

Control Systems

Laplace domain analysis

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outline

- introduce the Laplace unilateral transform
- define its properties
- show its advantages in turning ODEs into algebraic equations
- define an Input/Output representation of the system through the transfer function
- explore the structure of the transfer function
- solve a realization problem

Laplace transform

real valued function
in the real variable t

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$t \in \mathbb{R} \qquad \qquad \qquad s \in \mathbb{C}$

complex valued function
in the complex variable s

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

**Laplace (unilateral)
transform**

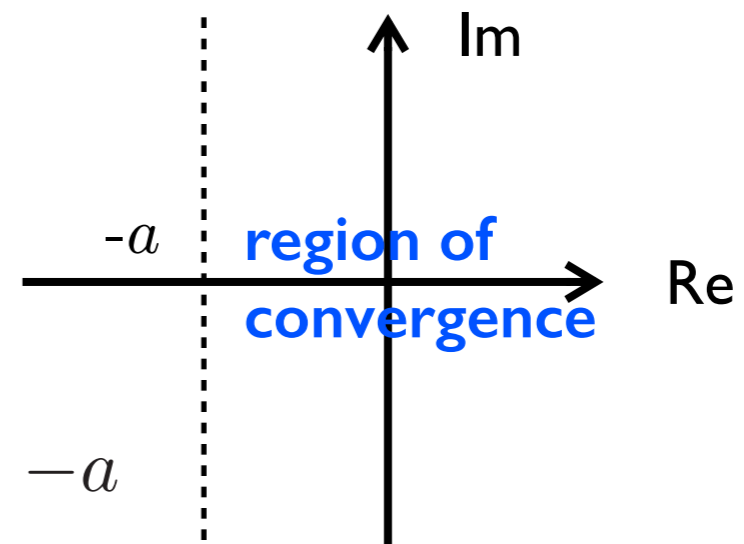
for $f(t)$ with no impulse and
no discontinuities at $t = 0$

$$F(s) = \mathcal{L}[f(t)]$$

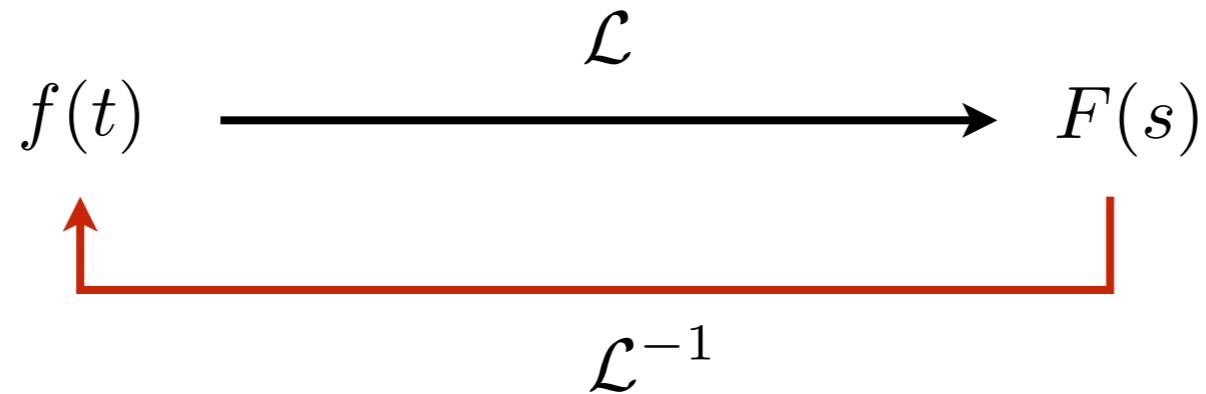
region of existence:

for all s with real part greater equal to
an **abscissa of convergence** σ_0

example $f(t) = e^{-at}, \quad a > 0, \quad \sigma_0 = -a$



inverse Laplace transform

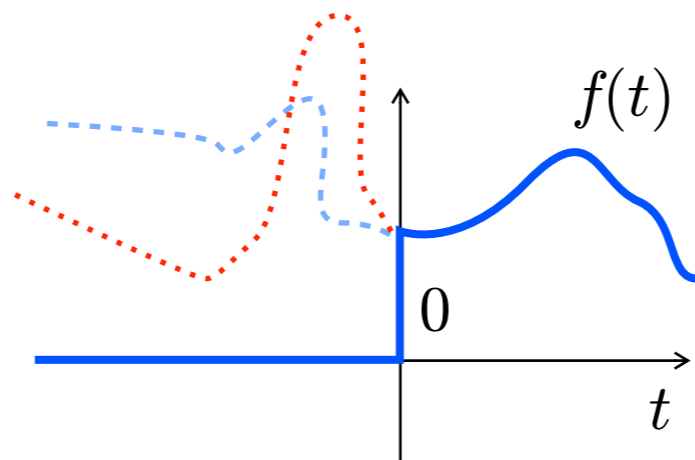
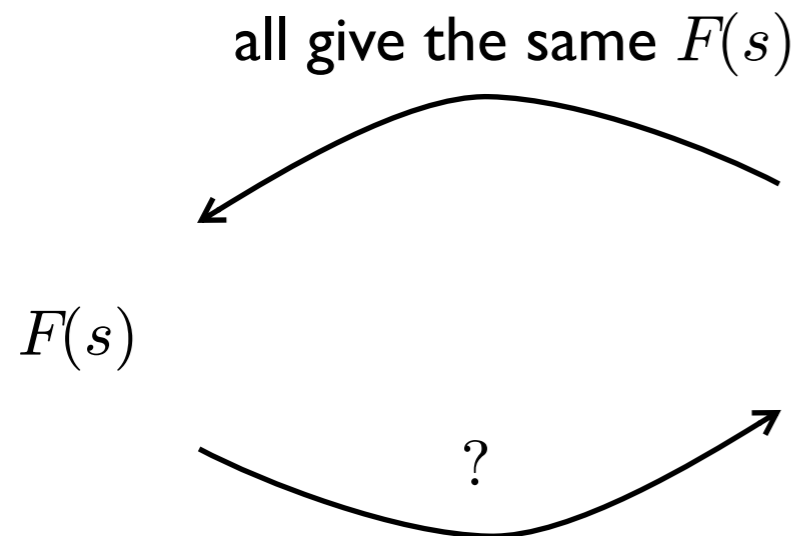


$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} F(s)e^{st} ds$$

unique for one-sided functions $f(t)$ defined for $t \geq 0$ or $f(t) = 0$ for $t < 0$

(one-to-one)

with this assumption the inverse Laplace transform is unique



Laplace transform

Laplace transform of the **impulse**

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

correct definition of the
Laplace transform

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st} dt = \int_{-\infty}^{\infty} \delta(t)e^{-st} dt = e^{-s0} = 1$$

Linearity

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

Derivative property

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\mathcal{L}[f(t)] - f(0)$$

can be applied iteratively

$$\mathcal{L}[\ddot{f}(t)] = s^2 \mathcal{L}[f(t)] - sf(0) - \dot{f}(0)$$

very useful in
model derivation

LTI systems

also useful here

$$\dot{x}(t) = Ax(t) + Bu(t) \quad x(0) = x_0$$

(proof): linearity + derivative property

algebraic solution

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B U(s)$$

and therefore

$$Y(s) = C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D] U(s)$$

LTI systems

state ZIR
transform

state ZSR
transform

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B U(s)$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

and therefore, comparing

$$\mathcal{L}[e^{At}] = (sI - A)^{-1}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \mathcal{L}[\delta_{-1}(t)] = \frac{1}{s}$$

Heaviside step function

$$\delta_{-1}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$




LTI systems


$$y(t) = C e^{At} x_0 + \int_0^t w(t - \tau) u(\tau) d\tau$$

with $w(t) = C e^{At} B + D \delta(t)$

transform $W(s) = C(sI - A)^{-1} B + D$


$$\begin{aligned} Y(s) &= C(sI - A)^{-1} x_0 + [C(sI - A)^{-1} B + D] U(s) \\ &= C(sI - A)^{-1} x_0 + W(s) U(s) \end{aligned}$$

general result



$$\mathcal{L} \left[\int_0^t w(t - \tau) u(\tau) d\tau \right] = W(s) U(s)$$

being $\mathcal{L}[\delta(t)] = 1$ the output response transform corresponding to $u(t) = \delta(t)$ is indeed $W(s)$

same for $H(s)$

transfer function

Input/Output behavior

$x_0 = 0$ (ZSR)

$$y_{\text{ZS}}(t) = \int_0^t w(t - \tau) u(\tau) d\tau$$

$$Y_{\text{ZS}}(s) = W(s) U(s)$$

Transfer function

$$\begin{aligned} W(s) &= C(sI - A)^{-1}B + D \\ &= \frac{Y_{\text{ZS}}(s)}{U(s)} \\ &= \mathcal{L}[w(t)] \end{aligned}$$

Input/Output behavior

independent from
state choice?

independent from state
space representation?

transfer function

$$(A, B, C, D) \xrightarrow{z = Tx \quad \det(T) \neq 0} (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$$

$$\begin{array}{ll} \dot{x} &= Ax + Bu & \dot{z} &= \tilde{A}z + \tilde{B}u \\ y &= Cx + Du & y &= \tilde{C}z + \tilde{D}u \end{array}$$

$$\tilde{A} = T A T^{-1} \quad \tilde{B} = T B \quad \tilde{C} = C T^{-1} \quad \tilde{D} = D$$

$$W(s) = C(sI - A)^{-1}B + D = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

for a given system, the transfer function is **unique**

(obvious since it's the Laplace transform of the impulsive response which is independent from the system representation)

shape of the transfer function

$$W(s) = C(sI - A)^{-1}B + D$$

inverse through the adjoint (transpose of the cofactor matrix)

$$\frac{1}{\det(sI - A)} C (\text{adjoint of } (sI - A)) B + D$$

$\text{cofactor}(i, j) = (-1)^{i+j} \text{minor}(i, j)$ polynomial of order $n - 1$

$$\frac{1}{\det(sI - A)} C (\text{cofactor of } (sI - A))^T B + D$$

polynomial of order $n - 1$

polynomial of order n

rational function
(strictly proper)

strictly proper rational function:

degree of numerator $<$ degree of denominator

proper rational function:

degree of numerator $=$ degree of denominator

we may have **cancellations of common factors** between the numerator and the denominator (final denominator degree may be less than n)

shape of the transfer function

$$W(s) = \underbrace{\text{strictly proper rational function}}_{\text{proper rational function}} + D$$

after
cancellations of
common terms

$$W(s) = \frac{N(s)}{D(s)}$$

$D = 0$	strictly proper rational function
$D \neq 0$	proper rational function

$$W(s) = \frac{N(s)}{D(s)} \xrightarrow{\text{roots}} \begin{matrix} \text{zeros} \\ \text{poles} \end{matrix} \quad \begin{matrix} \text{(for coprime } N(s) \text{ \& } D(s)) \\ \text{i.e. no common roots} \end{matrix}$$

from previous analysis the poles are a subset of the eigenvalues of A

$$\{\text{poles}\} \subseteq \{\text{eigenvalues}\}$$

more on this later

partial fraction expansion (distinct roots case)

Let $F(s) = \frac{N(s)}{D(s)}$ be a **strictly proper rational** function with **coprime** $N(s)$ and $D(s)$ and **distinct roots** of $D(s)$

i.e. $D(s) = a_n \prod_{i=1}^n (s - p_i) \Rightarrow F(s) = \frac{N(s)}{a_n \prod_{i=1}^n (s - p_i)}$

then $F(s)$ can be expanded as

$$F(s) = \sum_{i=1}^n \frac{R_i}{s - p_i}$$

with the **residues** R_i computed as

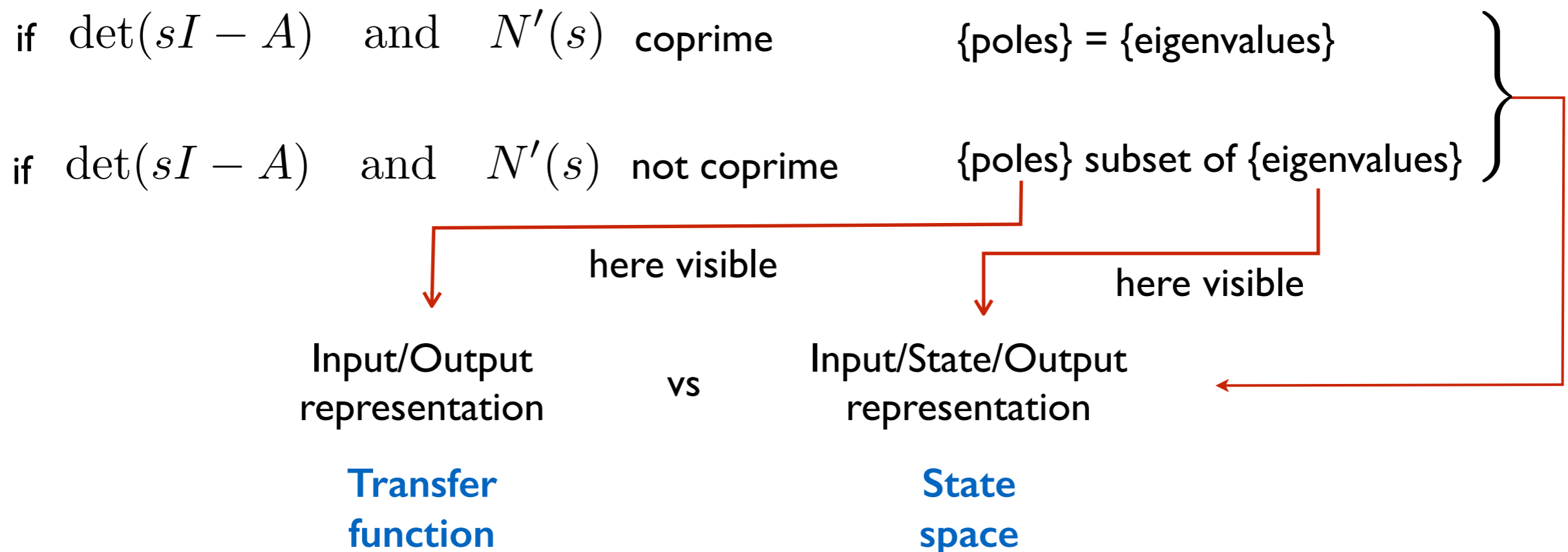
$$R_i = [(s - p_i)F(s)] \Big|_{s=p_i}$$

this result will be used for the

- transfer function $W(s)$
- output zero-state response $Y(s)$

poles & eigenvalues

$$W(s) = \frac{N(s)}{D(s)} \quad \xleftarrow{\quad ? \quad} \quad \text{from def} \quad W(s) = \frac{1}{\det(sI - A)} N'(s) \quad \xleftarrow{\quad \text{characteristic polynomial} \quad}$$



we need to understand when & why this happens
(so to understand when we can consider the transfer
function equivalent to a state space representation)

poles & eigenvalues

(distinct eigenvalues of A case)

n = state space dimension = dimension of A = number of eigenvalues

n_p = number of poles in $W(s)$

$D = 0$ case

partial fraction expansion

$$W(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^{n_p} (s - p_i)} = \sum_{i=1}^{n_p} \frac{R_i}{s - p_i}$$

done the same analysis
in t for $n = 2$

$$W(s) = \mathcal{L}[w(t)] = \mathcal{L}[C e^{At} B] = \mathcal{L}\left[C \left(\sum_{j=1}^n e^{\lambda_j t} u_j v_j^T \right) B\right] = \sum_{j=1}^n \frac{C u_j v_j^T B}{s - \lambda_j}$$

spectral form

if

$$\begin{array}{l} v_j^T B = 0 \\ \text{and/or} \\ C u_j = 0 \end{array}$$

the **eigenvalue** λ_j does **not** appear as a **pole**

we have a “**hidden mode**” associated to the eigenvalue λ_j
(see structural properties)

NB distinct eigenvalues is different from diagonalizable

poles & eigenvalues

(distinct eigenvalues of A case)

fact: all the natural modes appear in the state ZIR
for some generic initial condition $\longrightarrow e^{At}$

If for an eigenvalue λ_j we have that

$v_j^T B = 0$ implies the corresponding mode will not appear
in the state impulsive response $\longrightarrow e^{At} B = H(t)$

the corresponding mode (or eigenvalue) is said to be **uncontrollable**

$C u_j = 0$ implies the corresponding mode will not appear
in the output transition matrix $\longrightarrow C e^{At} = \Psi(t)$

the corresponding mode (or eigenvalue) is said to be **unobservable**

poles & eigenvalues

(distinct eigenvalues of A case)

Theorem

Every pole is an eigenvalue.

An eigenvalue λ_i becomes a pole if and only if it is both controllable and observable

$$v_i^T B \neq 0 \quad \text{and} \quad C u_i \neq 0$$

or equivalently the following two PBH rank tests are both verified

$$\text{rank} \left(\begin{array}{c|c} A - \lambda_i I & B \end{array} \right) = n$$

and

$$\text{rank} \left(\begin{array}{c} A - \lambda_i I \\ C \end{array} \right) = n$$

**Popov-Belevitch-Hautus
controllability test**

**Popov-Belevitch-Hautus
observability test**

(the PBH test could be tested for a generic λ but matrix $A - \lambda I$ loses rank only for $\lambda = \lambda_i$)

poles & eigenvalues

(distinct eigenvalues of A case)

Where does the PBH test comes from?

Observability (sketch):

$\text{rank} \begin{pmatrix} A - \lambda_i I \\ C \end{pmatrix} < n$ means that the rectangular $(n+1) \times n$ matrix has not full column rank and therefore it has a non-zero nullspace, that is there exists a n vector u_i such that

$$\begin{pmatrix} A - \lambda_i I \\ C \end{pmatrix} u_i = 0 \iff \begin{cases} (A - \lambda_i I) u_i = 0 \\ C u_i = 0 \end{cases} \iff \begin{cases} A u_i = \lambda_i u_i \\ C u_i = 0 \end{cases}$$

that is there exists an eigenvector which belongs to the nullspace of C (or the corresponding mode is unobservable)

example

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s-1)} \\ 0 & \frac{1}{s-1} \end{pmatrix} = M_1 \frac{1}{s+1} + M_2 \frac{1}{s-1}$$

fractional decomposition works also for rational matrices

with $M_1 = [(s+1)(sI - A)^{-1}]_{s=-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$

$$M_2 = [(s-1)(sI - A)^{-1}]_{s=1} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1 \end{pmatrix}$$

a different way to compute the matrix exponential

$$e^{At} = M_1 e^{-t} + M_2 e^t$$

both natural modes appear (as it should be) in the state transition matrix

example

$$(sI - A)^{-1}B = \begin{pmatrix} \frac{1}{s+1} \\ 0 \end{pmatrix} \longrightarrow \text{mode corresponding to } \lambda_2 \text{ does not appear} \longleftarrow e^{At}B$$

$$C(sI - A)^{-1} = \begin{pmatrix} 0 & \frac{1}{s-1} \end{pmatrix} \longrightarrow \text{mode corresponding to } \lambda_1 \text{ does not appear} \longleftarrow Ce^{At}$$

$W(s) = 0$
no poles

$$w(t) = Ce^{At}B = 0$$

forced response will always be zero
independently from the input applied
(look at the 2 first order ODE)

equivalently

$$u_1 \quad / \quad (A - \lambda_1 I)u_1 = 0 \quad \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} u_1 = 0 \quad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_1^T = [1 \quad -1/2]$$

$$u_2 \quad / \quad (A - \lambda_2 I)u_2 = 0 \quad \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} u_2 = 0 \quad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2^T = [0 \quad 1/2]$$

$$v_1^T B \neq 0 \quad v_2^T B = 0$$

$$Cu_1 = 0 \quad Cu_2 \neq 0$$

example

equivalently with the PBH rank test

controllability test

mode corresponding to λ_1 is controllable

$$\text{rank} \left(\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 2 & 0 \end{array} \right) = 2 = n$$

mode corresponding to λ_2 is uncontrollable

$$\text{rank} \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right) = 1 < n$$

observability test

$$\text{rank} \left(\begin{array}{cc} 0 & 1 \\ 0 & 2 \\ \hline 0 & 1 \end{array} \right) = 1 < n$$

mode corresponding to λ_1
is unobservable

$$\text{rank} \left(\begin{array}{cc} -2 & 1 \\ 0 & 0 \\ \hline 0 & 1 \end{array} \right) = 2 = n$$

mode corresponding to λ_2
is observable

Theorem

Every pole is an eigenvalue.

An eigenvalue λ_i becomes a pole with the multiplicity m_a (algebraic multiplicity) if and only if both PBH rank tests are verified

$$\text{rank} \left(\begin{array}{c|c} A - \lambda_i I & B \end{array} \right) = n$$

controllability

$$\text{rank} \left(\begin{array}{c} A - \lambda_i I \\ C \end{array} \right) = n$$

observability

NB - If one of the two conditions is not verified then the eigenvalue λ_i will appear as a pole with multiplicity strictly less than the algebraic multiplicity, possibly even 0 (in this case we will have a hidden eigenvalue). In particular the eigenvalue will appear **at most** as a pole with multiplicity equal to its **index** (dimension of the largest Jordan block).

NB - If for an eigenvalue λ_i the geometric $mg(\lambda_i) > 1$ then there exists a hidden dynamics.

example

$$A_0 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

PBH rank test
verified for

$$B = \begin{bmatrix} \forall \\ \forall \\ \neq 0 \end{bmatrix}$$

$$C = [\neq 0 \quad \forall \quad \forall]$$

easily seen from

$$A_0 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(sI - A_0)^{-1} = \frac{1}{(s - \lambda_1)^3} \begin{bmatrix} (s - \lambda_1)^2 & (s - \lambda_1) & 1 \\ 0 & (s - \lambda_1)^2 & (s - \lambda_1) \\ 0 & 0 & (s - \lambda_1)^2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_1 = [1 \quad 0 \quad 0]$$

$$\longrightarrow F_1(s) = \frac{1}{(s - \lambda_1)^3} \leftarrow \text{index of } \lambda_1$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = C_1 = [1 \quad 0 \quad 0]$$

$$\longrightarrow F_2(s) = \frac{(s - \lambda_1)^2}{(s - \lambda_1)^3} = \frac{1}{s - \lambda_1}$$

$$B_3 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = [0 \quad 0 \quad 1]$$

$$\longrightarrow F_3(s) = 0$$

example

$$A_4 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad \text{since} \quad A_4 - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the PBH rank test will never be satisfied independently from B and C . At most the eigenvalue will appear as a pole with multiplicity = index of $\lambda_1 = 2$

$$(sI - A_4)^{-1} = \frac{1}{(s - \lambda_1)^3} \begin{bmatrix} (s - \lambda_1)^2 & (s - \lambda_1) & 0 \\ 0 & (s - \lambda_1)^2 & 0 \\ 0 & 0 & (s - \lambda_1)^2 \end{bmatrix} = \frac{1}{(s - \lambda_1)^2} \begin{bmatrix} (s - \lambda_1) & 1 & 0 \\ 0 & (s - \lambda_1) & 0 \\ 0 & 0 & (s - \lambda_1) \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_4 = [1 \quad 0 \quad 0]$$

$$F_4(s) = \frac{1}{(s - \lambda_1)^2}$$

$$B_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = C_1 = [1 \quad 0 \quad 0]$$

$$F_5(s) = \frac{s - \lambda_1}{(s - \lambda_1)^2} = \frac{1}{s - \lambda_1}$$

$$B_6 = B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_6 = C_1 = [1 \quad 0 \quad 0]$$

$$F_6(s) = 0$$

example

$$A_7 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad (sI - A_7)^{-1} = \frac{1}{s - \lambda_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{etc ...}$$

the PBH rank test will never be satisfied independently from B and C . At most the eigenvalue will appear as a pole with multiplicity = index of $\lambda_1 = 1$

example

is the system characterized by the transfer function $\frac{1}{s^2}$ stable?

- when we start from the transfer function, we implicitly assume that the eigenvalues (and their algebraic multiplicity) coincide with the poles

fundamental assumption

- we have seen that an eigenvalue appears as a pole with multiplicity at most equal to its index (dimension of its larger Jordan block)
- for the pole multiplicity to be equal to both the index and the algebraic multiplicity of the eigenvalue, there must be only one Jordan block

thus the system has the eigenvalue in $\lambda = 0$ with geometric multiplicity = 1 and index = 2

system is **unstable**

can also be seen by computing the impulse response $\mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} = t \delta_{-1}(t)$

similarly for $\frac{1}{(s^2 + 1)^2}$

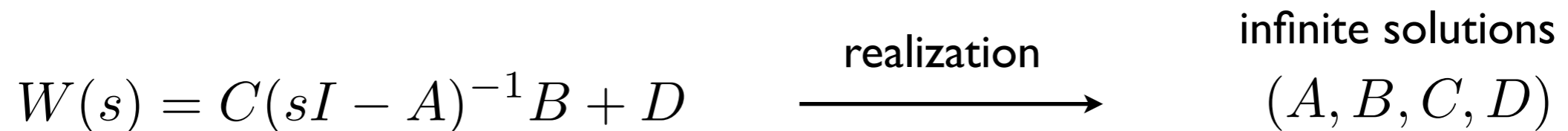
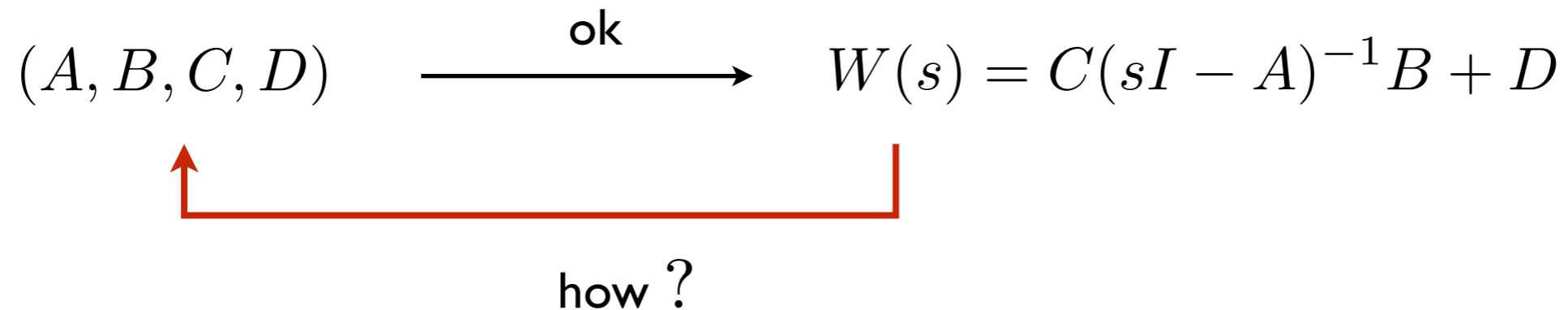
Laplace transform table

notation: in this list all the time functions start in $t = 0$, so when you read “Laplace transform of $\sin \omega t$ ” it is meant “Laplace transform of $(\sin \omega t)\delta_{-1}(t)$ ” and similarly for all other time functions.

$\delta(t)$	1
$\delta_{-1}(t)$	$\frac{1}{s}$
e^{at}	$\frac{1}{s - a}$
$\frac{t^k}{k!}$	$\frac{1}{s^{k+1}}$
$\frac{t^k}{k!} e^{at}$	$\frac{1}{(s - a)^{k+1}}$

$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2} = \frac{1/2j}{s - j\omega} - \frac{1/2j}{s + j\omega}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2} = \frac{1/2}{s - j\omega} + \frac{1/2}{s + j\omega}$
$\sin(\omega t + \varphi)$	$\frac{s \sin \varphi + \omega \cos \varphi}{s^2 + \omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$
$e^{at} \cos \omega t$	$\frac{(s - a)}{(s - a)^2 + \omega^2}$

realizations



- state dimension ?
- how can we easily find one state space representation (A, B, C, D) ?
- may be complicated for MIMO systems (here SISO)
- we see only one, obtainable directly from the coefficients of the transfer function (others are obtainable by simple similarity transformations) with state dimension = n (i.e. a **minimal realization**)

realizations

given $W(s) = \frac{N(s)}{D(s)}$ with $N(s)$ & $D(s)$ **coprime**

- first we determine D if $W(s)$ is proper strictly proper a non-zero D leads to $W(s)$ proper

$$W(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_1s + a_0} + D$$

- the state has dimension n and therefore the dynamic matrix is $n \times n$
- one possible choice for A, B, C (D has already been determined) is

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = [b_0 \quad b_1 \quad b_2 \quad \dots \quad b_{n-1}]$$

controller canonical form
(useful for eigenvalue assignment)

realizations

the matrix A_c is called a **companion matrix** and has as characteristic polynomial

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \rightarrow p_{A_c}(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

the pair (A_c, B_c) has, by construction, all its natural modes **controllable**

we then say that the pair (A_c, B_c) is, by construction, **controllable**

recall that the poles of the transfer function are also eigenvalues and therefore the matrix A_c of the realization has the minimum necessary number of eigenvalues

realizations (examples)

$$P(s) = \frac{1}{3s^2 + 2s + 6} = \frac{\frac{1}{3}}{s^2 + \frac{2}{3}s + 2}$$

$$\rightarrow A_c = \begin{bmatrix} 0 & 1 \\ -2 & -2/3 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_c = [1/3 \quad 0] \quad D_c = 0$$

$$P(s) = \frac{s^2 + 3}{2s^2 + 6s + 2} = \frac{s^2 + 3}{2(s^2 + 3s + 1)} = \frac{1}{2} \frac{s^2 + 3}{s^2 + 3s + 1} = \frac{1}{2} \left(1 + \frac{-3s}{s^2 + 3s + 1} \right) = \frac{-3/2s}{s^2 + 3s + 1} + \frac{1}{2}$$

$$\rightarrow A_c = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_c = [0 \quad -3/2] \quad D_c = \frac{1}{2}$$

$$P(s) = \frac{2s^4 + 13s^3 + 6s^2 + 6s + 7}{s^4 + 5s^3 + 3s^2 + 2s + 1} = \frac{3s^3 + 2s + 5}{s^4 + 5s^3 + 3s^2 + 2s + 1} + 2$$

$$\rightarrow A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -5 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C_c = [5 \quad 2 \quad 0 \quad 3] \quad D_c = 2$$