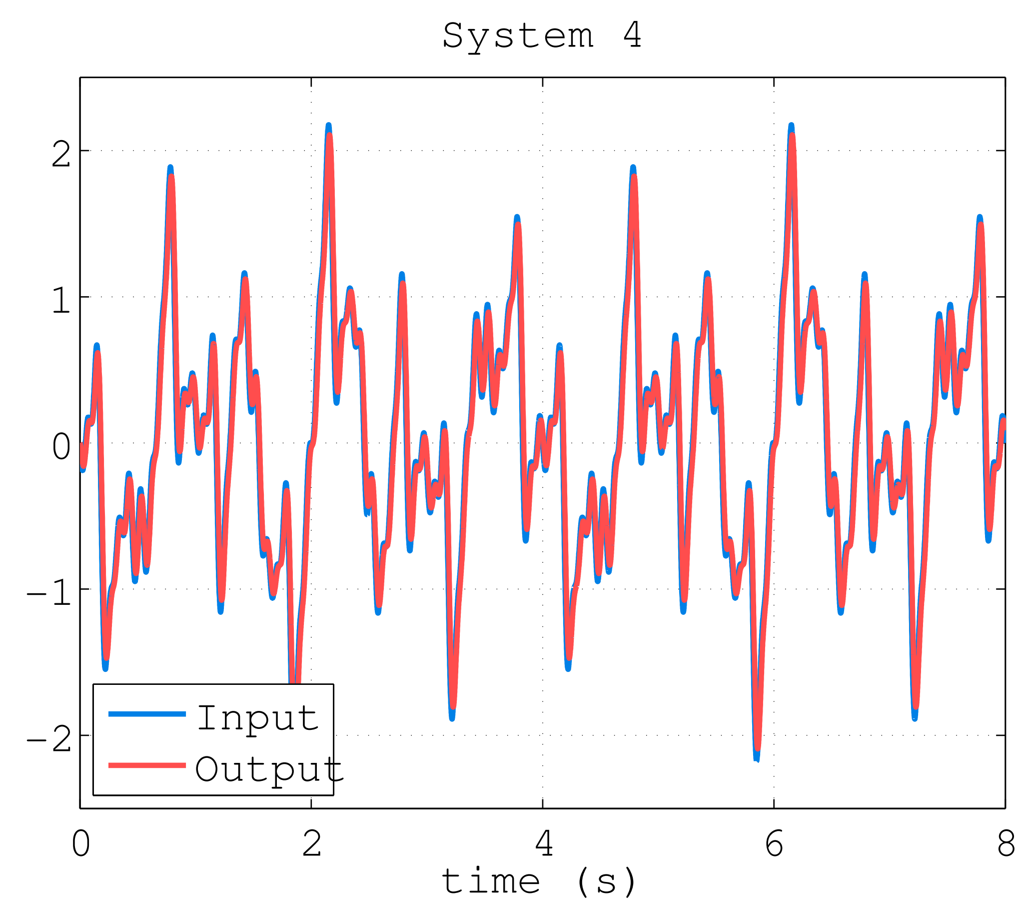

System 1
$F_1(s) = \frac{1}{1+10s}$
Magnitude
frequency (rad/s)
frequency spectrum of the output
these are plotted for unit magnitude sinusoids
System 1
Input
Output
time (s)
System 2
$F_2(s) = \frac{1}{1+s}$
Magnitude
frequency (rad/s)
System 2
Input
Output
time (s)





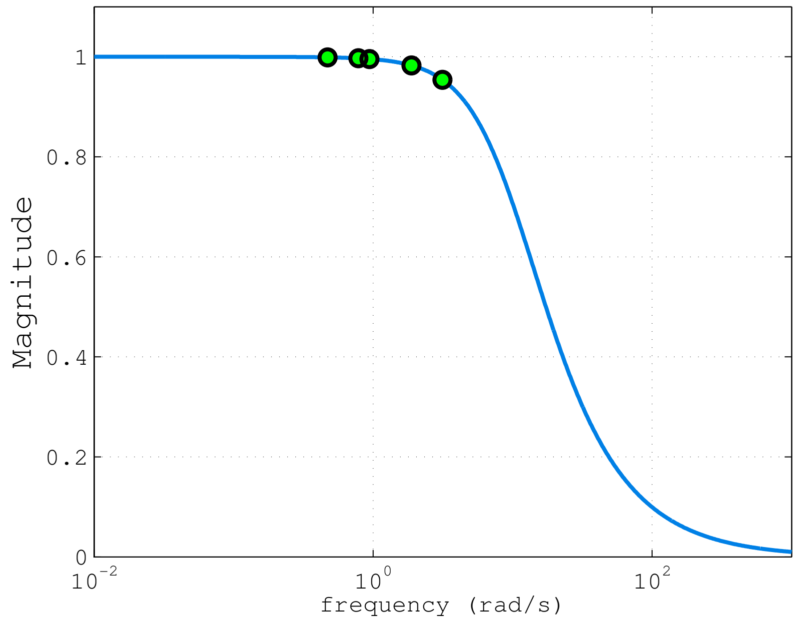
it just shows at which frequencies the input has contributions





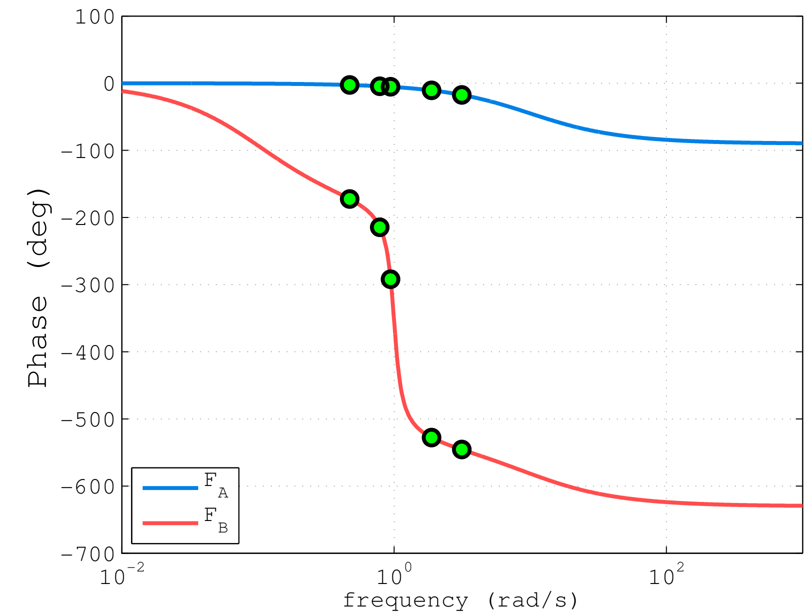
magnitude vs phase

2 systems with same magnitude but different phase

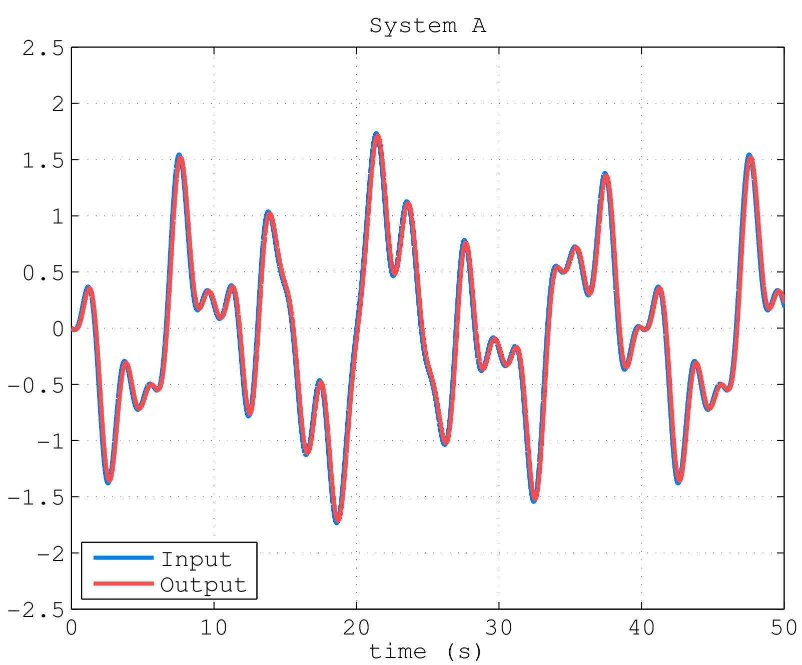


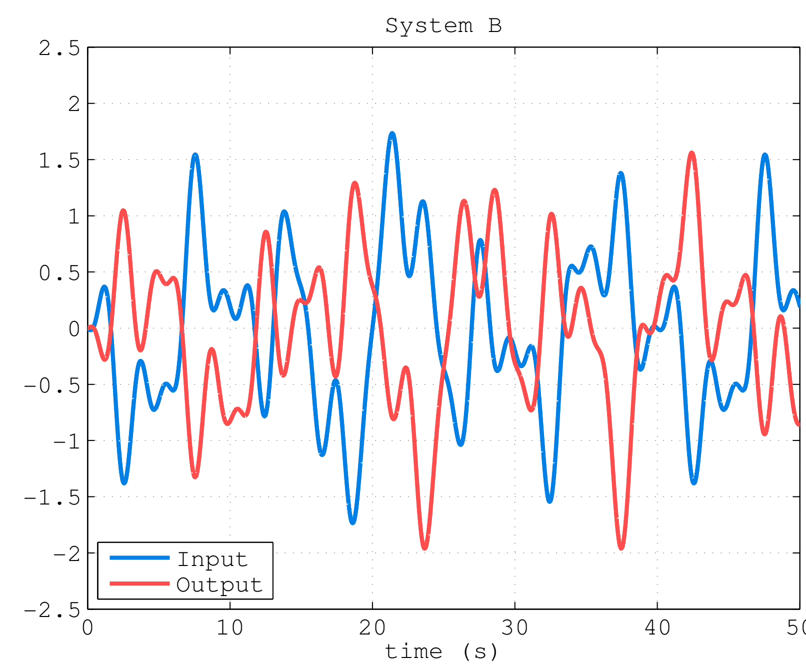
$$F_A(s) = \frac{1}{1 + s/10}$$

$$F_B(s) = \frac{(s^2 - 0.2s + 1)(1 - 10s)}{(s^2 + 0.2s + 1)(1 + 10s)(1 + s/10)}$$



differences in the system phase can lead to noticeable output difference



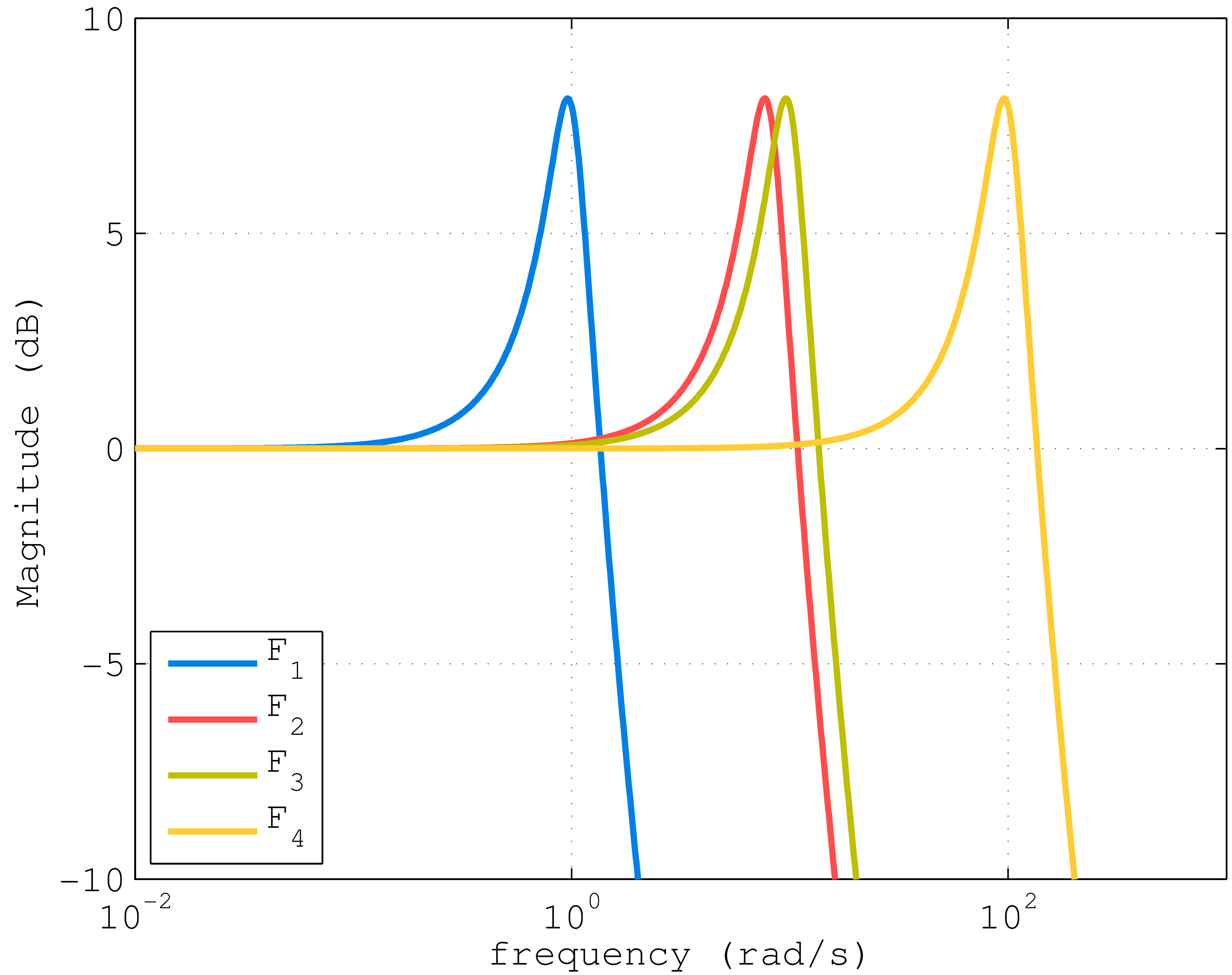


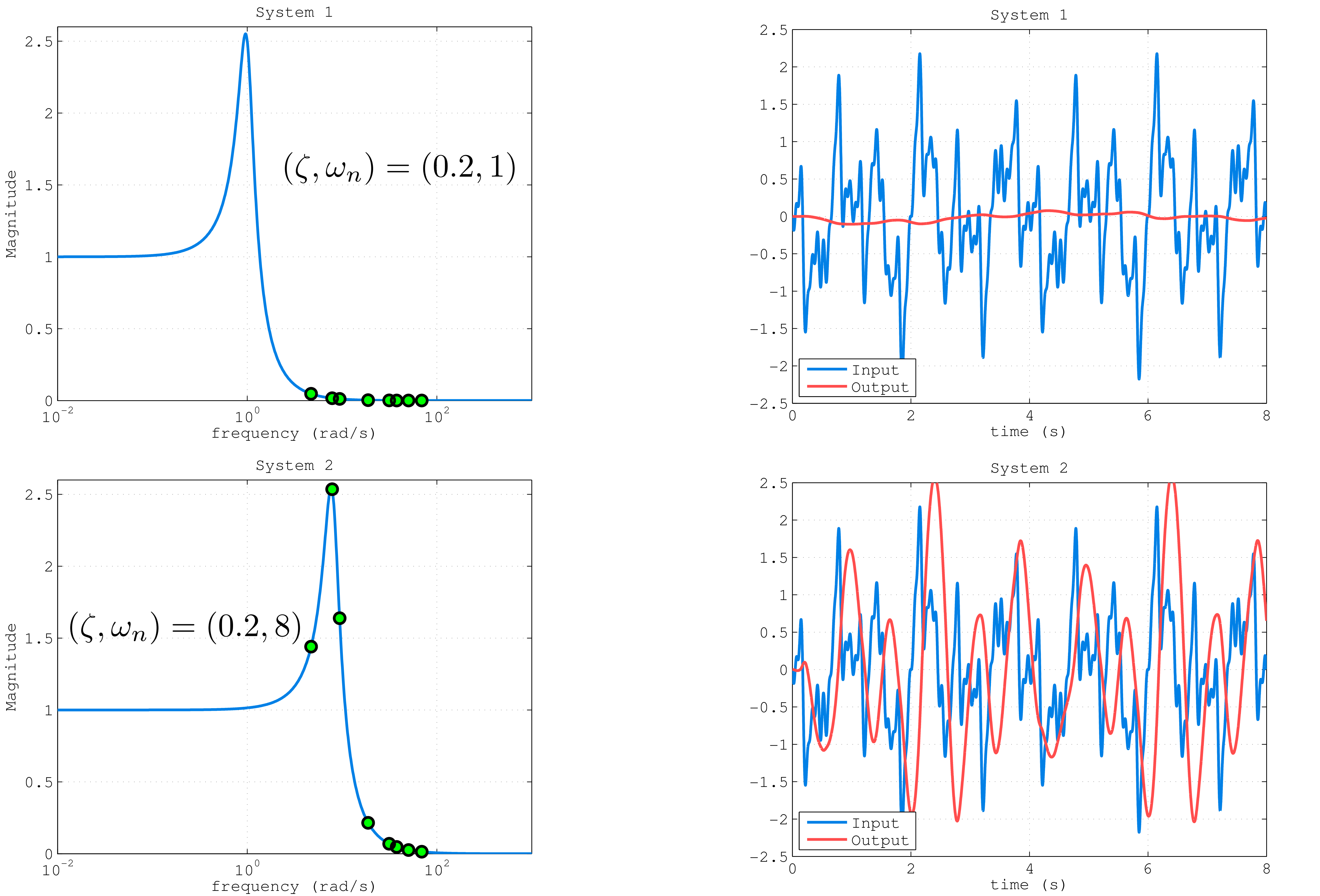
to replicate an input signal at the output (at steady state) it is not sufficient to require that the system has unitary magnitude at the input frequencies

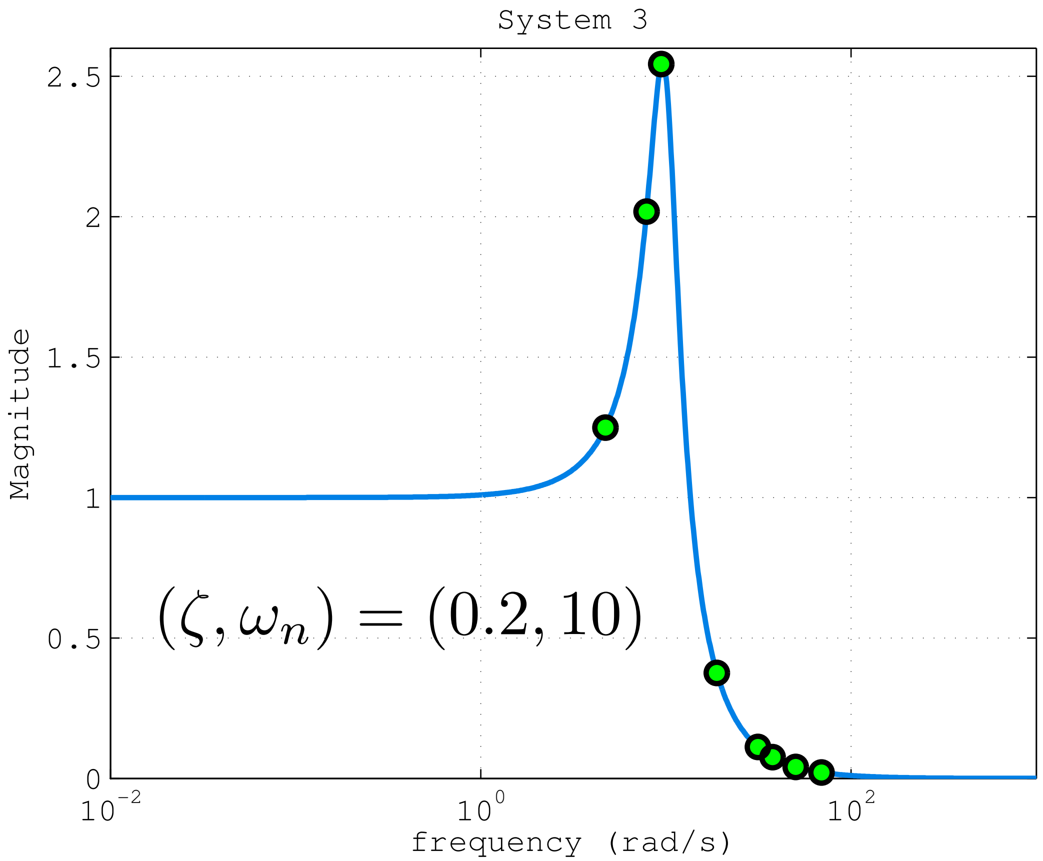
behavior at steady state: example 2

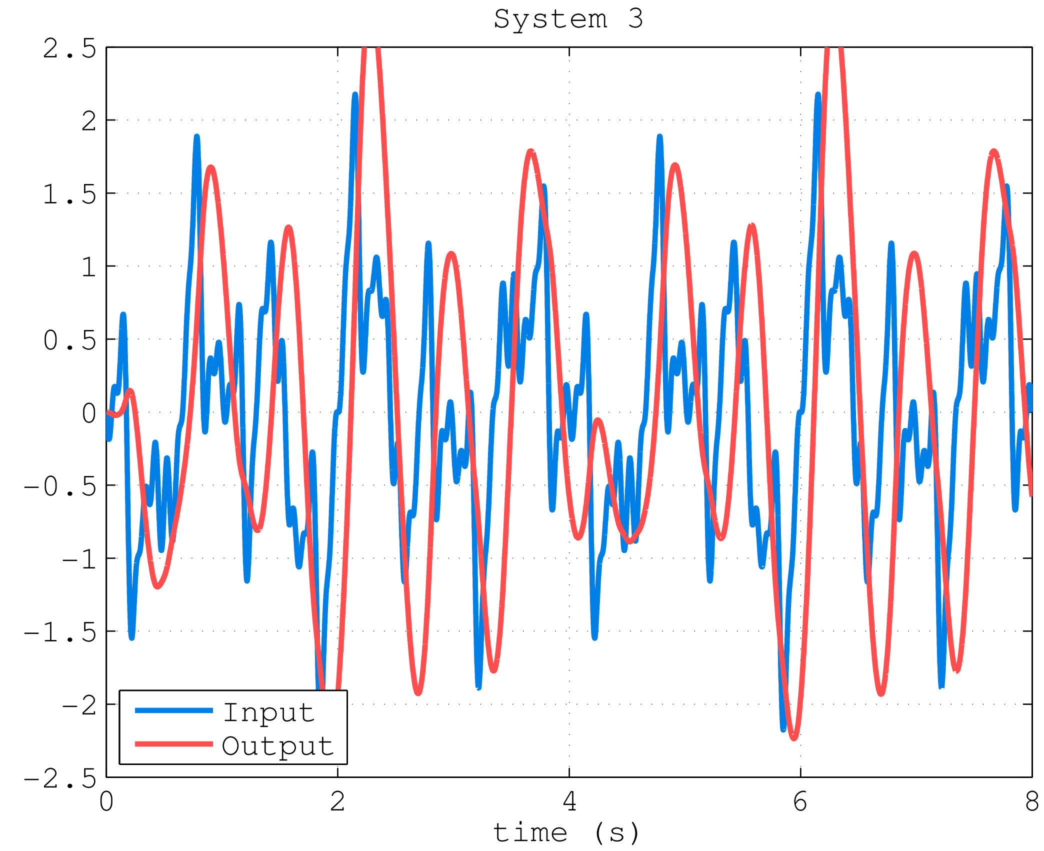
4 different systems
(all second order with same damping and unit gain)

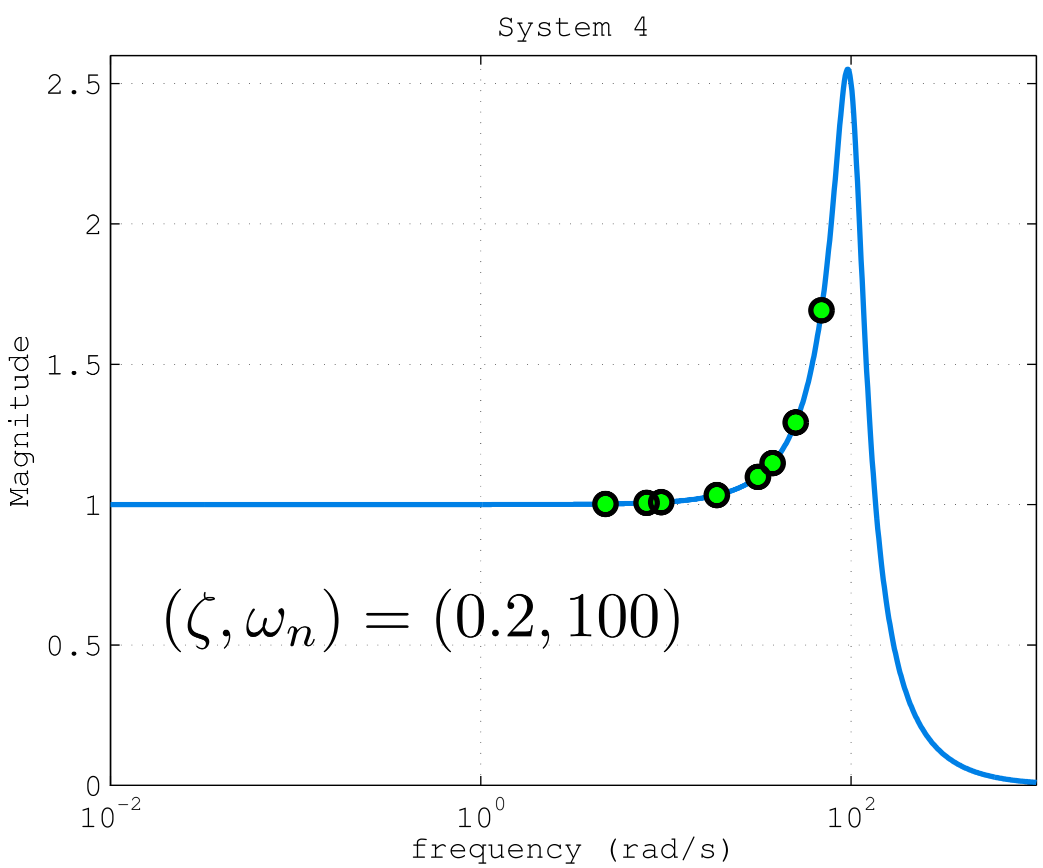
$$F(s) = \frac{1}{(1 + 2\zeta s/\omega_n + s^2/\omega_n^2)} \qquad \zeta = 0.2 \qquad \omega_n = \{1,\ 8,\ 10,\ 100\}$$

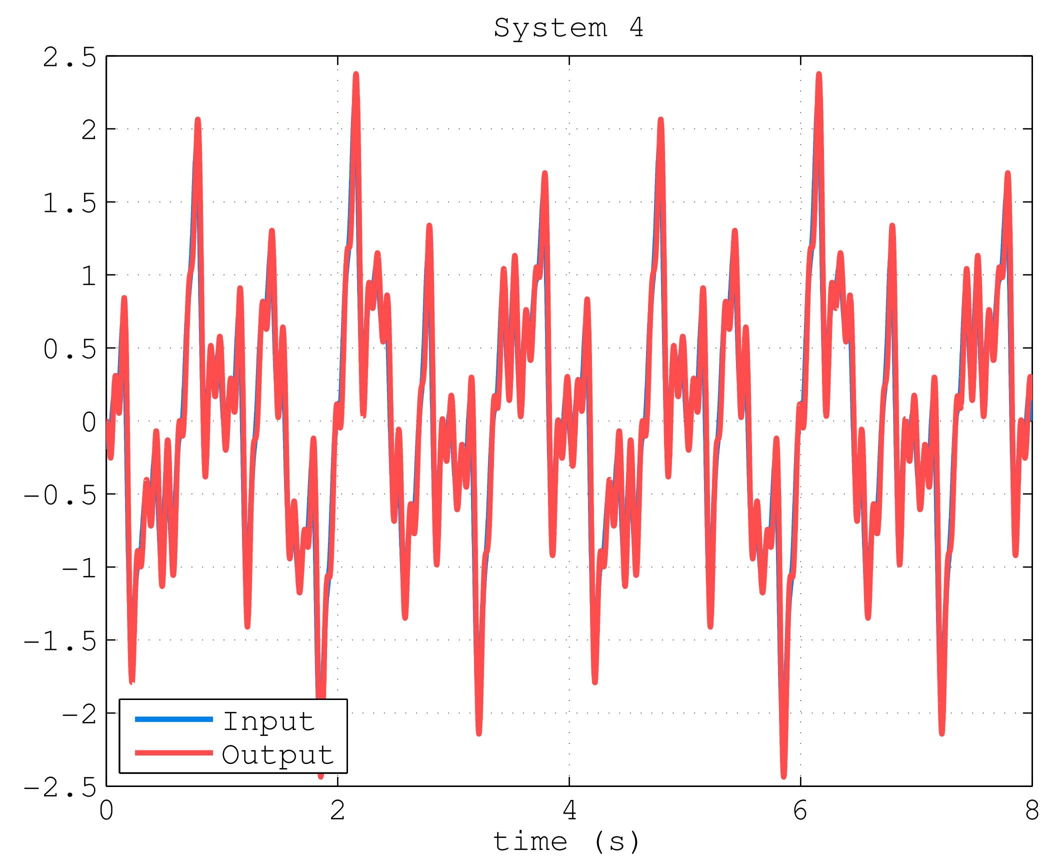



System 1
Magnitude
frequency (rad/s)
$(\zeta, \omega_n) = (0.2, 1)$
System 2
$(\zeta, \omega_n) = (0.2, 8)$
time (s)
Input
Output


System 3
Magnitude
frequency (rad/s)
$(\zeta, \omega_n) = (0.2, 10)$
2.5
2
1.5
1
0.5
0
10^{-2}
10^{0}
10^{2}


System 3
time (s)
Input
Output
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
2
4
6
8


System 4
Magnitude
frequency (rad/s)
$(\zeta, \omega_n) = (0.2, 100)$
2.5
2
1.5
1
0.5
0
10^{-2}
10^{0}
10^{2}


System 4
time (s)
Input
Output
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
2
4
6
8

transient

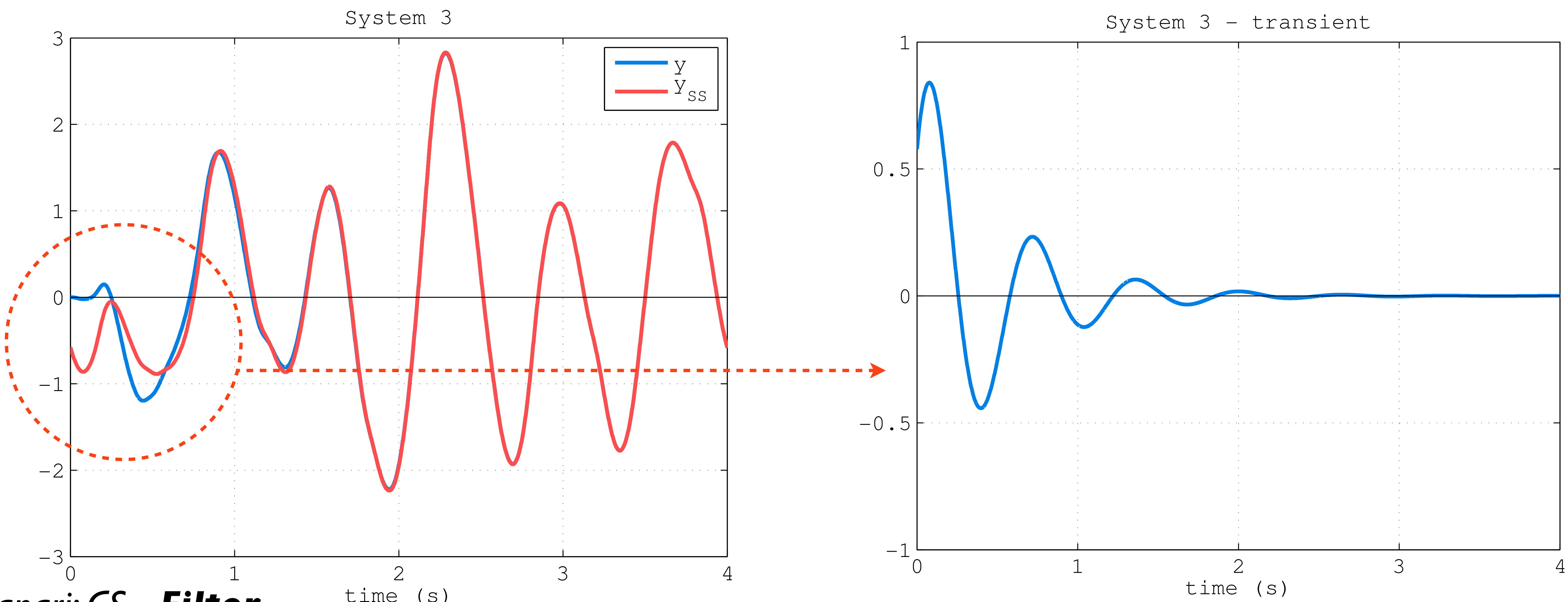
If a system is asymptotically stable it admits a steady state (not necessarily constant) to any persistent input: for example ramp, parabola, sinusoid. In this case we can also define the transient as the difference between the forced and the steady state response, that is transient exists also for inputs which differ from the step.

However we decided to characterize the transient with specific quantities on the step response.

example: transient for a sinusoidal input

forced response and steady state

transient as the difference between the forced response and the steady state



transient: bandwidth

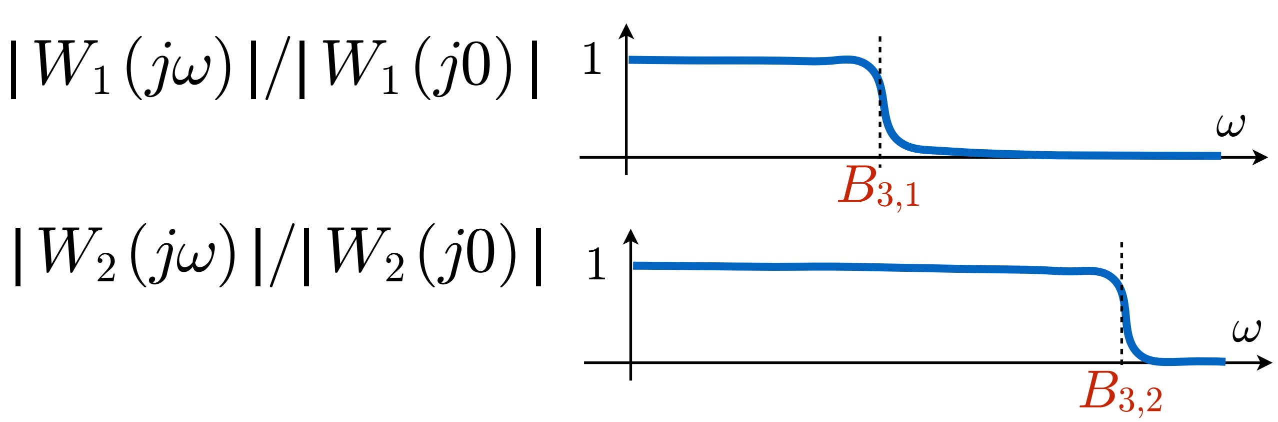
for the typical magnitude plots encountered so far, we define **bandwidth** B_3 of an asymptotically stable system as the first frequency such that for all frequencies greater than B_3 the magnitude is attenuated by a factor greater than $1/\sqrt{2}$ w.r.t. its value in $\omega = 0$. Recall that $1/\sqrt{2} \approx 0.707$

at B_3 : $\quad |W(jB_3)| = \dfrac{|W(j0)|}{\sqrt{2}}$ $\quad$ or $\quad \dfrac{|W(jB_3)|}{|W(j0)|} = \dfrac{1}{\sqrt{2}}$

and being $20 \log_{10} \left(\dfrac{1}{\sqrt{2}} \right) \approx -3\, dB$

at B_3 : $\quad |W(jB_3)|_{dB} = |W(j0)|_{dB} - 3$

- characterizes the filtering capacities of the dynamical system with transfer function $W(s)$



$B_{3,1} < B_{3,2}$

the first system $W_1(j\omega)$ cuts off more frequencies than the second

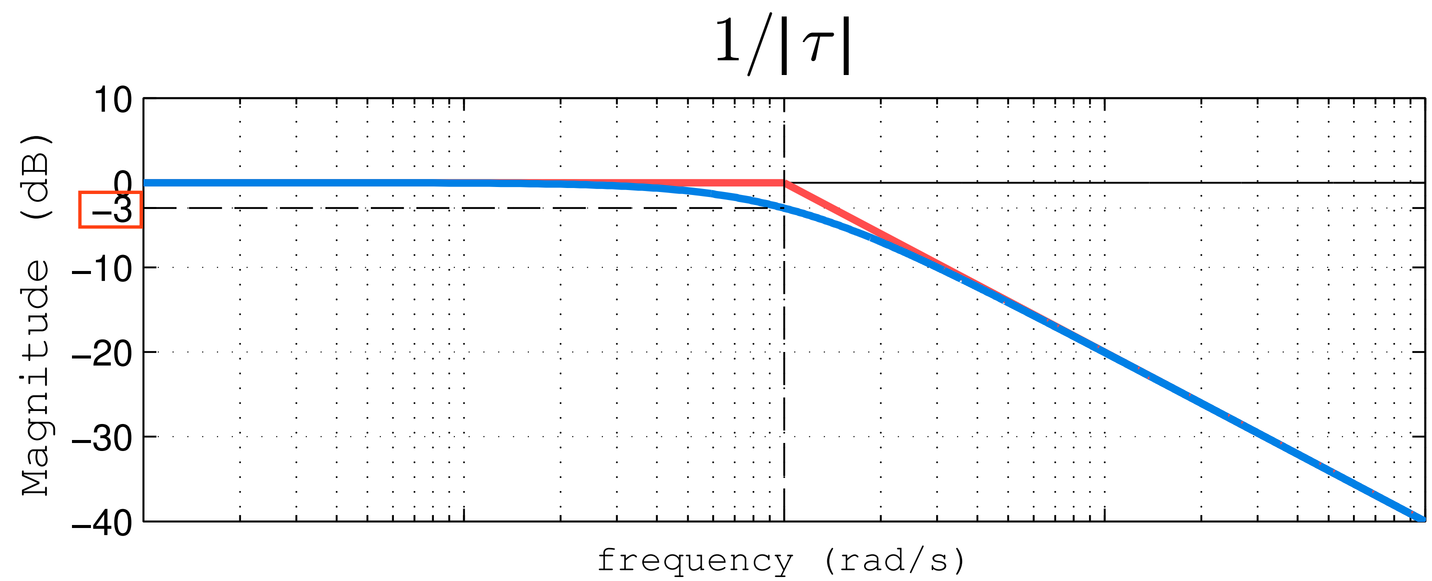
- relative to the static gain $|W(j0)|$

transient: simplest example

$W(s) = \dfrac{K}{1+\tau s}$ asymptotically stable system (therefore $\tau > 0$)

magnitude plot

normalized w.r.t. $|K|_{dB}$



being
$$\begin{aligned} |W(j\omega)|_{dB} - |W(j0)|_{dB} &= |W(j\omega)|_{dB} - |K|_{dB} \\ &= |K|_{dB} + |1/(1+j\omega\tau)|_{dB} - |K|_{dB} \\ &= |1/(1+j\omega\tau)|_{dB} \end{aligned}$$

and
$$|1 + j\tau/|\tau|\,|_{dB} = 20\log_{10}\sqrt{2} \approx 3\,dB$$

for a first order system, the bandwidth coincides with the cutoff frequency $B_3 = \dfrac{1}{\tau}$

- similarly for higher order systems in the presence of a dominant pole

transient: resonant peak

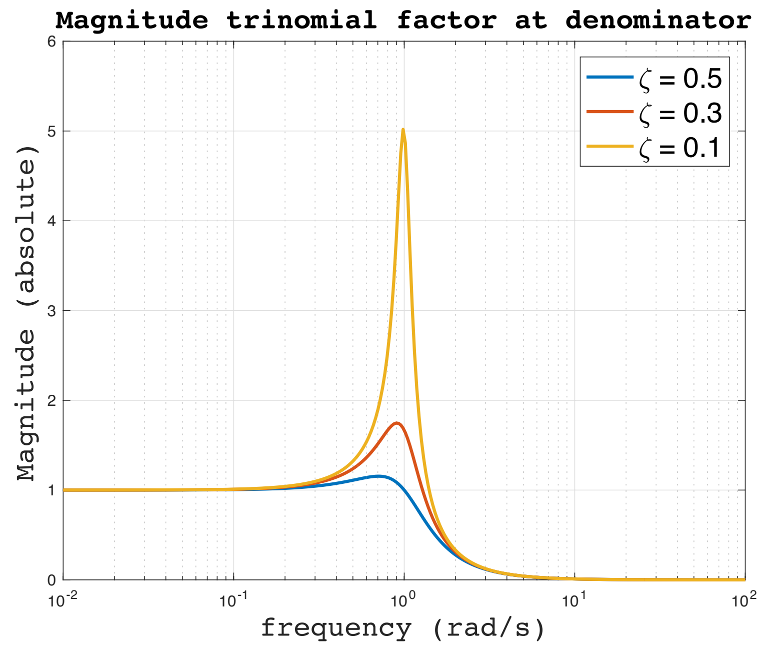
for an asymptotically stable system, we define **resonant peak** M_r as the maximum value of the frequency response magnitude referred to its value in $\omega = 0$

$$M_r = \frac{\max |W(j\omega)|}{|W(j0)|}$$

or in dB

$$M_r|_{dB} = \max |W(j\omega)|_{dB} - |W(j0)|_{dB}$$

a high resonant peak indicates that the system behaves similarly to a second order system with low damping coefficient



transient: resonant peak

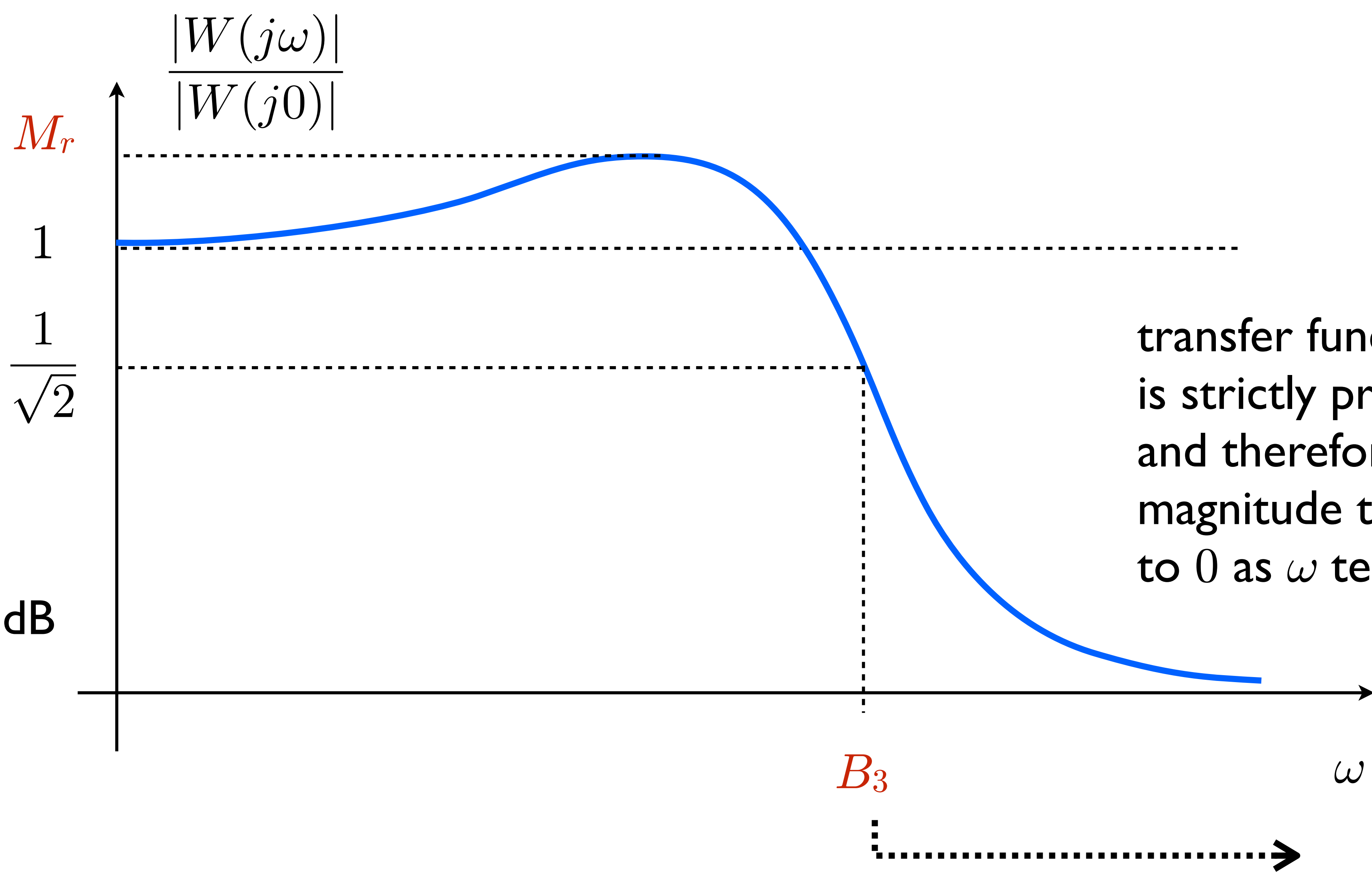
- Note that the resonant peak is defined w.r.t. the value of the magnitude in $\omega = 0$ and it is not just the maximum value (a constant gain $F(s) = K$ would not give any resonant peak)

- since the presence of a peak in a frequency response is similar to the peak of a second order system with complex conjugate poles and low values of the damping coefficient, the higher the peak the smaller the "equivalent damping" value and therefore the higher the overshoot in the step response

- from the frequency response we get not only information on the steady state but also on the transient

transient: frequency domain characterization

characterization of the transient in the frequency domani on a plot of the normalized magnitude (not in dB)



magnitude plot not in dB

transfer function is strictly proper and therefore the magnitude tends to 0 as ω tends to ∞

any sinusoidal signal with frequency greater than B_3 will be attenuated at steady state by more than 0.707

- a similar plot can be drawn when the magnitude is in dB

transient: relationships in t and ω

typically (with some exceptions)

in frequency   in time

$$B_3 \, t_r \approx \text{constant}$$

- higher bandwidth B_3 (higher frequency components of the input signal are not attenuated and therefore are allowed to go through) leads to smaller rise time t_r (faster system response)

in time (numerator), in frequency (denominator)

$$\frac{1 + M_p}{M_r} \approx \text{constant}$$

- higher resonant peak M_r (as if we had a second order system with lower damping coefficient) leads to higher overshoot M_p (the oscillation damps out slower)

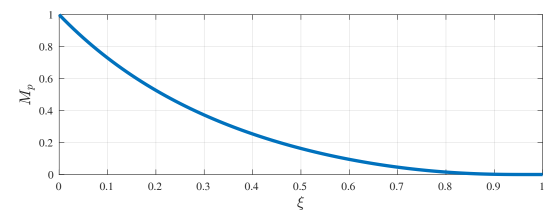
- very useful relationships in order to understand the connections between time and frequency domain response characteristics

transient: explicit relations for second order system

$$W(s) = \frac{1}{1 + 2\zeta \dfrac{s}{\omega_n} + \dfrac{s^2}{\omega_n^2}} \qquad 0 < \zeta < 1$$

for a second order system some explicit expressions can be obtained (as an example)

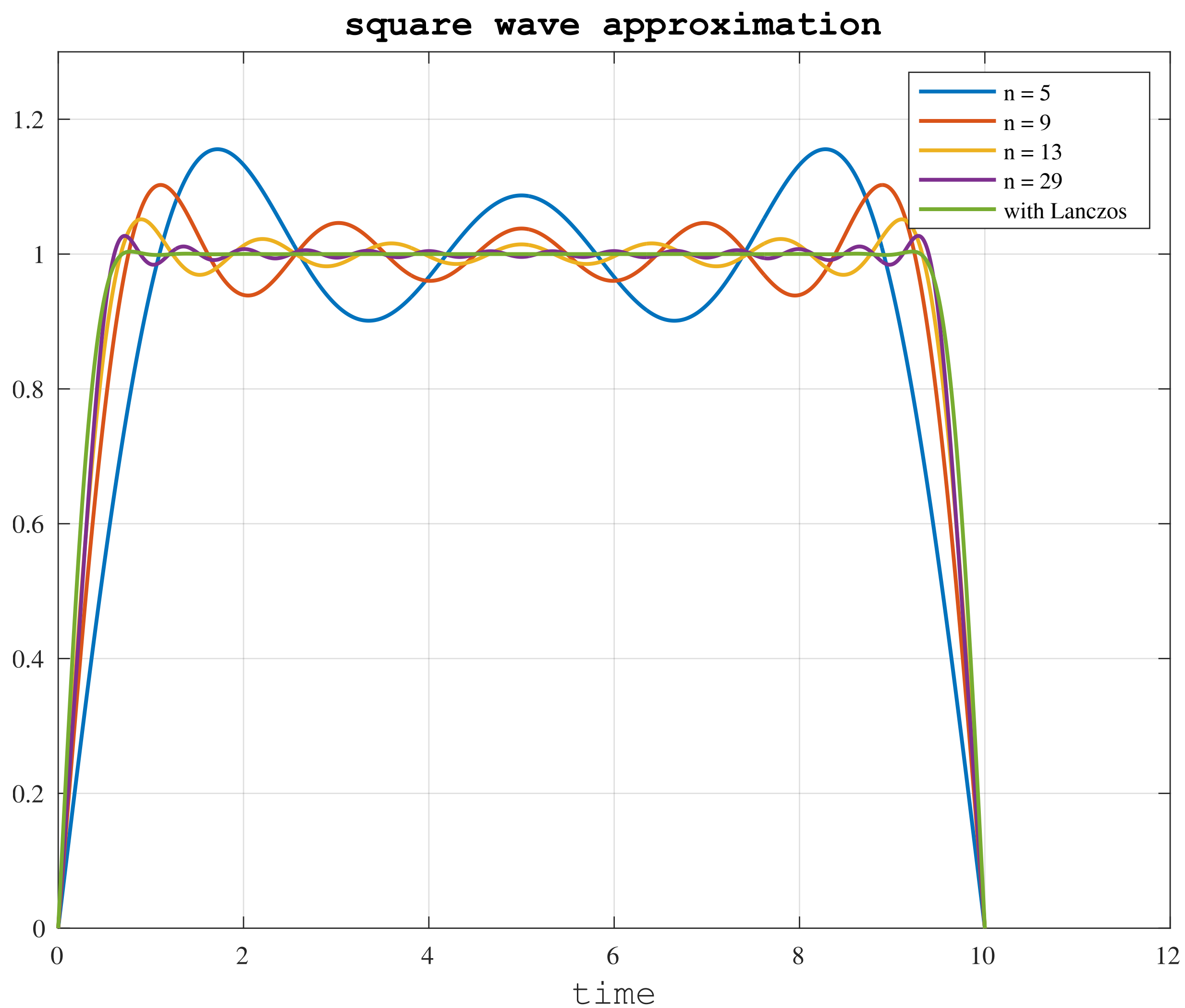
- step response $$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \left[\sqrt{1-\zeta^2}\, \omega_n t + \arctan \frac{\sqrt{1-\zeta^2}}{\zeta} \right]$$
- overshoot $$M_p = e^{-\dfrac{\pi \zeta}{\sqrt{1-\zeta^2}}}$$



(1 being = 100%)

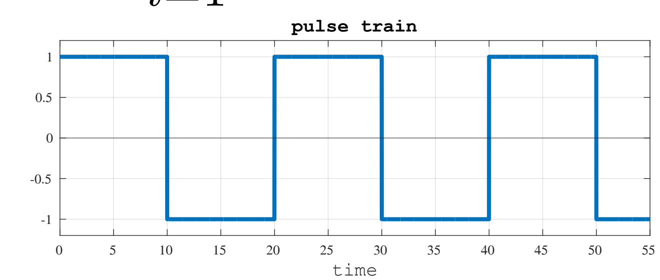
- resonance peak $$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ valid for $\zeta \leq \dfrac{1}{\sqrt{2}}$
- bandwidth $$B_3 = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$
- rise time (up to roughly $\zeta = 0.7$) $$t_r = \frac{1}{\omega_n} \frac{1}{\sqrt{1-\zeta^2}} \left[\pi - \arctan \frac{\sqrt{1-\zeta^2}}{\zeta} \right]$$

example: a discontinuous signal



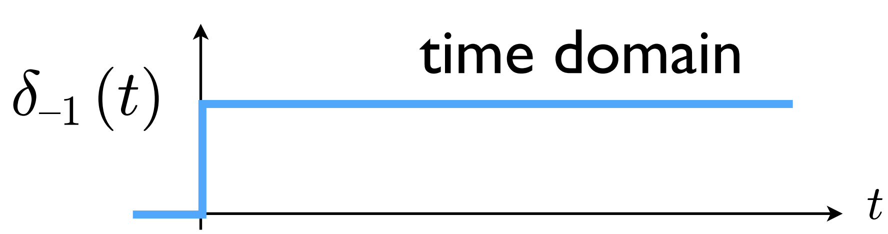
detail of the truncated Fourier expansion of a pulse train (square wave): the more components with higher frequency we include in the sum the better the approximation is.

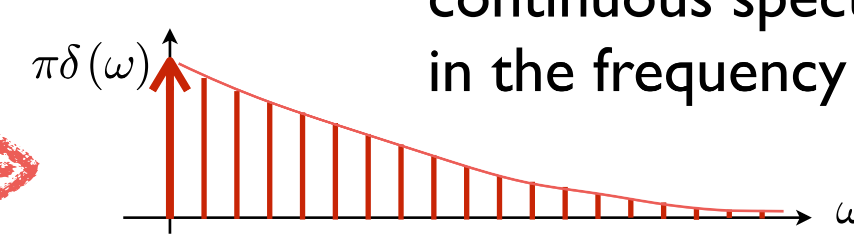
$$1 + 2 \sum_{i=1} (-1)^i \delta_{-1}(t - iT)$$



the discontinuous signal (pulse train) is made of infinite sinusoidal components

almost similarly, the **step function** has an infinite frequency content

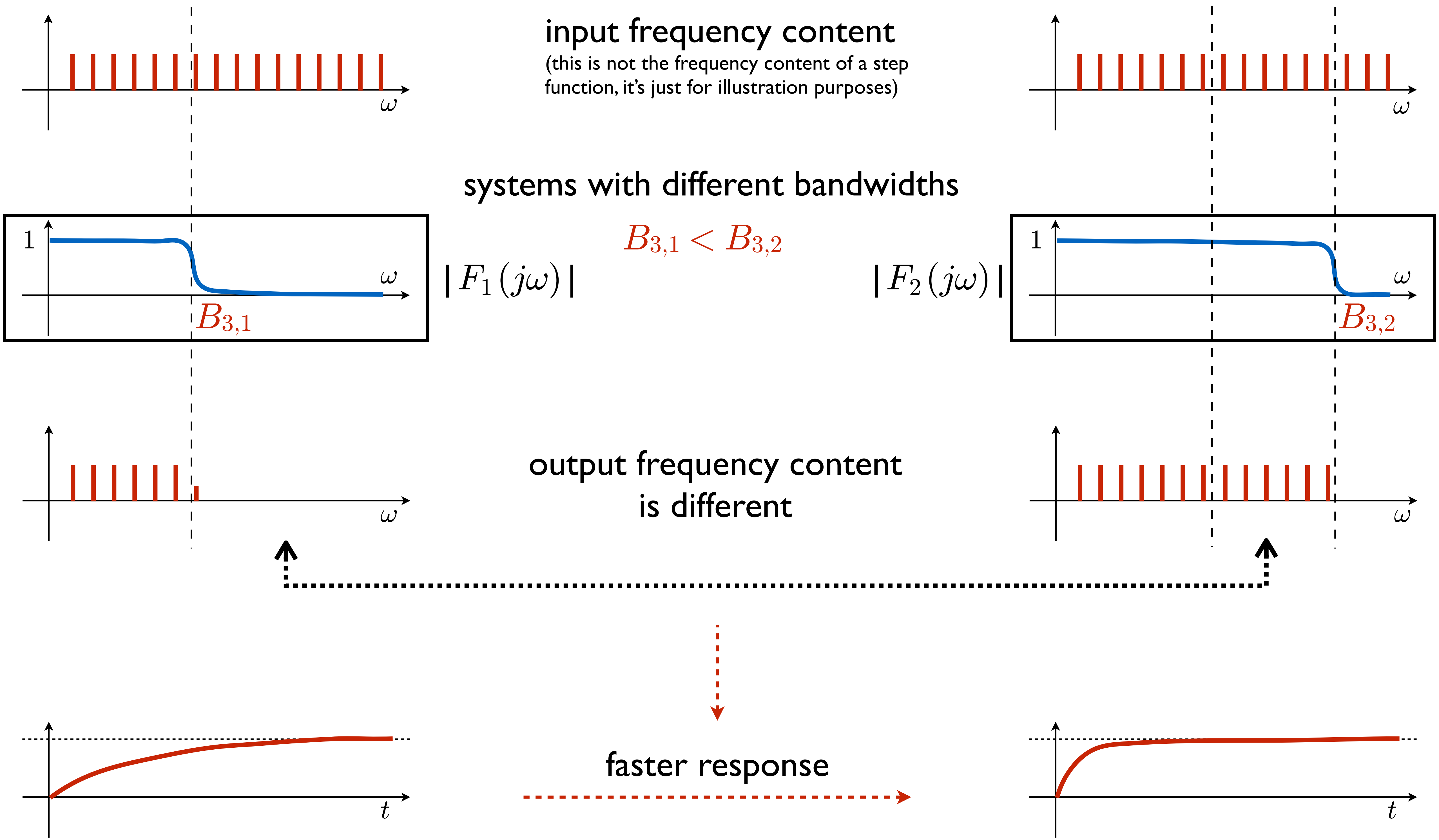




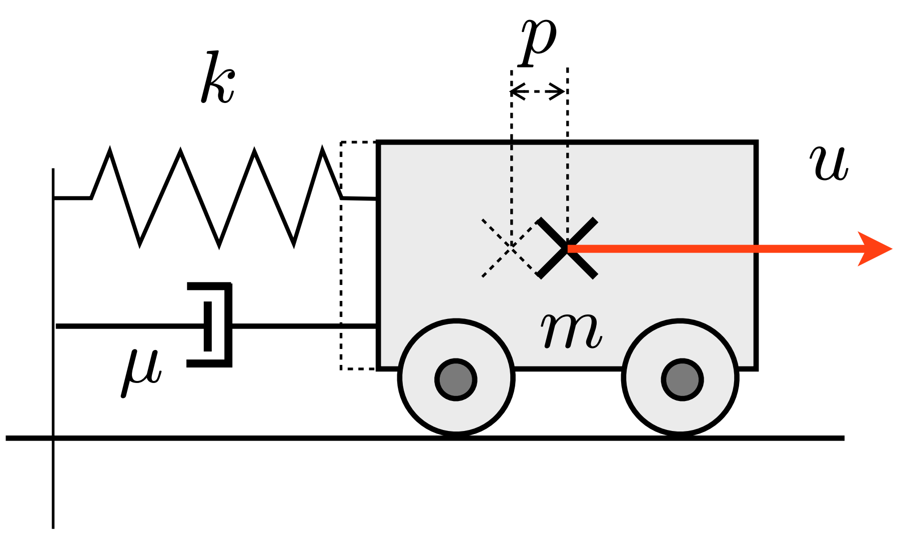
continuous spectrum
in the frequency domain

we can therefore see in the frequency domain the filtering effect of a system on a step input

system as a filter (transient)



Mass - Spring - Damper



transfer function

$$F(s) = \frac{1}{ms^2 + \mu s + k}$$

asymptotically stable system for $\mu > 0$

- under-damped $\quad 0 < \mu < 2\sqrt{km}$

  complex conjugate poles

$$p_{1,2} = \frac{-\frac{\mu}{m} \pm j\sqrt{4\left(\frac{k}{m}\right) - \left(\frac{\mu}{m}\right)^2}}{2}$$

- critically-damped $\quad \mu = 2\sqrt{km}$

  real coincident poles

$$p_{1,2} = -\frac{\mu}{2m}$$

- over-damped $\quad \mu > 2\sqrt{km}$

  real distinct poles

$$p_{1,2} = \frac{-\frac{\mu}{m} \pm \sqrt{\left(\frac{\mu}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2}$$

Mass - Spring - Damper

gain = $1/k$

- under-damped $0 < \mu < 2\sqrt{km}$

  trinomial factor

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{\mu}{2\sqrt{km}}$$

- critically-damped $\mu = 2\sqrt{km}$

  two coincident binomial factors

$$\frac{1}{\tau} = -p_{1,2} = \frac{\mu}{2m} = \sqrt{\frac{k}{m}}$$

- over-damped $\mu > 2\sqrt{km}$

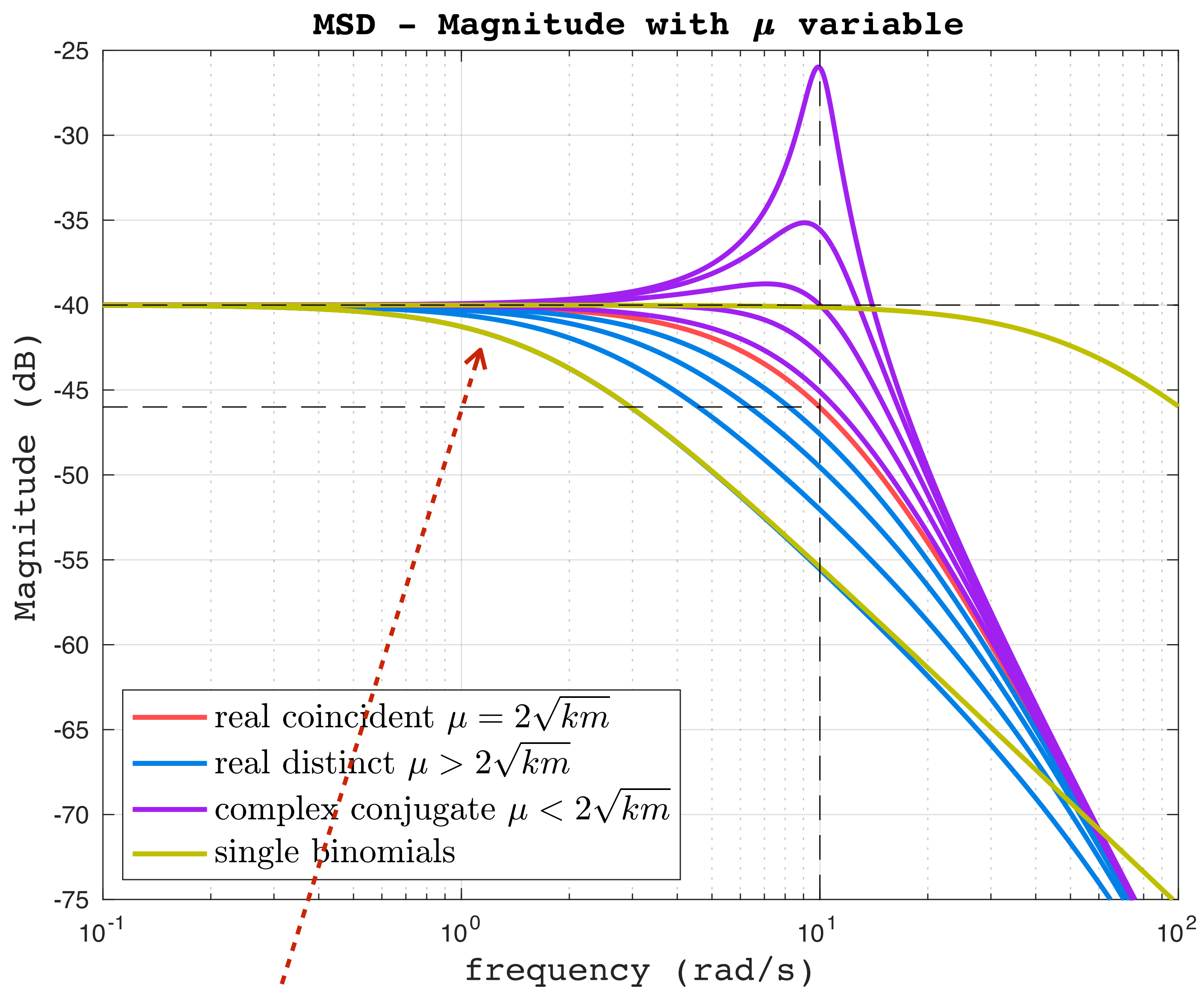
  two distinct binomial factors

$$\frac{1}{\tau_1} = -p_1 > \sqrt{\frac{k}{m}} \qquad \frac{1}{\tau_2} = -p_2 < \sqrt{\frac{k}{m}}$$

Mass - Spring - Damper

$m = 1$ kg
$k = 100$ N/m

magnitude in terms of the damping factor μ

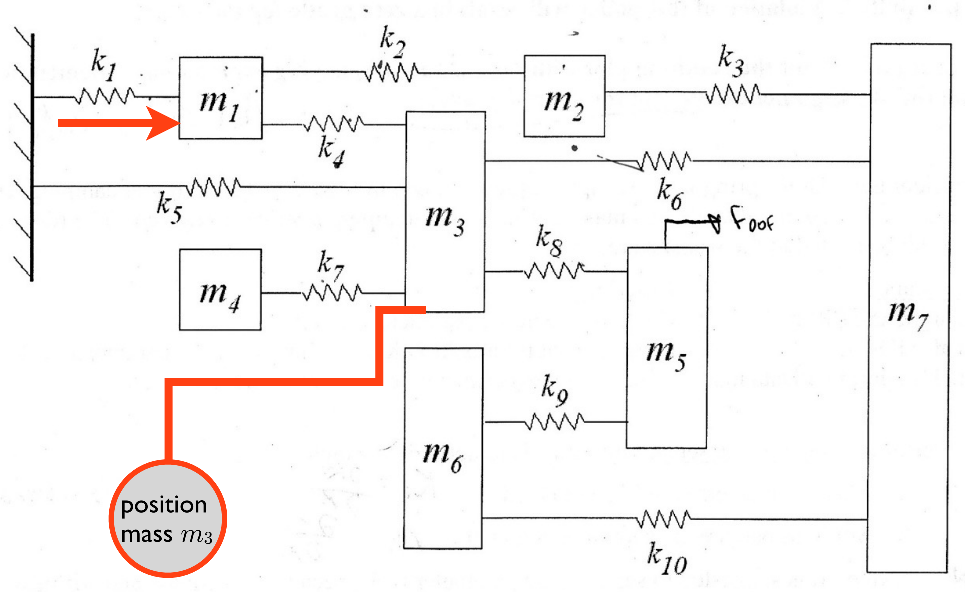


the two single binomials are also shown (over-damped case) in order to put in evidence the dominant pole (corresponding to the real pole closest to the origin)

7-mass

input force on mass 1

output position on mass 3

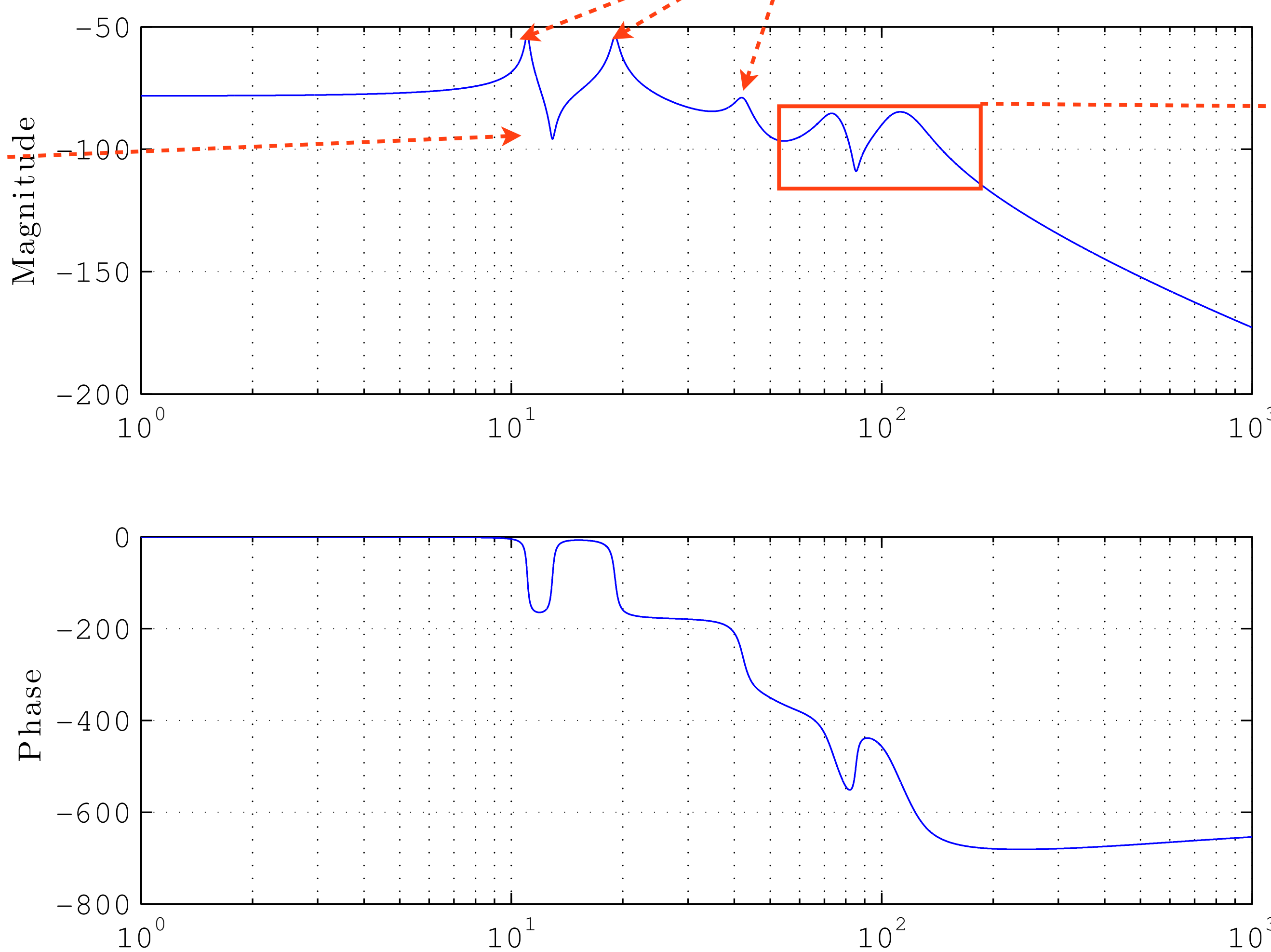


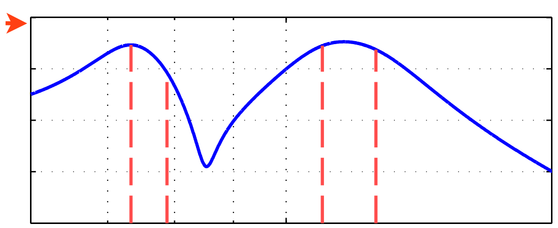
natural frequencies

119.6311
107.5098
78.7957
73.3411
42.2283
19.0596
11.0478

resonance peaks

anti-resonance peak

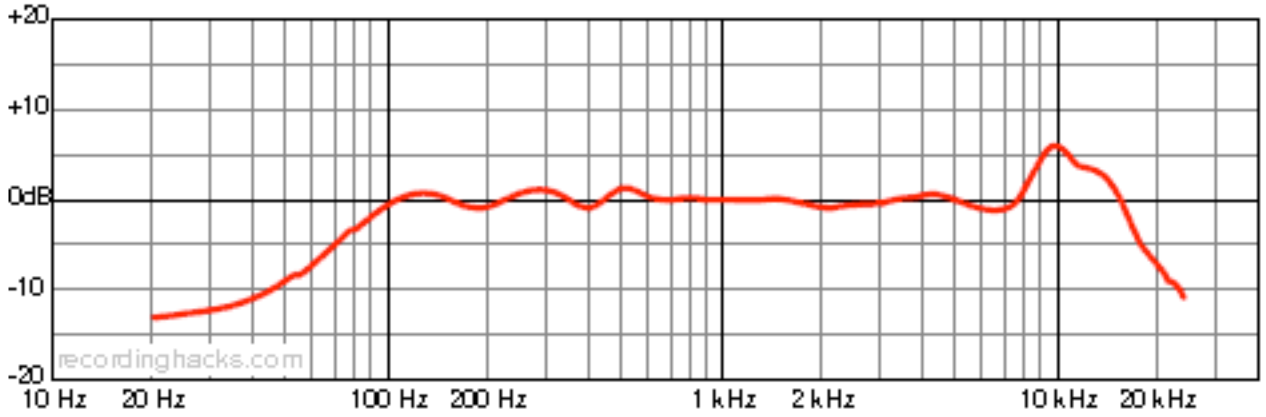




natural frequencies very close to each other

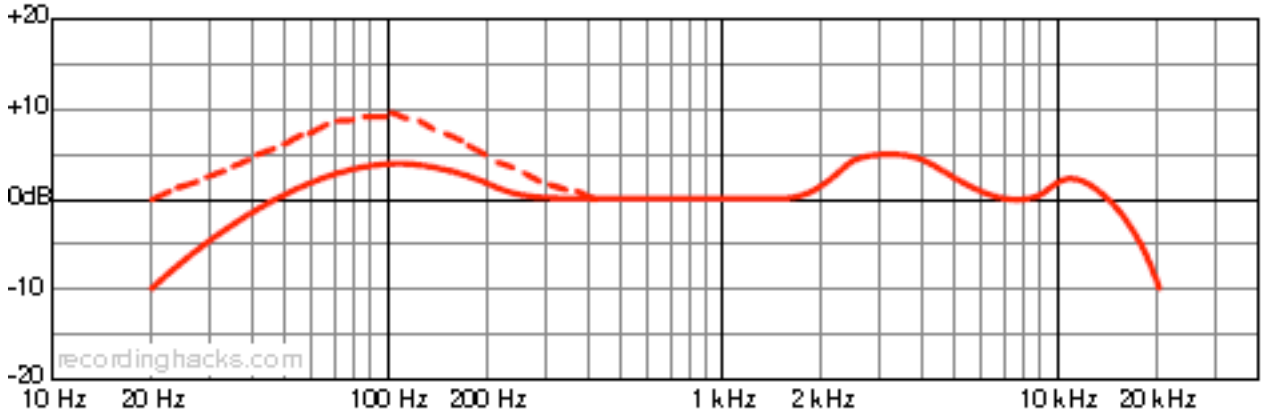
7 - MSD system (with **damping**)

microphones

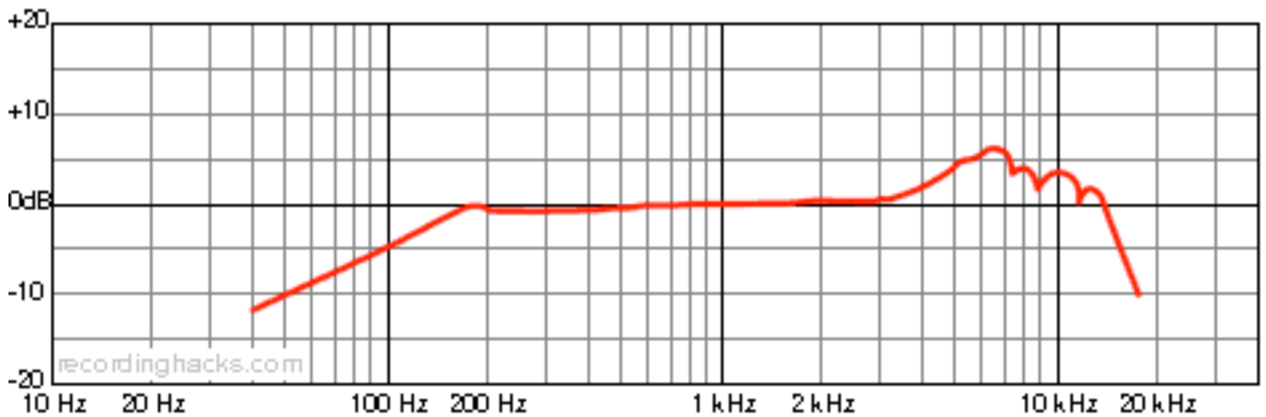


a 10.000 € voice microphone ...

recall that 1 Hz = 2π rad/s and that voice is in the frequency rance roughly from 300 to 3000 Hz

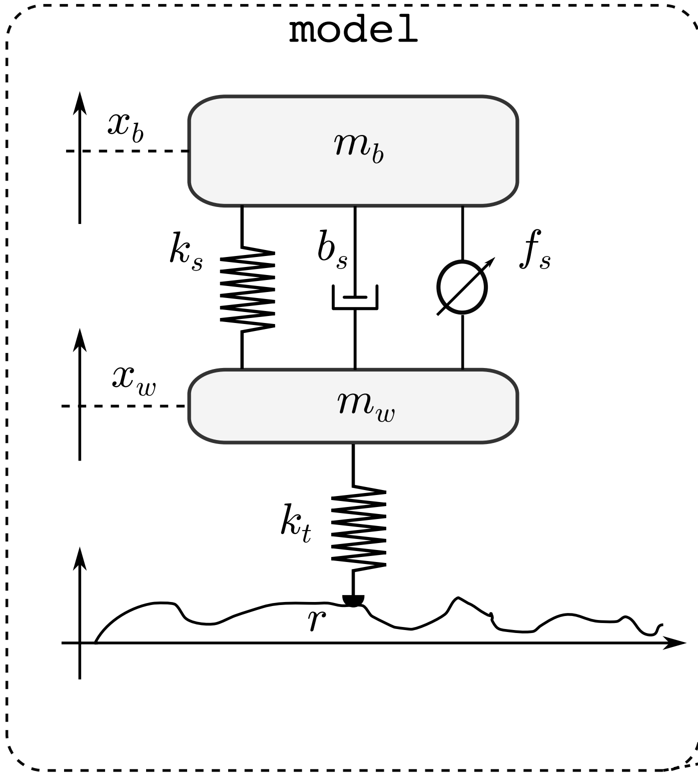


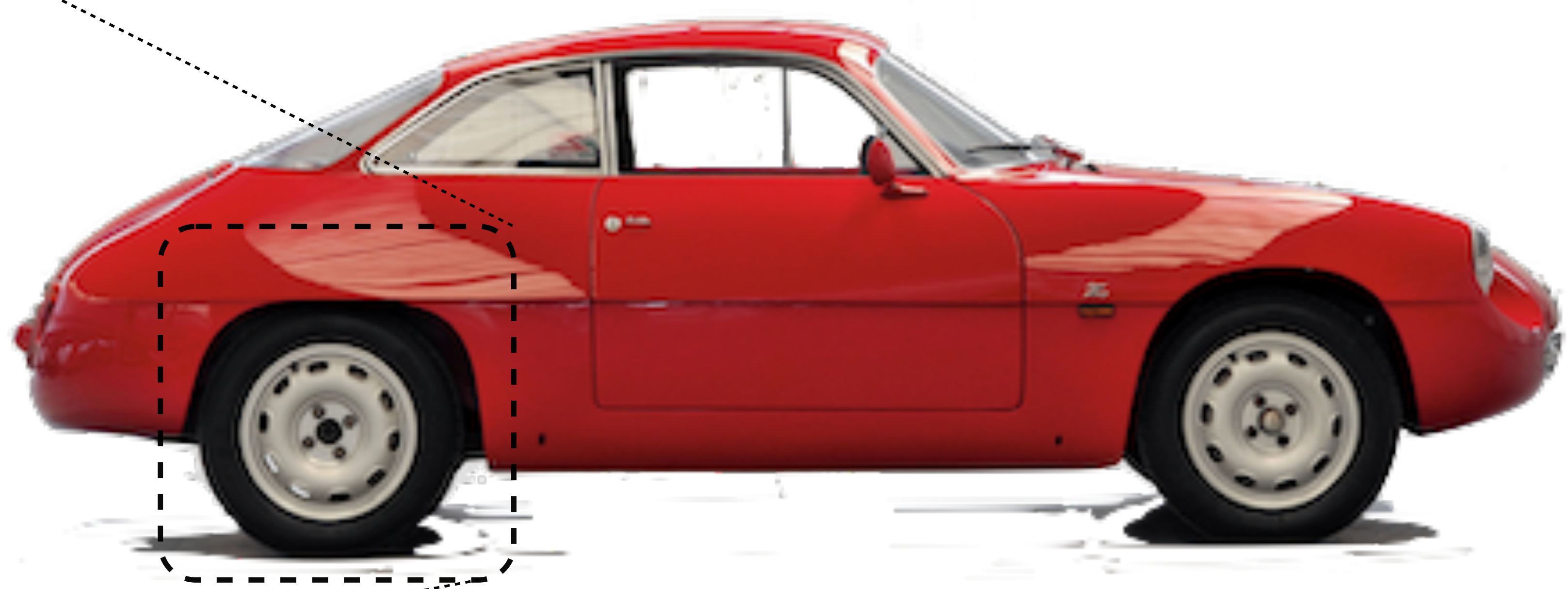
dynamic bass microphone (tailored for kick drum, works well with any low frequency instrument, low frequency peak at 100 Hz)

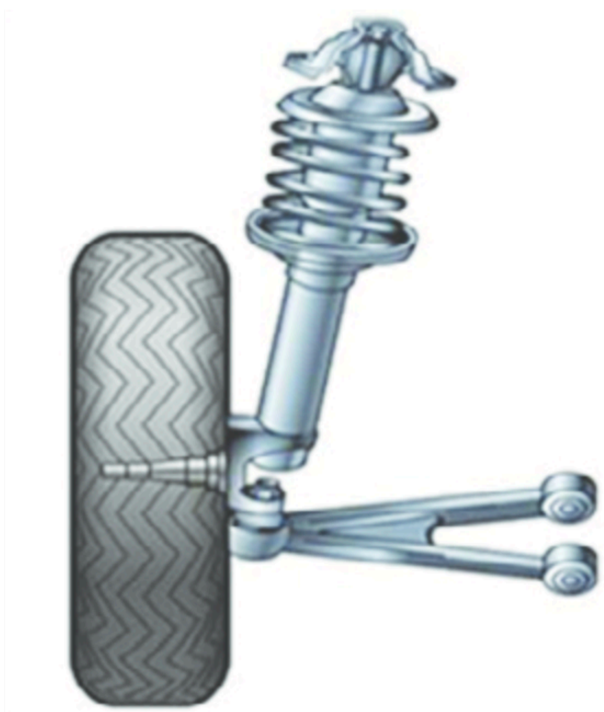


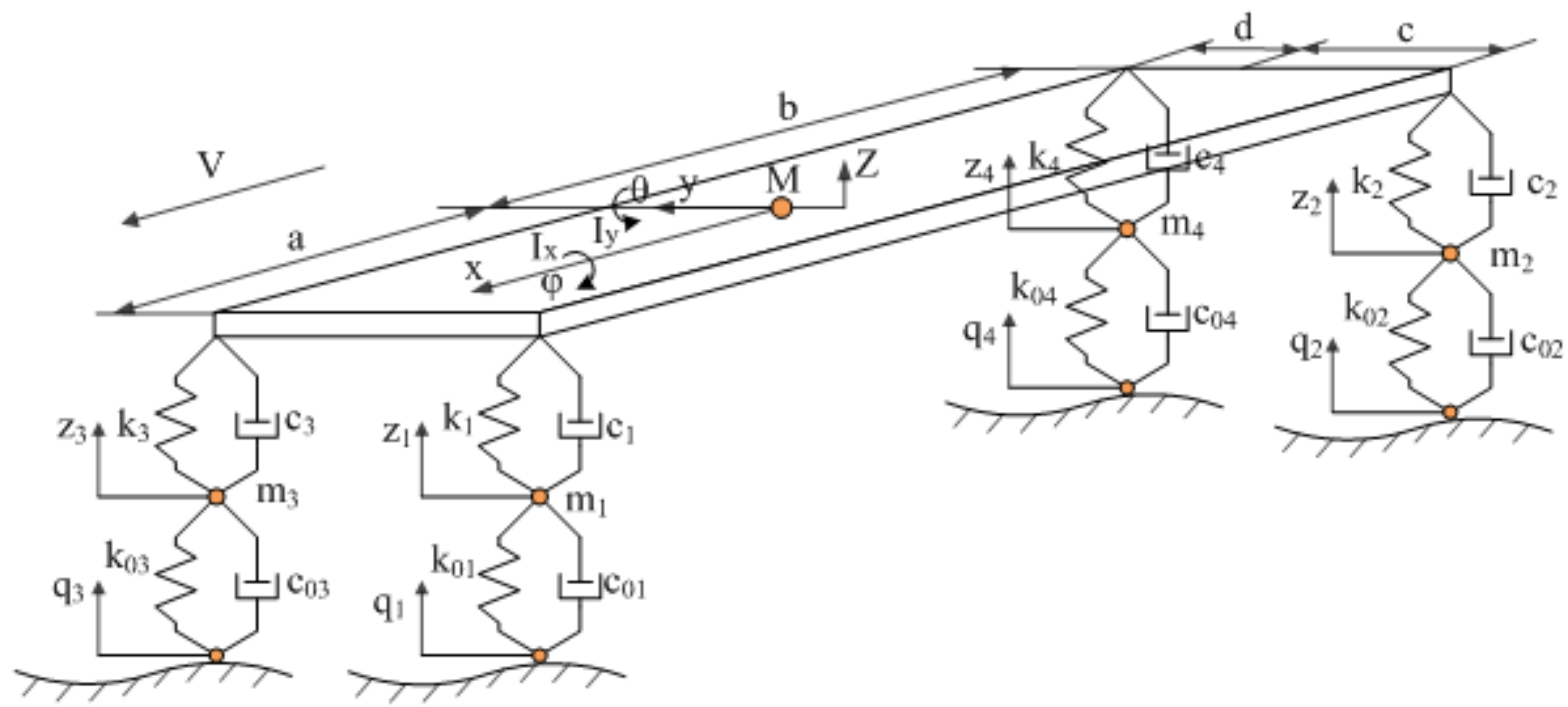
electric guitar microphone

quarter-car suspension model

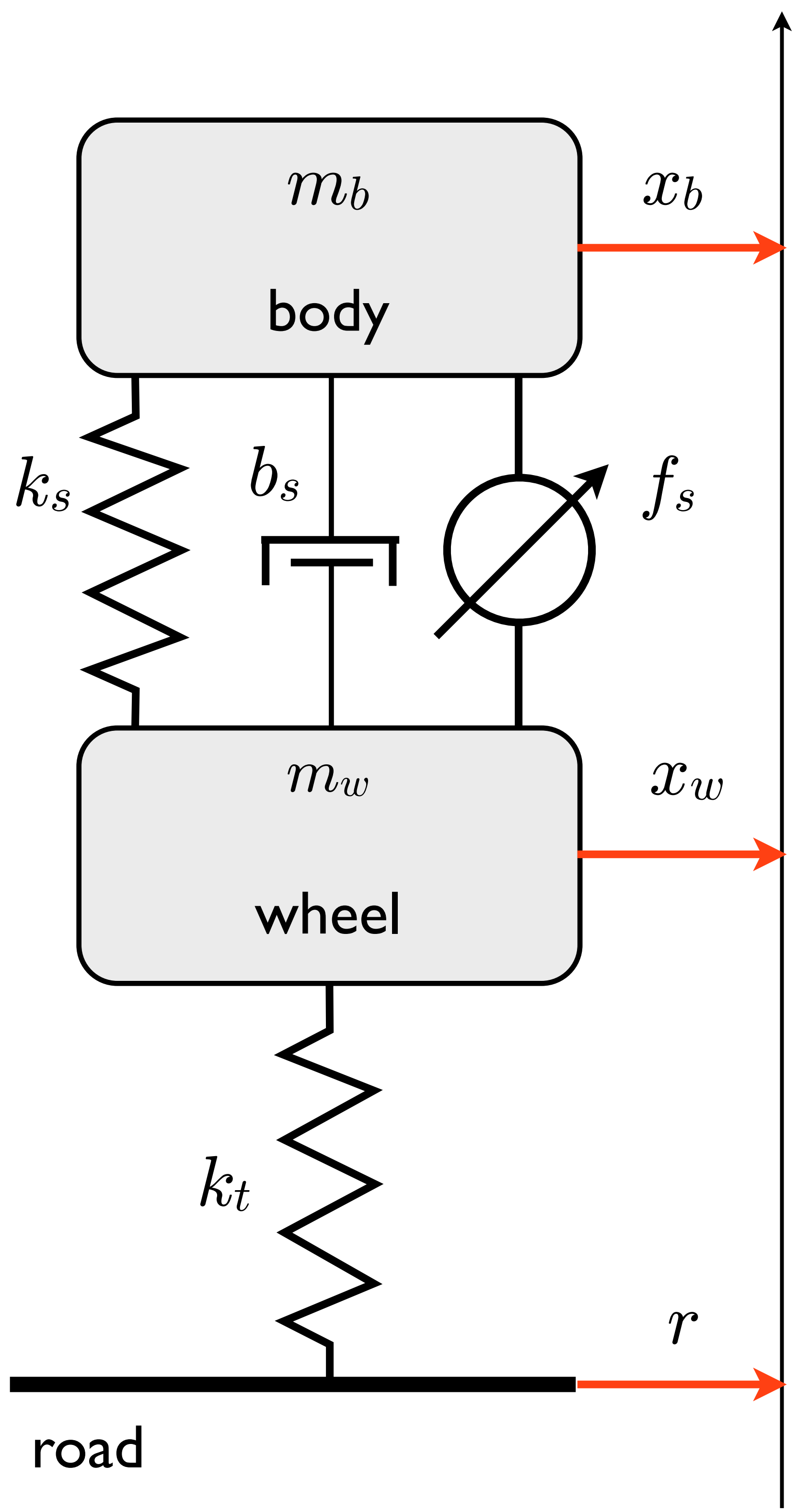








quarter-car suspension model



Model of a quarter part of a car with its wheel and tire

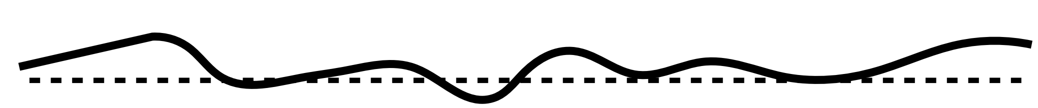
The body with mass m_b represents the car chassis connected to the wheel by a passive spring (k_s), and a shock absorber represented by a damper (b_s).
The spring (k_t) models the compressibility of the tire pneumatic.

In an **active suspension** a hydraulic actuator (f_s) between the chassis and wheel assembly may help in balancing conflicting objectives as passenger comfort, road handling and suspension deflection.

$$
\begin{aligned}
m_b \ddot{x}_b + k_s(x_b - x_w) + b_s(\dot{x}_b - \dot{x}_w) &= f_s \\
m_w \ddot{x}_w - k_s(x_b - x_w) + b_s(\dot{x}_b - \dot{x}_w) + k_t(x_w - r) &= -f_s
\end{aligned}
$$

f_s acts on both the body and the wheel assembly

r can be seen as an input affecting the evolution of the system through the tire (disturbance)



$r(t)$ generic terrain profile

- state vector $\begin{bmatrix} x_b & \dot{x}_b & x_w & \dot{x}_w \end{bmatrix}^T$
- several outputs of interest

$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ body (passenger) position

$C_2 = \begin{bmatrix} -k_s/m_b & -b_s/m_b & k_s/m_b & b_s/m_b \end{bmatrix}$ body (passenger) acceleration

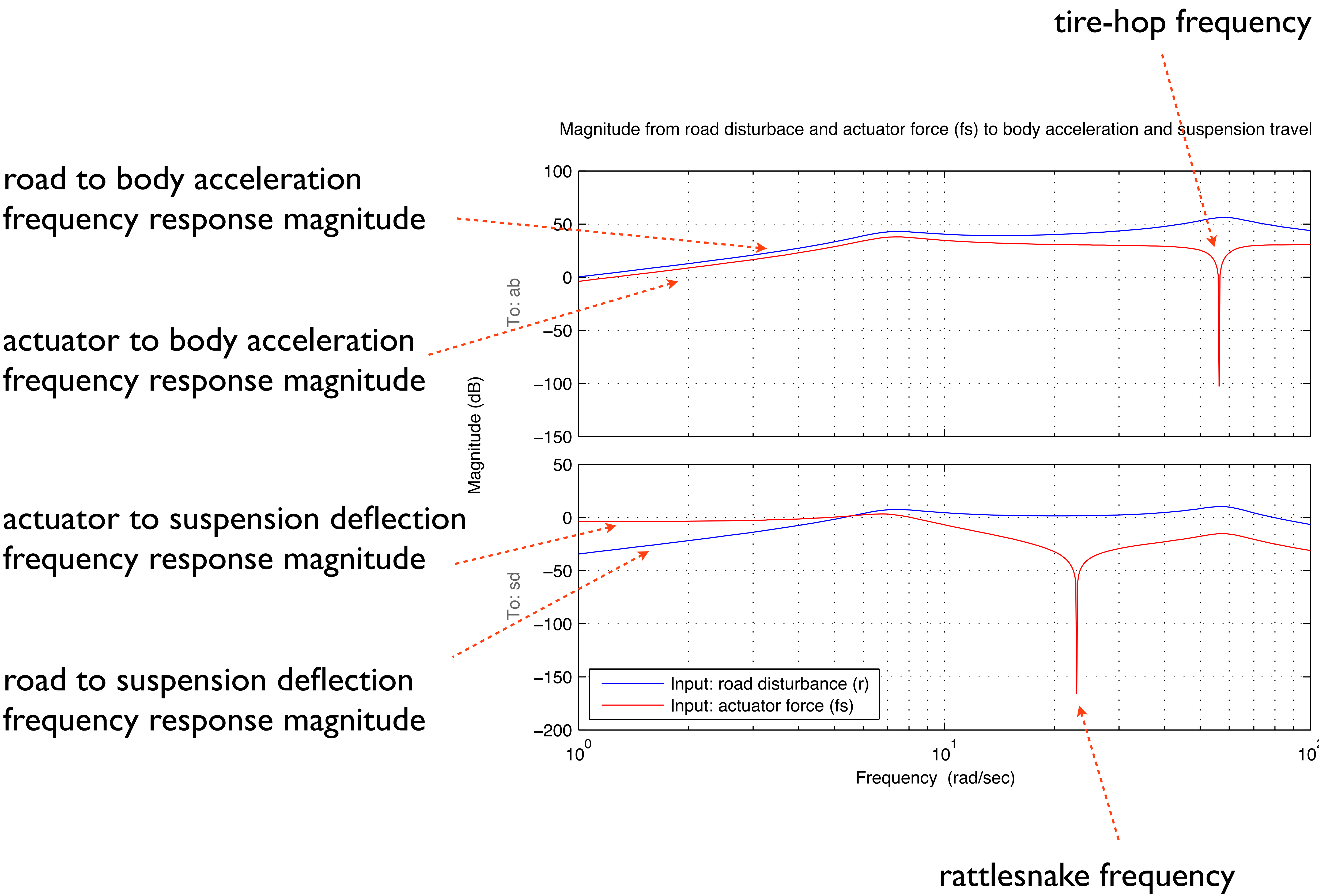
$C_3 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$ suspension deflection

- two inputs (one, f_s, can be controlled, the other is the disturbance r)

by setting one of the two inputs to zero and choosing the output of interest, we have a SISO system with corresponding transfer function

- Passenger comfort is associated to small passenger acceleration
- Physical limitation of the actuator (limits on maximum displacements) defines a constraint

these are the two considered outputs



tire-hop frequency
Magnitude from road disturbace and actuator force (fs) to body acceleration and suspension travel
road to body acceleration frequency response magnitude
actuator to body acceleration frequency response magnitude
actuator to suspension deflection frequency response magnitude
road to suspension deflection frequency response magnitude
To: ab
To: sd
Magnitude (dB)
100
50
0
−50
−100
−150
−200
Input: road disturbance (r)
Input: actuator force (fs)
10^0
10^1
10^2
Frequency (rad/sec)
rattlesnake frequency

tire-hop frequency: pure imaginary zeros in the transfer function from the actuator to the body acceleration (also from actuator to body position), anti-resonance at 56.27 rad/s

rattlesnake frequency: pure imaginary zeros in the transfer function from the actuator to the suspension deflection, anti-resonance at 22.97 rad/s

at these frequencies it is difficult to counteract any effect of the road on acceleration or on the suspension deflection (no control "authority")