Control Systems

Controllability & Observability

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Outline

- introduce the concept of reachable states
- the controllability matrix P
- Kalman decomposition w.r.t. controllability
- undistinguishable and unobservable states
- observability matrix
- Kalman decomposition w.r.t. observability
- case study: 2-mass and linear actuator system

Reachable states: an example

Let us consider a 2-dimensional system characterized by the matrices $(A,\ B)$ with distinct eigenvalues λ_1 and λ_2 and their corresponding right $(u_1,\ u_2)$ and left $(v_1^{\mathrm{T}},\ v_2^{\mathrm{T}})$ eigenvectors such that $v_i^{\mathrm{T}} u_j = \delta_{ij}$.

The state impulsive response is then

$$e^{At}B = e^{\lambda_1 t} u_1 v_1^T B + e^{\lambda_2 t} u_2 v_2^T B$$

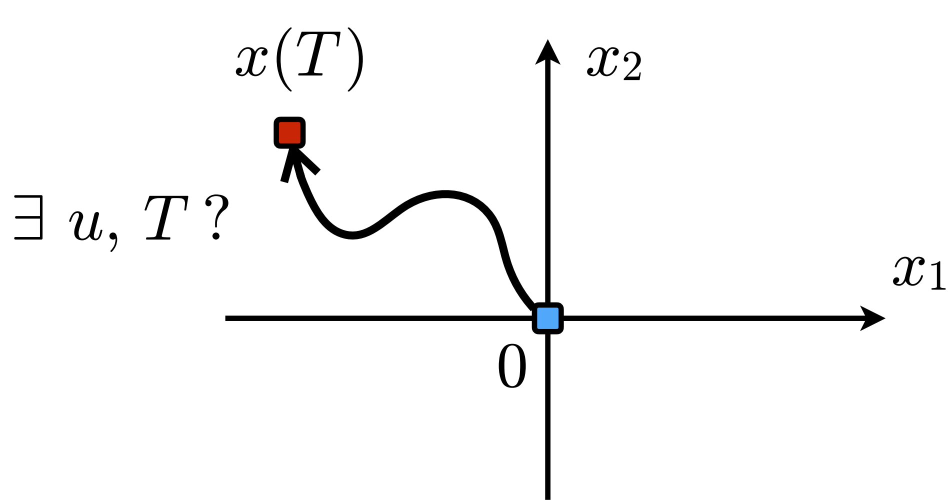
Hyp: Let us assume also that only λ_1 is controllable and therefore $v_2^{\mathrm{T}} B = 0$.

Problem:

we want to determine the set of states x_R that are reachable from the origin at some (finite) time interval T or, equivalently, the states x_R for which there exists an input $u(t)$ such that at some finite time T the state, starting from the zero initial condition, satisfies $x(T) = x_R$

reachable set at T

$$\left\{ x_R \ : \ \exists\, u(\cdot),\ x_R = \int_0^T e^{A(T-\tau)} B u(\tau) d\tau \right\}$$



Reachable states: an example

Since we start from the zero initial condition the state response is the ZSR

$$x(T) = \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

which for the considered system, being $v_1^{\mathrm{T}} B$ and $u(\tau)$ scalars, can be written

$$x_R = x(T) = \int_0^T e^{\lambda_1 (T-\tau)} u_1 v_1^T B u(\tau) d\tau = v_1^T B \left(\int_0^T e^{\lambda_1 (T-\tau)} u(\tau) d\tau \right) u_1$$

and since $v_1^{\mathrm{T}} B$ and the integral are nonzero (for a nonzero input) the set of reachable states is the eigenspace (consisting in all the vectors parallel to u_1) relative to the controllable eigenvector λ_1

$\Longrightarrow$ the set of reachable states is a linear subspace

The input $u(t)$ that steers the state from the origin to a state $x_R = \alpha u_1$ parallel to u_1 is such that

$$\alpha = v_1^T B \left(\int_0^T e^{\lambda_1 (T-\tau)} u(\tau) d\tau \right)$$

N.B. the result is independent from the specific value of T, therefore

$\Longrightarrow$ if a state is reachable at T it is also reachable at any finite time

Reachable states: an example

we could also ask which states x_C are **controllable to the origin**, that is for which there exists an input that drives the system from the state x_C to the origin in time T

$$0 = e^{AT} x_C + \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

$$x_C = -e^{-AT} \int_0^T e^{A(T-\tau)} B u(\tau) d\tau = -\int_0^T e^{-A\tau} B u(\tau) d\tau$$

in our example

$$x_C = -\int_0^T e^{-\lambda_1 \tau} u_1 v_1^T B u(\tau) d\tau = \underbrace{-v_1^T B \left(\int_0^T e^{-\lambda_1 \tau} u(\tau) d\tau \right)} u_1$$

we can generate any point along u_1
therefore this is the subspace of
reachable states

that is

- if a state is reachable from the origin it is also controllable to the origin
- controllability studies how the input influences the state and is independent of the choice of the output

Controllability matrix

- the subspace of controllable states is given by $\mathrm{Im}(P)$ where P is the **controllability matrix**

$$P = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

- the image of a matrix M is a subspace generated by the linearly independent columns of M
- if all the states are controllable the system is said to be controllable
- system is **controllable** $\Leftrightarrow$ $\mathrm{rank}(P) = n$
  $\Leftrightarrow$ $\det(P) \neq 0$
  $\Leftrightarrow$ (A,B) controllable
  $\Leftrightarrow$ all eigenvalues are controllable
  $\Leftrightarrow$ the controllability PBH test is verified for all eigenvalues
- what happens to the controllability matrix when we perform a change of coordinates $z = Tx$?

$$(A, B) \quad \rightarrow \quad \tilde{A} = TAT^{-1}, \quad \tilde{B} = TB \quad \rightarrow \quad \tilde{A}^k = TA^kT^{-1}$$

$$\longrightarrow \quad \tilde{P} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} TB & TAB & \dots & TA^{n-1}B \end{bmatrix} = TP$$

being T invertible, the rank of $\tilde{P}$ is the same as that of P.

- controllability is a structural property of the system (independent from the particular coordinates)

Basis of Image of P - example

example

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

controllability matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

therefore $\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ the same subspace is generated by $\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can be generated as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -0.5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Controllability Grammian (optional)

the following matrix $M(T)$ is called **Controllability Grammian**

$$M(T) = \int_0^T e^{A(T-\tau)} B B^T e^{A^T(T-\tau)} d\tau$$

we then have the following result (no proof)

$$\boxed{\text{the system is controllable if and only if } M(T) \text{ is nonsingular}}$$

if the system is controllable, an input that steers any state x_a to any state x_b at T is

$$u(t) = B^T e^{A^T(T-t)} M^{-1}(T) \left(x_b - e^{AT} x_a \right)$$

Proof

$$\begin{aligned} x(T) &= e^{AT} x_a + \int_0^T e^{A(T-\tau)} B B^T e^{A^T(T-\tau)} M^{-1}(T) \left(x_b - e^{AT} x_a \right) d\tau \\ &= e^{AT} x_a + M(T) M^{-1}(T) \left(x_b - e^{AT} x_a \right) = x_b \end{aligned}$$

■

this is the minimum energy input that steers x_a to x_b at T

Kalman decomposition w.r.t. controllability

what happens when the system S is **not controllable**?

- $\text{rank}(P) = m < n$ $\longrightarrow$ $\dim \text{Range}\,(P) = \dim \text{Image}\,(P) = \dim \text{Im}\,(P) = m$

  $\longrightarrow$ $\text{Im}\,(P) =$ subspace generated by m independent vectors

  we can choose as basis of $\text{Im}\,(P)$ the m linearly independent columns of P which we call $\{v_1, v_1, ..., v_m\}$

$$\text{Im}\,(P) = \text{gen}\,\{v_1, v_1, ..., v_m\}$$

define a change of coordinates T (nonsingular) such that

$$T^{-1} = [\underbrace{v_1 \quad v_2 \quad \cdots \quad v_m}_{\text{basis of } P} \quad \underbrace{v_{m+1} \quad \cdots \quad v_n}_{\text{completion}}] \; \Big] \; n \text{ components}$$

completion:
choose the remaining n-m n-dimensional vectors such that all the columns of T^{-1} are linearly independent

note that, by construction,
no vector v_k, for $k = m{+}1, \ldots, n$ (vector of the completion) belongs to $\text{Im}\,(P)$

Basis of Image of P - example

example

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

controllability matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

therefore $\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ the same subspace is generated by $\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can be generated as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -0.5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ we can choose different changes of coordinates

$$T^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

different basis of $\mathrm{Im}(P)$

different completion but always such that the matrix T is nonsingular

Kalman decomposition w.r.t. controllability

under this change of coordinates $z = Tx$ matrices A and B have a special structure

$$\tilde{A} = TAT^{-1} = \begin{array}{cc} \begin{array}{cc} m & n-m \end{array} & \\ \left[\begin{array}{c:c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hdashline 0 & \tilde{A}_{22} \end{array}\right] & \begin{array}{c} m \\ n-m \end{array} \end{array} \qquad \tilde{B} = TB = \left[\begin{array}{c} \tilde{B}_1 \\ \hdashline 0 \end{array}\right] \begin{array}{c} m \\ n-m \end{array}$$

$$\tilde{C} = CT^{-1} = \left[\begin{array}{c:c} \tilde{C}_1 & \tilde{C}_2 \end{array}\right]$$

no special structure
(controllability depends only upon A and B)

such that

- $\mathrm{rank}\left[\tilde{B}_1 \quad \tilde{A}_{11}\tilde{B}_1 \quad \tilde{A}_{11}^2\tilde{B}_1 \quad \ldots \quad \tilde{A}_{11}^{m-1}\tilde{B}_1\right] = m$
- that is $(\tilde{A}_{11}, \tilde{B}_1)$ controllable
- $\mathrm{eig}\left\{\tilde{A}\right\} = \mathrm{eig}\left\{\tilde{A}_{11}\right\} \bigcup \mathrm{eig}\left\{\tilde{A}_{22}\right\}$
- all the m eigenvalues of $\tilde{A}_{11}$ are controllable
- all the $n-m$ eigenvalues of $\tilde{A}_{22}$ are unncontrollable
- eigenvalues of $\tilde{A}_{22}$ will not be poles of the transfer function (independently from the choice of C)

Kalman decomposition w.r.t. controllability (partial proof)

- property of $\mathrm{Im}\,(P)$: **invariance** w.r.t. A

if v belongs to $\mathrm{Im}\,(P)$ then Av also belongs to $\mathrm{Im}\,(P)$

$$v \in \mathrm{Im}(P) \quad \Rightarrow \quad A\,v \in \mathrm{Im}(P)$$

- the subspace of controllable states is A-invariant in any coordinates $z = T\,x$

$$w \in \mathrm{Im}(\tilde{P}) = \mathrm{Im}(TP) \quad \Rightarrow \quad \tilde{A}w \in \mathrm{Im}(\tilde{P}) = \mathrm{Im}(TP)$$

- if v belongs to $\mathrm{Im}\,(P)$ it can be expressed as a linear combination of the basis of $\mathrm{Im}\,(P)$

$$v \in \mathrm{Im}(P) \quad \Rightarrow \quad v = \sum_{i=1}^{m} \beta_i v_i$$

in the new coordinates, $w = T\,v$ is controllable and being

$$v = T^{-1} w \in \mathrm{Im}(P) \quad \text{and} \quad v = \sum_{i=1}^{m} \beta_i v_i \quad \Rightarrow \quad w = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

recall that the first m columns of T^{-1} are the v_i

i.e., if w belongs to $\mathrm{Im}(\tilde{P})$ then its last n-m components are equal to 0 in the new coordinates $w = T\,v$

Kalman decomposition w.r.t. controllability (partial proof)

if $\mathrm{Im}\,(P)$ is **invariant** w.r.t. A then $\mathrm{Im}\,(TP)$ is invariant w.r.t. $A_c = TA\,T^{-1}$ (coordinate independent)

- in the new particular coordinates $w = T\,v$ if w belongs to $\mathrm{Im}\,(TP)$ it has the last n-m components equal to 0

$$w \in \mathrm{Im}(\tilde{P}) = \mathrm{Im}(TP) \quad \Rightarrow \quad w = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

- if w belongs to $\mathrm{Im}\,(TP)$ then also $A_c w$ belongs to $\mathrm{Im}\,(TP)$ since $\mathrm{Im}\,(TP)$ is A_c-invariant and therefore also $A_c w$ has the last n-m components equal to 0

$$A_c\, w \in \mathrm{Im}(P) \quad \Rightarrow \quad A_c\, w = \begin{bmatrix} \star \\ 0 \end{bmatrix}$$

- then, for a generic A_c the invariance of w belonging to $\mathrm{Im}\,(TP)$ w.r.t. A_c means that the following relationship must hold for **any** vector $[\,]$

$$\overbrace{\begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}}^{A_c} \begin{bmatrix} [\,] \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ 0 \end{bmatrix} \quad \Rightarrow \quad A_{c21} \cdot [\,] = 0 \quad \Rightarrow \quad A_{c21} = 0$$

- note that since B belongs to $\mathrm{Im}\,(P)$ then $B_c = T\,B$ also belongs to $\mathrm{Im}\,(TP)$ and therefore B_c has its last n-m components equal to 0

Kalman decomposition w.r.t. controllability

Let's partition z accordingly

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{array}{l} m \\ n - m \end{array} \quad \text{dimensions}$$

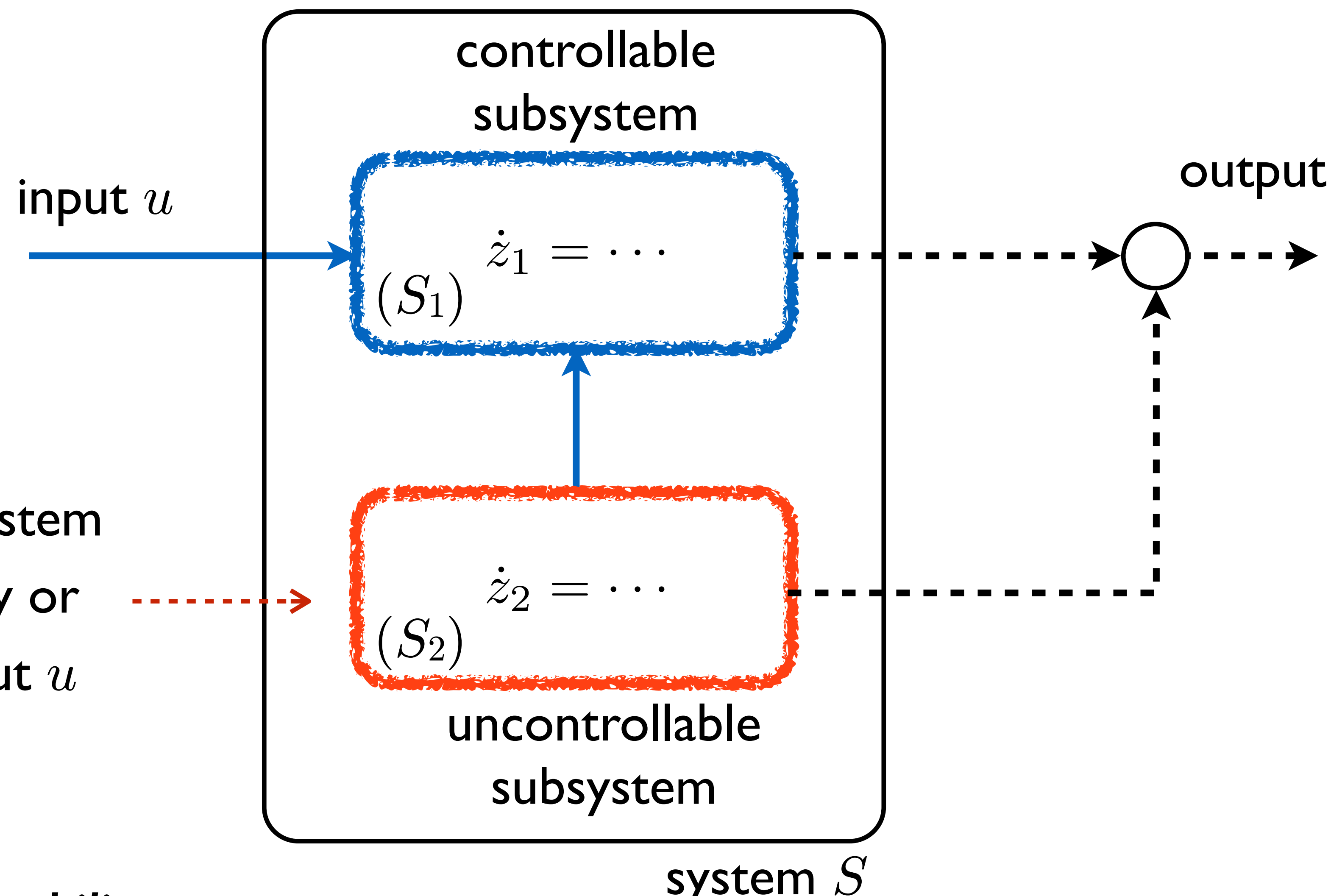
the two sets of equations can be viewed as **two interconnected subsystems**, (S_1) with state z_1 and (S_2) with state z_2

$$\begin{cases} \dot{z}_1 = \tilde{A}_{11} z_1 + \boxed{\tilde{A}_{12} z_2 + \tilde{B}_1 u} \\ \dot{z}_2 = \tilde{A}_{22} z_2 \\ y = \tilde{C}_1 z_1 + \tilde{C}_2 z_2 \end{cases}$$

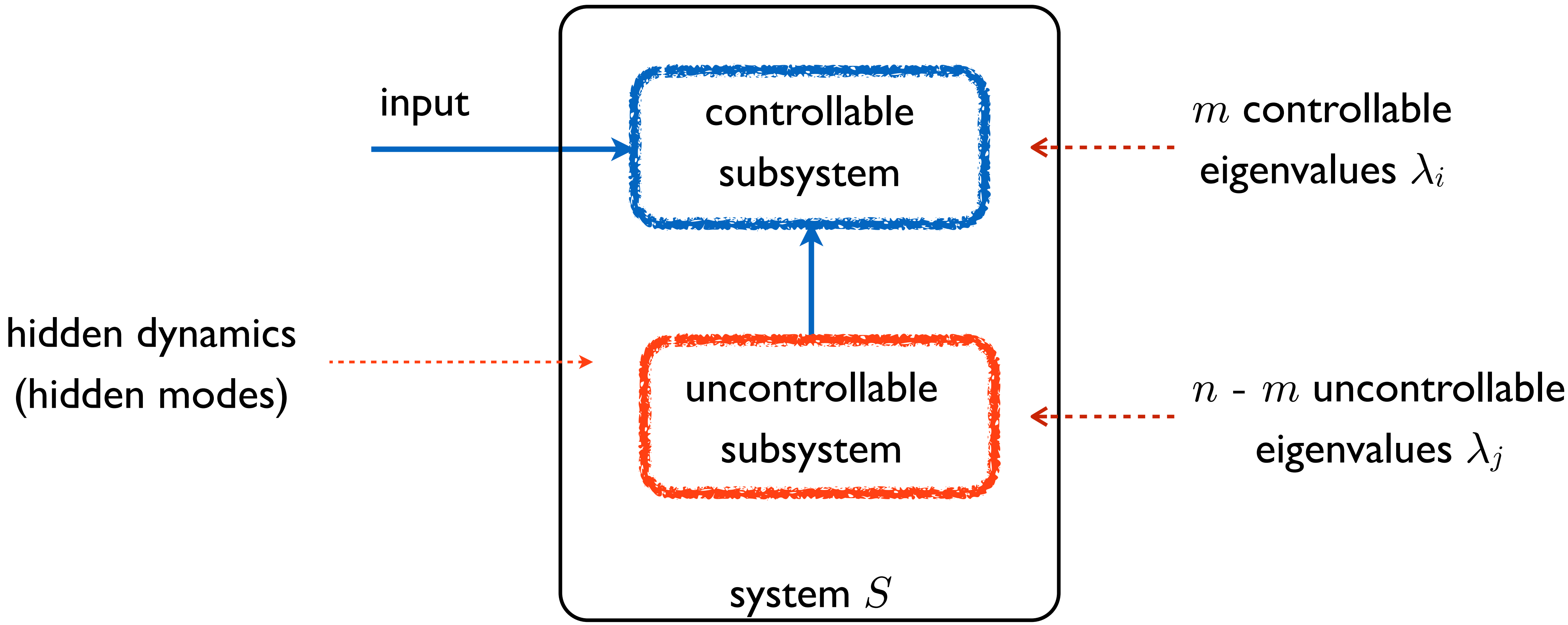
inputs of (S_1)

(S_2) is autonomous (no inputs)

the state evolution of this subsystem is **not influenced**, either directly or indirectly through z_1, by the input u



Kalman decomposition w.r.t. controllability



this decomposition plays a fundamental role in control

Undistinguishable and unobservable states

Consider the output evolution from two initial states x_a and x_b under the action of the same input

$$y_a(t) = Ce^{At}x_a + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \quad \text{and} \quad y_b(t) = Ce^{At}x_b + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

the two states x_a and x_b are said to be **indistinguishable** if $y_a(t) = y_b(t)$ for all t that is

$$y_b(t) - y_a(t) = Ce^{At}(x_b - x_a) = 0 \qquad \Leftrightarrow \qquad (x_b - x_a) \in \mathrm{Ker}\left[Ce^{At}\right], \quad \forall\, t \geq 0$$

and being $x_b - x_a = (x_b - x_a) - 0$, we can equivalently say that $x_b - x_a$ is indistinguishable from 0

Reminder: the kernel of a matrix M is the linear subspace $\quad \mathrm{Ker}(M) = \{v : Mv = 0\}$

A state which is indistinguishable from 0 is said to be **unobservable**

- the set of unobservable states is the subspace $\mathrm{Ker}(O)$ with O being the **observability matrix**

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

n x n matrix (for systems with one-dimensional output)

depends only on A and C and not on B, that is not on how the input affects the state dynamics

observability studies the state-to-output interaction

Observability matrix - properties

- the set of unobservable states is a linear subspace, the set of observable states not (in general)
- if all the states are observable the system is said to be observable
- system is **observable** $\Leftrightarrow$ $\text{rank}(O) = n$

  $\Leftrightarrow$ $\det(O) \neq 0$

  $\Leftrightarrow$ (A,C) observable

  $\Leftrightarrow$ all eigenvalues are observable

  $\Leftrightarrow$ the observability PBH test is verified for all eigenvalues
- what happens to the observability matrix when we perform a change of coordinates $z = Tx$?

$$(A, C) \quad \to \quad \tilde{A} = TAT^{-1}, \quad \tilde{C} = CT^{-1} \quad \to \quad \tilde{A}^k = TA^kT^{-1}$$

$$\tilde{O} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \cdots = \begin{bmatrix} CT^{-1} \\ CAT^{-1} \\ \vdots \\ CA^{n-1}T^{-1} \end{bmatrix} = OT^{-1}$$

being T invertible, the rank of $\tilde{O}$ is the same as that of O.

- Observability/unobservability is a structural property of the system

Observability Grammian (optional)

Note that only the free (or unforced) output response is involved in the observability property

Alternative definition of observable system

An unforced system is said to be observable if and only if it is possible to determine any (arbitrary) initial state $x_0 = x(0)$ using only a finite record $y(\tau)$ for $0 \leq \tau \leq t_f$ of the output.

In other words, if we observe the output ZIR for any nonzero time interval, we can determine the initial state from where the evolution originated.

a system is observable if and only if the observability grammian $G(t_f)$ is nonsingular

$$G(t_f) = \int_0^{t_f} e^{A^T\tau} C^T C e^{A\tau} d\tau$$ **observability grammian**

For every $0 \leq \tau \leq t_f$ we have $y(\tau) = Ce^{A\tau}x_0$, multiply from the left by $e^{A^T\tau}C^T$

and integrate from 0 to t_f

$$\int_0^{t_f} e^{A^T\tau} C^T y(\tau) d\tau = \int_0^{t_f} e^{A^T\tau} C^T C e^{A\tau} d\tau \, x_0 = G(t_f) x_0$$

$$\Rightarrow \qquad x_0 = G(t_f)^{-1} \int_0^{t_f} e^{A^T\tau} C^T y(\tau) d\tau$$

Kalman decomposition w.r.t. observability

What happens if the system S is **not observable**?

- the observability matrix is singular and is therefore not full rank: $\mathrm{rank}(O) = m < n$

  therefore the kernel or nullspace of O has dimension $n - m$

  $\mathrm{Ker}(O) = \mathrm{gen}\{v_1,\, v_2,\, \dots\, ,\, v_{n-m}\}$ is a linear subspace

- choose the particular change of coordinates T such that

$$T^{-1} = \begin{bmatrix} \underbrace{w_1 \quad \cdots \quad w_m}_{\substack{\text{completion} \\ \text{(such that all the columns} \\ \text{are linearly independent)}}} & \underbrace{v_1 \quad v_2 \quad \cdots \quad v_{n-m}}_{\text{basis of } \mathrm{Ker}(O)} \end{bmatrix}$$

- the subspace $\mathrm{Ker}(O)$ is A-invariant: if $v \in \mathrm{Ker}(O)$ then $A\, v \in \mathrm{Ker}(O)$

- in these new coordinates an unobservable state has its first m coordinates equal to zero

  therefore if $z = Tx$ is such that $z \in \mathrm{Ker}(\tilde{O})$ then $z = \begin{array}{c} \begin{bmatrix} 0 \\ * \end{bmatrix} \end{array} \begin{array}{l}]\; m \\ ]\; n-m \end{array}$

  and also z is $\tilde{A} = TAT^{-1}$ invariant so $\tilde{A}z$ will have the same structure with the last n-m elements equal to 0

Kalman decomposition w.r.t. observability

this change of coordinates $z = Tx$ decomposes the original system into two interconnected systems: one fully observable while the other is fully unobservable, the new A and C have special structures

$$\boxed{\tilde{A} = TAT^{-1} = \begin{array}{cc} \begin{array}{cc} m & n-m \end{array} & \\ \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} & \begin{array}{c} m \\ n-m \end{array} \end{array} \qquad \tilde{C} = CT^{-1} = \begin{array}{c} \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \\ \begin{array}{cc} m & n-m \end{array} \end{array}}$$

$$\tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$$ no special structure

- $(\tilde{A}_{11}, \tilde{C}_1)$ observable $\Leftrightarrow$ $\text{rank} \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_1 \tilde{A}_{11} \\ \cdots \\ \tilde{C}_1 \tilde{A}_{11}^{m-1} \end{bmatrix} = m$
- $\text{eig}\left\{\tilde{A}\right\} = \text{eig}\left\{\tilde{A}_{11}\right\} \bigcup \text{eig}\left\{\tilde{A}_{22}\right\}$
- all the m eigenvalues of $\tilde{A}_{11}$ are observable
- all the $n-m$ eigenvalues of $\tilde{A}_{22}$ are unobservable
- eigenvalues of $\tilde{A}_{22}$ will not be poles of the transfer function (independently from the choice of B)

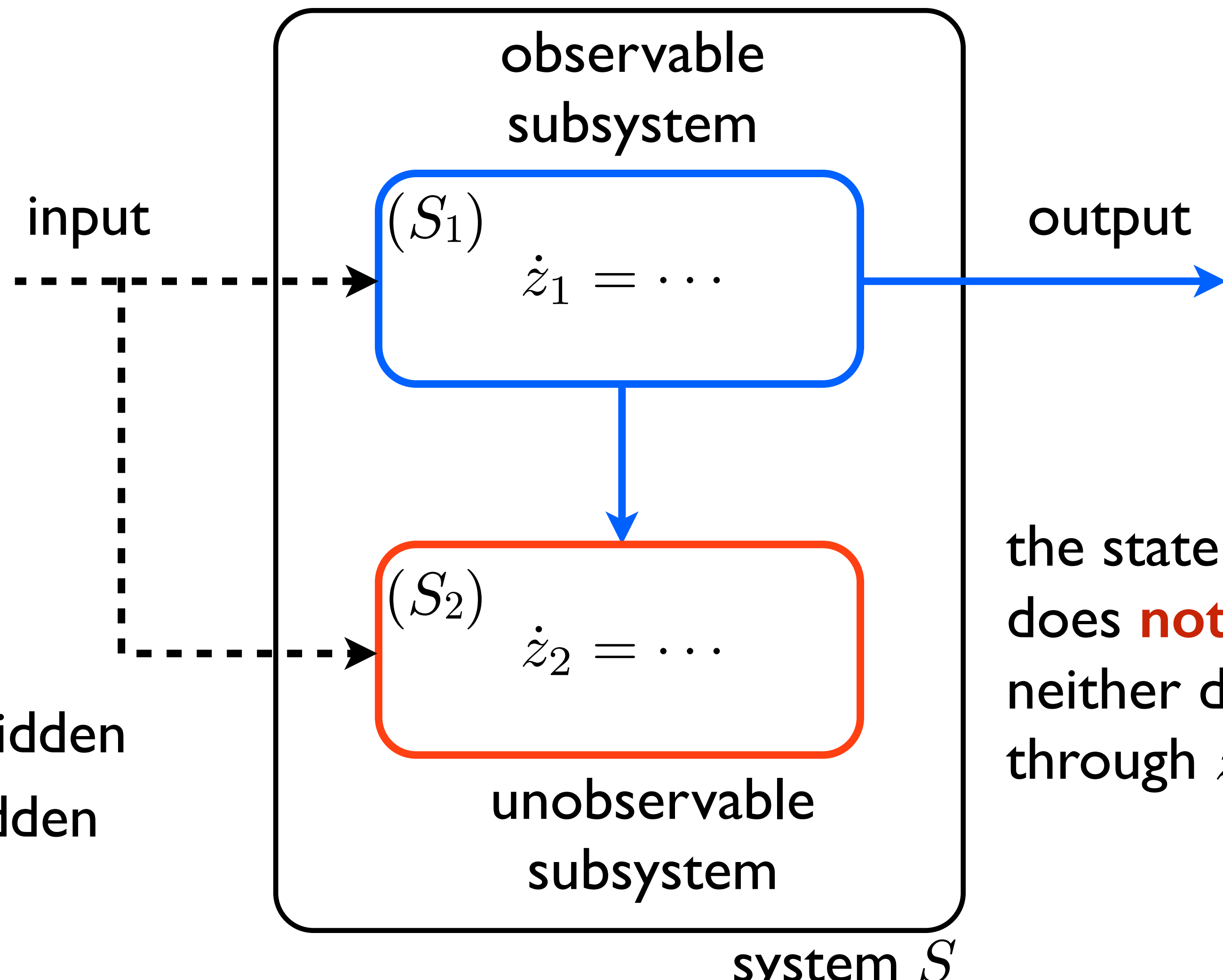
Kalman decomposition w.r.t. observability

Let's partition z accordingly

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{array}{l} m \\ n - m \end{array}$$

the two sets of equations can be viewed as **two interconnected subsystems**, (S_1) with state z_1 and (S_2) with state z_2

$$\begin{cases} \dot{z}_1 &= \tilde{A}_{11} z_1 + \tilde{B}_1 u \\ \dot{z}_2 &= \tilde{A}_{21} z_1 + \tilde{A}_{22} z_2 + \tilde{B}_2 u \\ y &= \tilde{C}_1 z_1 \end{cases}$$

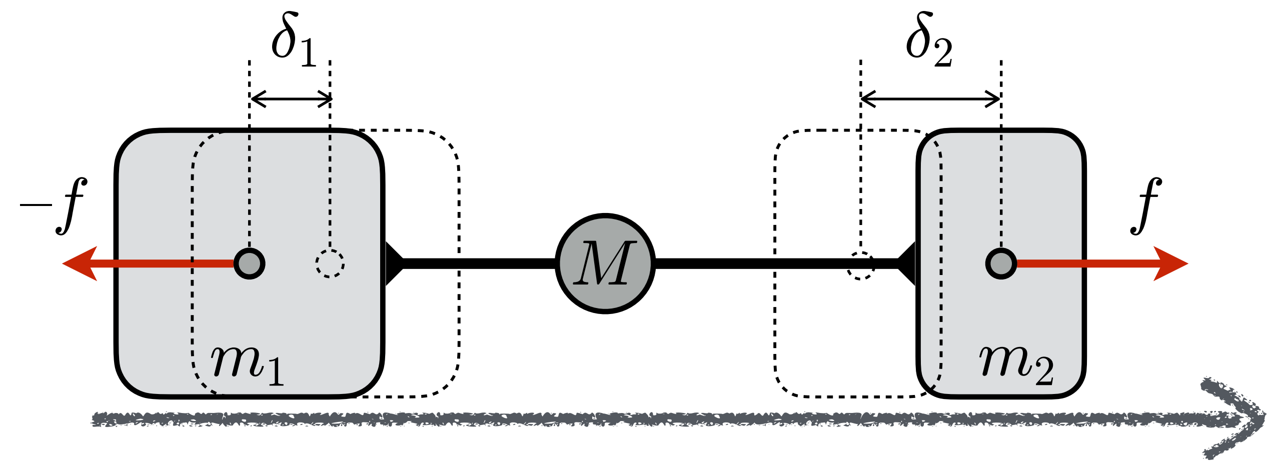


the state evolution of this subsystem does **not influence** the output y, neither directly nor indirectly through z_1

(S_2) represents the hidden dynamics with its hidden natural modes

Example

Consider two masses m_1 and m_2 with no friction with the ground and moved by a single linear actuator M which acts on both masses with equal intensity



Newton equations

$$m_1\ddot{\delta}_1 = -f$$
$$m_2\ddot{\delta}_2 = f$$

state (a possible choice)

$$x = \begin{bmatrix} \delta_1 \\ \dot{\delta}_1 \\ \delta_2 \\ \dot{\delta}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\rightarrow \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{bmatrix}$$

controllability matrix

$$P = \begin{bmatrix} 0 & -1/m_1 & 0 & 0 \\ -1/m_1 & 0 & 0 & 0 \\ 0 & 1/m_2 & 0 & 0 \\ 1/m_2 & 0 & 0 & 0 \end{bmatrix}$$

subspace of controllable states

$$\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{pmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{pmatrix}, \begin{pmatrix} -1/m_1 \\ 0 \\ 1/m_2 \\ 0 \end{pmatrix} \right\}$$

since the dimension of $\mathrm{Im}(P)$ is 2, the original system can be decomposed into two subsystems each 2-dimensional, that is each with a state vector of dimension 2.

Before proceeding to the decomposition, let's try to give some physical insight into the controllable and uncontrollable states of this system (or equivalently reachable/unreachable states from 0).

Example (continued)

- the set of reachable states is a 2-dimensional linear subspace. Recall also that if a state is reachable from the origin in time T it is reachable in any finite time

$$\mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{pmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{pmatrix}, \begin{pmatrix} -1/m_1 \\ 0 \\ 1/m_2 \\ 0 \end{pmatrix} \right\}$$

- let's interpret the first vector of $\mathrm{Im}(P)$: it is parallel to $\begin{pmatrix} 0 & 1 & 0 & -m_1/m_2 \end{pmatrix}^T$ and any vector belonging to the same 1-dimensional subspace (and therefore parallel to it) must have the second (x_2) and fourth (x_4) components proportional with $x_4 = -\left(m_1/m_2\right) x_2$ and $x_1 = x_3 = 0$.
  In terms of the physical coordinates this means that a reachable state at a generic instant T should be such that

$$x_4(T) = -(m_1/m_2)x_2(T) \;\longrightarrow\; m_1\dot{\delta}_1(T) + m_2\dot{\delta}(T) = 0$$

and $\quad x_1(T) = x_3(T) = 0 \;\longrightarrow\; \delta_1(T) = \delta_2(T) = 0$

or equivalently in terms of the center of mass velocity $\dot{c}$, from this first basis vector any reachable state at time T is such that

$$\delta_1(T) = \delta_2(T) = 0 \quad \text{and} \quad \dot{c}(T) = \frac{m_1\dot{\delta}_1(T) + m_2\dot{\delta}_2(T)}{m_1 + m_2} = 0$$

Example (continued)

- similarly for the second vector of $\mathrm{Im}(P)$: since it is parallel to $\begin{pmatrix}1 & 0 & -m_1/m_2 & 0\end{pmatrix}^T$ any vector parallel to it must have the first (x_1) and third (x_3) components proportional with $x_3 = -(m_1/m_2)\, x_1$ or, equivalently, $\delta_2 = -(m_1/m_2)\, \delta_1$. But it should also have the two velocities equal to 0 (second (x_2) and fourth (x_4) components of the vector).
  Therefore any reachable state (deriving from this vector of the basis of $\mathrm{Im}(P)$) at time T should be such that

$$\delta_2(T) = -\frac{m_1}{m_2}\delta_1(T) \;\rightarrow\; m_1\delta_1(T) + m_2\delta_2(T) = 0 \quad \Rightarrow \quad c(T) = \frac{m_1\delta_1(T) + m_2\delta_2(T)}{m_1 + m_2} = 0$$

  and that $\dot\delta_1(T) = \dot\delta_2(T) = 0$

  that is at the generic T the center of mass c is in 0 and the two masses have zero velocity.

- as a summary we have

  from the first vector $\left\{(\delta_1, \delta_2, \dot\delta_1, \dot\delta_2) : \; \dot c(T) = 0, \delta_1(T) = \delta_2(T) = 0\right\}$

  from the second vector $\left\{(\delta_1, \delta_2, \dot\delta_1, \dot\delta_2) : \; c(T) = 0, \dot\delta_1(T) = \dot\delta_2(T) = 0\right\}$

  any reachable state at $t = T$ is a linear combination of these two conditions that is

$$\left\{(\delta_1, \delta_2, \dot\delta_1, \dot\delta_2) : \; c(T) = 0 \text{ and } \dot c(T) = 0\right\}$$

Example (continued)

- note that
$$\delta_1(T) = \delta_2(T) = 0 \quad \Rightarrow \quad c(T) = 0$$
$$\dot{\delta}_1(T) = \dot{\delta}_2(T) = 0 \quad \Rightarrow \quad \dot{c}(T) = 0$$
but not necessarily vice-versa. However in characterizing the whole set of reachable states we have combined all conditions.

- from the subspace of reachable states at time T and recalling that if a state is reachable at T it is reachable in any instant (with a different input), we can therefore also state that when we apply any input (the force $f(t)$) we will always have $c(t) = 0$ and $\dot{c}(t) = 0$ for all $t \geq 0$ along the state trajectory.

  To clarify this, imagine that we have found an input $f(t)$ with t in $[0, T]$ that drives the state from the origin to a reachable state at time T, that is to a state such that $c(T) = 0$ and $\dot{c}(T) = 0$. We are stating that along the whole trajectory we should also have $c(t) = 0$ and $\dot{c}(t) = 0$. If this was not true, it means there would be a time $\bar{t} \in (0,T)$ when the state would have $c(\bar{t}) \neq 0$ and/or $\dot{c}(\bar{t}) \neq 0$ and this state has been reached using the input $f(t)$ with $t \in [0,\bar{t}]$ (the same input that drives the system in T at a reachable state but truncated at $\bar{t}$). This then means that this state at $\bar{t}$ is reachable which is not possible since it does not satisfy the reachability conditions $c(\bar{t}) = \dot{c}(\bar{t}) = 0$.

Example (continued)

- we have just shown that forced any trajectory of the state (starting from 0 initial conditions) will have the center of mass and its derivative remain in 0 at all times.

$$c(t) = \dot{c}(t) = 0, \qquad t \geq 0$$

- Note that the condition $\dot{c}(t) = 0$ is a direct consequence of the conservation of the linear momentum

$$m_1 \dot{\delta}_1(t) + m_2 \dot{\delta}_2(t) = \text{constant} = m_1 \dot{\delta}_1(0) + m_2 \dot{\delta}_2(0) = 0$$

Moreover, if we integrate the momentum conservation equation, we obtain

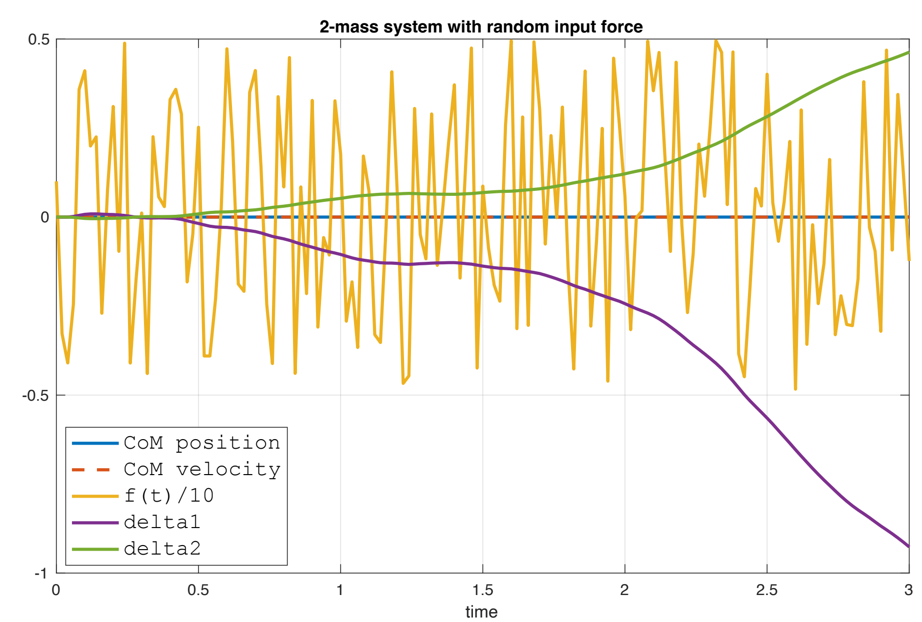
$$m_1(\delta_1(t) - \delta_1(0)) + m_2(\delta_2(t) - \delta_2(0)) = 0$$

which, being $\delta_1(0) = \delta_2(0) = 0$, gives

$$m_1 \delta_1(t) + m_2 \delta_2(t) = 0, \qquad \forall t$$

and therefore $\mathrm{c}(t) = 0$.

- Simulation with a random force: the relative CoM and its derivative remain 0 (one line is blue continuous the other is dashed in red and partially covers the blue line) while the displacements vary. The input force has been scaled by 10 to allow proper reading of the plot.



Example (continued)

- a more direct way to prove that any reachable state at time T is such that

$$c(T) = 0 \qquad \text{and} \quad \dot{c}(T) = 0$$

is writing the generic reachable x_R state as a linear combination of the base of $\mathrm{Im}(P)$

$$x_R = \alpha \begin{pmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{pmatrix} + \beta \begin{pmatrix} -1/m_1 \\ 0 \\ 1/m_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta/m_1 \\ -\alpha/m_1 \\ \beta/m_2 \\ \alpha/m_2 \end{pmatrix}$$

with α and β generic real numbers. So any reachable state at time T will have its components as

$$\delta_{1R} = -\frac{\beta}{m_1} \qquad \dot{\delta}_{1R} = -\frac{\alpha}{m_1} \qquad \delta_{2R} = \frac{\beta}{m_2} \qquad \dot{\delta}_{2R} = \frac{\alpha}{m_2}$$

therefore we have

$$m_1\delta_{1R} + m_2\delta_{2R} = m_1\left(-\frac{\beta}{m_1}\right) + m_2\left(\frac{\beta}{m_2}\right) = -\beta + \beta = 0, \quad \Longrightarrow \quad c = 0$$

$$m_1\dot{\delta}_{1R} + m_2\dot{\delta}_{2R} = m_1\left(-\frac{\alpha}{m_1}\right) + m_2\left(\frac{\alpha}{m_2}\right) = -\alpha + \alpha = 0 \quad \Longrightarrow \quad \dot{c} = 0$$

Example (continued)

Now that we have explored the controllability (or reachability) subspace, we can proceed with the Kalman decomposition w.r.t. controllability.
The first two columns of T^{-1} will form the basis of $\mathrm{Im}(P)$ while the remaining 2 will be a completion so that the matrix T is nonsingular.

There are infinite choices for the completion in T, all equivalent. For example:

- a first possible choice for the completion is

$$T^{-1} = \begin{bmatrix} 0 & -1/m_1 & 1 & 0 \\ -1/m_1 & 0 & 0 & 1 \\ 0 & 1/m_2 & 0 & 0 \\ 1/m_2 & 0 & 0 & 0 \end{bmatrix} \quad \longrightarrow \quad T = \begin{bmatrix} 0 & 0 & 0 & m_2 \\ 0 & 0 & m_2 & 0 \\ 1 & 0 & m_2/m_1 & 0 \\ 0 & 1 & 0 & m_2/m_1 \end{bmatrix}$$

we know that the last two components z_3 and z_4 of z, from $z = T\,x$, represent the state of the uncontrollable subsystem. These are

$$z_3 = \delta_1 + (m_2/m_1)\,\delta_2 \qquad z_4 = \dot{\delta}_1 + (m_2/m_1)\dot{\delta}_2$$

which are clearly proportional to the CoM and its derivative $(c, \dot{c})$.

Recall that, in the new coordinates, a controllable state has necessarily the structure $z = \begin{bmatrix} * \\ 0 \end{bmatrix}$ and therefore we can reach from the origin any position and velocity of the second mass (first two components of a controllable z) provided that the CoM and its velocity remain 0.

Example (continued)

Since for a controllable state we have $c = 0$, if with a force f the mass m_2 reaches at a given instant a relative position $\delta_2 = d$, the position of the mass m_1 at the same instant will be such that $c = 0$ is guaranteed, i.e., the mass m_1 should be at $\delta_1 = -\left(m_2/m_1\right) d$.

- A different possible choice for the completion is

$$T^{-1} = \begin{bmatrix} 0 & -1/m_1 & 0 & 0 \\ -1/m_1 & 0 & 0 & 0 \\ 0 & 1/m_2 & 1 & 0 \\ 1/m_2 & 0 & 0 & 1 \end{bmatrix} \longrightarrow T = \begin{bmatrix} 0 & -m_1 & 0 & 0 \\ -m_1 & 0 & 0 & 0 \\ m_1/m_2 & 0 & 1 & 0 \\ 0 & m_1/m_2 & 0 & 1 \end{bmatrix}$$

  the uncontrollable states are still given by the CoM and its derivative (or a linear combination of the two), while we can reach any position or velocity of the first mass.

- A third example for the completion is

$$T^{-1} = \begin{bmatrix} 0 & -1/m_1 & 0 & 0 \\ -1/m_1 & 0 & -1 & 1 \\ 0 & 1/m_2 & 1 & 0 \\ 1/m_2 & 0 & 0 & 0 \end{bmatrix} \longrightarrow T = \begin{bmatrix} 0 & 0 & 0 & m_2 \\ -m_1 & 0 & 0 & 0 \\ m_1/m_2 & 0 & 1 & 0 \\ m_1/m_2 & 1 & 1 & m_2/m_1 \end{bmatrix}$$

  again the uncontrollable states are the CoM and its derivative (or a linear combination of the two), while the controllable ones are, in this case, any position of the first mass and any velocity of the second mass

Example (continued)

All this is even more evident if we choose as new coordinates (state vector) the CoM c, its derivative $\dot{c}$, the relative distance r between the two masses and its derivative $\dot{r}$. This change of coordinates is represented by the following matrix T where $M = m_1 + m_2$

$$z = \begin{pmatrix} c \\ \dot{c} \\ r \\ \dot{r} \end{pmatrix} = \begin{pmatrix} m_1/M & 0 & m_2/M & 0 \\ 0 & m_1/M & 0 & m_2/M \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \dot{\delta}_1 \\ \delta_2 \\ \dot{\delta}_2 \end{pmatrix} = Tx$$

In these new coordinates, setting $H = 1/m_1 + 1/m_2$, we obtain

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ H \end{pmatrix}$$ and the controllability matrix $$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 \\ H & 0 & 0 & 0 \end{pmatrix}$$

- In these coordinates, the reachable (and controllable) states are more evident since a basis for $\mathrm{Im}(P_1)$ is given by a vector having only a nonzero relative distance r (second column of P_1) and another with only a nonzero relative velocity $\dot{r}$ (first column of P_1).

  We can, for example, reach any relative distance $\delta_2 - \delta_1 = d$. Then, since we also have the constraint $c = m_1\,\delta_1 + m_2\,\delta_2 = 0$, solving for δ_2 and δ_1 leads to

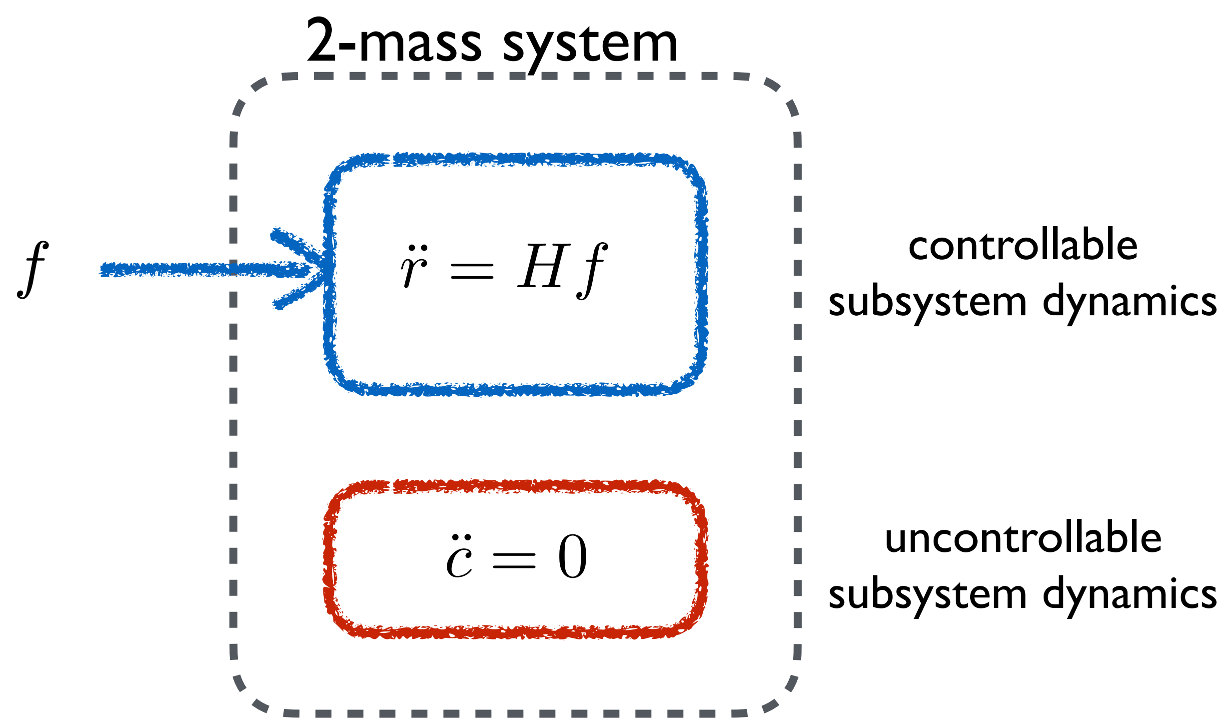
$$\delta_1 = -\frac{m_2}{M}d \qquad \delta_2 = \frac{m_1}{M}d$$

  A similar reasoning holds for the derivative of the relative distance, $\dot{r}$.

Example (continued)

These coordinates clearly already highlight the controllable and uncontrollable subsystems,

In this system there is absolutely no interaction between the two particular subsystems: relative distance dynamics and CoM dynamics



The Kalman decomposition, starting from these new coordinates, highlights similar controllable and uncontrollable subsystems (the new coordinates are practically a rearrangement of the old ones)

controllable subsystem dynamics

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

obtained with

$$T^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ H & 0 & 0 & 0 \\ 0 & H & 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 1/H & 0 \\ 0 & 0 & 0 & 1/H \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

uncontrollable subsystem dynamics

Example (continued)

We can now study the observability of the 2-mass system (in the original coordinates) but we need first to define the output.

- Let's assume we choose the relative position of the first mass as output $y_1 = \delta_1$ i.e.,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow O_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \mathrm{Ker}(O_1) = \mathrm{gen}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

  which clearly shows that the position and velocity of the second mass are unobservable

  Recall that we are considering the output ZIR so with no actuation, both masses move independently. Knowing the position of the first mass does not allow us to know anything about the other mass. We cannot say that since $c = 0$ we can deduce the position of the second mass.

- Let's assume we choose the relative velocity of the first mass as output $y_2 = \dot{\delta}_1$ i.e.,

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow O_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \mathrm{Ker}(O_2) = \mathrm{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

  now not only the position and velocity of the second mass are unobservable but also the position of the first mass

Example (continued)

- As a last example of choice output, let's assume we measure the CoM position c

$$C_3 = \begin{bmatrix} m_1/M & 0 & m_2/M & 0 \end{bmatrix} \Rightarrow O_3 = \begin{bmatrix} \frac{m_1}{M} & 0 & \frac{m_2}{M} & 0 \\ 0 & \frac{m_1}{M} & 0 & \frac{m_2}{M} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{Ker}(O_3) = \mathrm{gen}\left\{ \begin{pmatrix} \frac{1}{m_1} \\ 0 \\ -\frac{1}{m_2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ -\frac{1}{m_2} \end{pmatrix} \right\}$$

considering the first vector of $\mathrm{Ker}(O_3)$ which is parallel to $\begin{pmatrix} 1 & 0 & -m_1/m_2 & 0 \end{pmatrix}^T$, we see that an unobservable state should have its components verifying

$$\delta_2 = -\frac{m_1}{m_2}\delta_1 \quad \text{that is} \quad m_1\delta_1 + m_2\delta_2 = 0 \quad \Rightarrow \quad c = \frac{m_1\delta_1 + m_2\delta_2}{m_1 m_2} = 0$$

and similarly for the second vector parallel to $\begin{pmatrix} 0 & 1 & 0 & -m_1/m_2 \end{pmatrix}^T$ one has $m_1\dot{\delta}_1 + m_2\dot{\delta}_2 = 0$ which is the linear momentum conservation law

In other words, if we measure only the center of mass (in the absence of an input) we cannot distinguish between all those positions (δ_1, δ_2) which give a center of mass in 0.

Similarly, we cannot distinguish between all the masses velocities which satisfy the linear momentum conservation law

Example (continued)

- A common alternative choice (in particular for nonlinear mechanical systems) for the state is using the linear momentum instead of the velocity, that is choosing as state the following vector

$$z = \begin{bmatrix} \delta_1 & m_1\dot{\delta}_1 & \delta_2 & m_2\dot{\delta}_2 \end{bmatrix}$$

which corresponds to the following change of variables (trivial)

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix} \longrightarrow \qquad A_1 = \begin{bmatrix} 0 & 1/m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/m_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

However there is no real advantages for our system.