Control Systems

Dynamic response in the time domain (natural modes)

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outline

- A (real) diagonalizable
  - real eigenvalues (aperiodic natural modes)
  - complex conjugate eigenvalues (pseudoperiodic natural modes)
  - phase plots
- A (real) not diagonalizable
  - Jordan blocks and corresponding natural modes both for real and complex conjugate eigenvalues
  - special case: $\mathrm{Re}(\lambda_i) = 0$

what we know

We start from a state space representation

$$\mathcal{S} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$x \in \mathbb{R}^n$

$u \in \mathbb{R}^p$

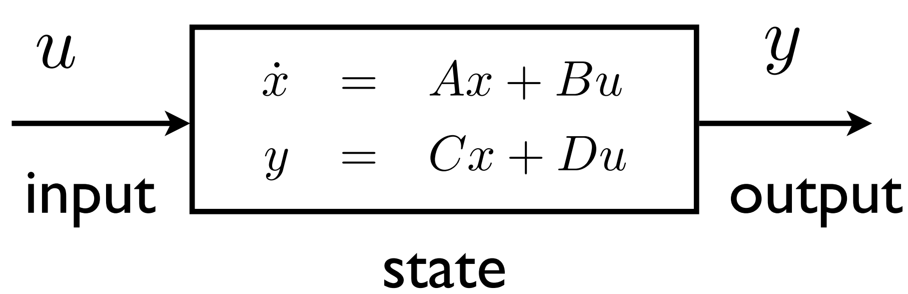
$y \in \mathbb{R}^q$

$A \; : \; n \times n$

$B \; : \; n \times p$

$C \; : \; q \times n$

$D \; : \; q \times p$



solution

$$x(t) = \Phi(t)x_0 + \int_0^t H(t-\tau)u(\tau)d\tau$$

← state ZIR + ZSR

$$y(t) = \Psi(t)x_0 + \int_0^t W(t-\tau)u(\tau)d\tau$$

← output ZIR + ZSR

with

$$\Phi(t) = e^{At} \qquad H(t) = e^{At}B$$

$$\Psi(t) = Ce^{At} \qquad W(t) = Ce^{At}B + D\delta(t)$$

what we need

we want to analyze the general solution $x(t)$ (and obviously also $y(t)$) so to be able to qualitatively describe the motion of our system and understand some of its basic properties (for example convergence/divergence of the state evolution, characteristics of the output time behavior, asymptotic behavior $\ldots$)

we need to be able to easily compute the exponential (or at least understand the important time functions that will be displayed in it)

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

note that the matrix exponential appears in all the following 4 terms

$$\Phi(t) = e^{At} \qquad H(t) = e^{At}B$$

$$\Psi(t) = Ce^{At} \qquad W(t) = Ce^{At}B + D\delta(t)$$

and thus in the free and forced evolutions for both the state and the output

how can linear algebra help?

we start from
the original system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

if $\exists T :$
$z = Tx$
$\det(T) \neq 0$

we obtain a different representation
of the same system
(same system - different state)

$$\begin{aligned}\dot{z} &= \widetilde{A}z + \widetilde{B}u \\ y &= \widetilde{C}z + \widetilde{D}u\end{aligned}$$

with $e^{\widetilde{A}t}$

easier to compute

$$e^{\widetilde{A}t} = e^{TAT^{-1}t} = Te^{At}T^{-1}$$

then

$$e^{At} = T^{-1}e^{\widetilde{A}t}T$$

- we want to find (if it exists) T such that $e^{\widetilde{A}t}$ is "easier to compute"
- if $e^{\widetilde{A}t}$ is easier to compute then also e^{At} is easier to compute
- what special structure should $\widetilde{A}$ in order to make $e^{\widetilde{A}t}$ easier to compute?

easiest case

Let's assume we have $\widetilde{A} = \Lambda$, a diagonal matrix, and compute its exponential

$$\Lambda = \text{diag}\{\lambda_i\} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad \Lambda^k = \text{diag}\{\lambda_i^k\} = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

$$e^{\Lambda t} = \sum_{k=0}^{\infty} \Lambda^k \frac{t^k}{k!} = \cdots = \begin{bmatrix} \sum_{k=0}^{\infty} \lambda_1^k t^k/k! & & & \\ & \sum_{k=0}^{\infty} \lambda_2^k t^k/k! & & \\ & & \ddots & \\ & & & \sum_{k=0}^{\infty} \lambda_n^k t^k/k! \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} = \text{diag}\{e^{\lambda_i t}\}$$

all off-diagonal terms are 0

matrix exponential of a diagonal matrix is immediate

how can linear algebra help?

diagonalizable case

- if the matrix is diagonal then its matrix exponential is immediate

- we found that a square matrix A could be diagonalized if and only if

$$mg(\lambda_i) = ma(\lambda_i) \quad \text{for all eigenvalues } \lambda_i$$

with the diagonalizing change of coordinates T defined as

$$T^{-1} = \mathcal{U}$$

- therefore we have $$e^{At} = T^{-1}\, e^{\Lambda t}\, T = \mathcal{U}\, e^{\Lambda t}\, \mathcal{U}^{-1}$$

→ $e^{\Lambda t}$ is straightforward and therefore also e^{At} is easy to compute

how can linear algebra help?

non-diagonalizable case

if $\widetilde{A} = \text{diag}\{J_i\}$ is **block diagonal**, is the matrix exponential also simplified?

yes

$$e^{\text{diag}\{J_i\}t} = \sum_{k=0}^{\infty} \text{diag}\{J_i\}^k \frac{t^k}{k!} = \cdots = \begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_r t} \end{bmatrix} = \text{diag}\{e^{J_i t}\}$$

- the exponential of a block diagonal matrix is still a block diagonal matrix with the exponentials of the single submatrices (blocks) on the diagonal

- moreover $\text{diag}\{e^{J_i t}\}$ has a special structure that we are going to explore (being J_i a Jordan block)

first summary

we want to compute explicitly the matrix exponential e^{At} and we understood that, in the proper coordinates, this reduces to the computation of the exponential of a diagonal matrix or a particular block diagonal matrix

e^{At}

- A diagonalizable $\longrightarrow$ $e^{At} = T^{-1}\, e^{\Lambda t}\, T = \mathcal{U}\, e^{\Lambda t}\, \mathcal{U}^{-1}$
- A non-diagonalizable $\longrightarrow$ $e^{At} = T^{-1}\, \text{diag}\left\{e^{J_i t}\right\}\, T$

what's next?

we are going to explore these two cases and understand the different **time functions** that are present so that we will be able to predict how, for example, the ZIR behaves qualitatively

NB. we will also need to consider the particular cases when the elements on the diagonal or the Jordan blocks correspond to complex and conjugate eigenvalues.

matrix exponential: A diagonalizable

A first result allows to move from the definition of matrix exponential involving an infinite sum to a spectral form of the matrix exponential which uses a finite sum of simple terms. Moreover these terms, in the real eigenvalue case, will also directly describe the type of motions which can be obtained in the state ZIR.

From $e^{At} = T^{-1} e^{\Lambda t} T = \mathcal{U} e^{\Lambda t} \mathcal{U}^{-1}$ being $T^{-1} = \mathcal{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$

we can rewrite explicitly

$$e^{At} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

hyp: the left eigenvectors v_i^T have been chosen so that $v_i^T u_j = \delta_{ij}$

from which we obtain the spectral form of the matrix exponential valid when A is diagonalizable (both real and/or complex eigenvalues)

spectral form of the matrix exponential

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} u_i v_i^T$$

matrix exponential: ZIR (A diagonalizable)

Since $x_{ZIR}(t) = e^{At}x_0$ we can explicitly write the Zero Input Response (ZIR) as

$$x_{ZIR}(t) = \sum_{i=1}^{n} e^{\lambda_i t} \boxed{u_i \, v_i^T \, x_0}$$

natural modes ← only functions of time

projection of the initial condition on the subspace generated by u_i } constant

NB $v_i^T x_0$ is a scalar

the time functions appearing in the matrix exponential will define the natural modes of the system which qualitatively represent the system behavior, in particular during the state ZIR, but also in all other three terms (state ZSR, output ZIR and ZSR)

→ How the zero-input response (free or unforced response) and more in general the whole response varies in time depends upon the **eigenvalues**

To acquire the **qualitative behavior** of the system motion, we need to distinguish the two cases:

- λ_i is real
- (λ_i, λ_i^*) complex conjugate

matrix exponential: A diagonalizable

we distinguish between real and complex eigenvalues

when A is diagonalizable

- if **real** eigenvalue λ_i the corresponding time function is $e^{\lambda_i t}$
- if **complex** eigenvalues $(\lambda_i, \lambda_i{}^*)$ with $\lambda_i = \alpha_i + j\omega_i$ the corresponding matrix exponential in the proper coordinates will be:

$$\begin{bmatrix} e^{\lambda_i t} & 0 \\ 0 & e^{\lambda_i^* t} \end{bmatrix}$$

but instead of having the time functions $e^{(\alpha_i + j\omega_i)t}$ and $e^{(\alpha_i - j\omega_i)t}$ we want real functions of time so we need to go through the real system representation of the matrix exponential

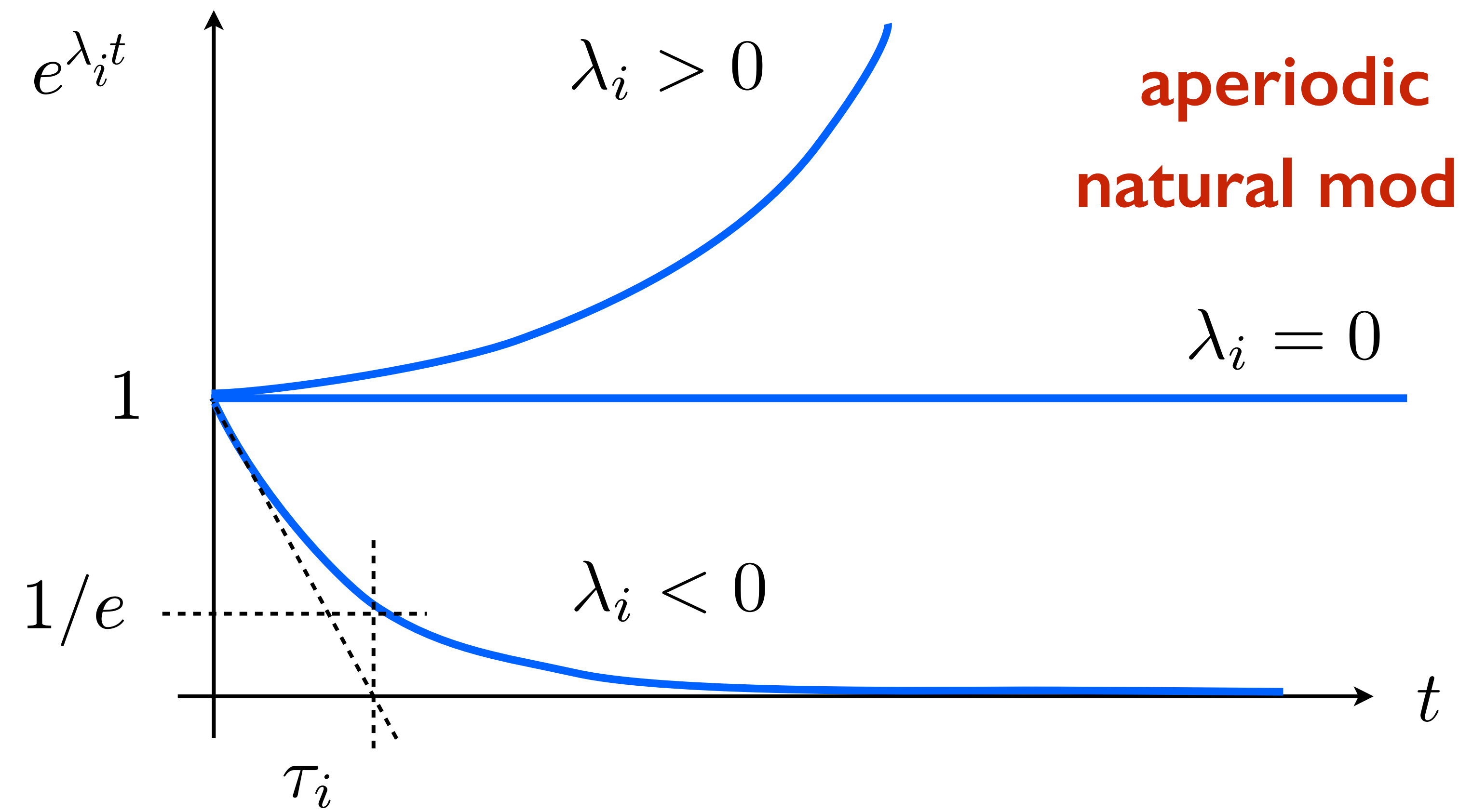
$$e^{\begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} t}$$

→ we need to expand this exponential

A diagonalizable - real eigenvalues

a real eigenvalue λ_i generates the natural mode $e^{\lambda_i t}$ which is defied as an **aperiodic natural mode**

depending on the sign of the real eigenvalue, we obtain completely different time evolutions



when the eigenvalue λ_i is negative, it is common to describe the decaying exponential through the time interval it takes to go from 1 to $1/e$

$$e^{\lambda_i t} = e^{-t/\tau_i} \qquad \text{with} \qquad \tau_i = -\frac{1}{\lambda_i} \qquad \textbf{time constant}$$

the smaller the time constant τ_i the faster the natural mode decays to 0

initial conditions

When A is diagonalizable and the eigenvalues are real, from the spectral representation of the matrix exponential we can interpret the effect of the matrices $u_i\, v_i^T$ on the initial condition

$$x_{ZIR}(t) = \sum_{i=1}^{n} e^{\lambda_i t} \boxed{u_i\, v_i^T\, x_0}$$

how to see the contribution of an initial condition to each natural mode

$$x_0 \begin{cases} u_i v_i^T = P_i \\ \text{or} \\ x_0 = \sum_{i=1}^{n} c_i u_i \end{cases}$$

use the projection matrices

$$u_i v_i^T x_0 = P_i x_0 = c_i u_i$$

express the initial condition in the base given by the eigenvectors

- $A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$ $\quad u_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad v_1^T = \frac{1}{3}\begin{pmatrix} 2 & 1 \end{pmatrix} \Rightarrow u_1 v_1^T = \begin{pmatrix} 4/3 & 2 \\ 2/3 & 1 \end{pmatrix}$

  $\lambda_1 = -1 \qquad \lambda_2 = 2$ $\quad u_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad v_2^T = \frac{1}{3}\begin{pmatrix} -1 & -2 \end{pmatrix} \Rightarrow u_2 v_2^T = \begin{pmatrix} -1/3 & -2 \\ 2/3 & 4 \end{pmatrix}$

A diagonalizable - real eigenvalues

example: real eigenvalue ($n = 2$) the 2D plot in the (x_1, x_2) plane displays the state trajectories for different initial conditions

case with $\lambda_1 > 0$ and $\lambda_2 < 0$

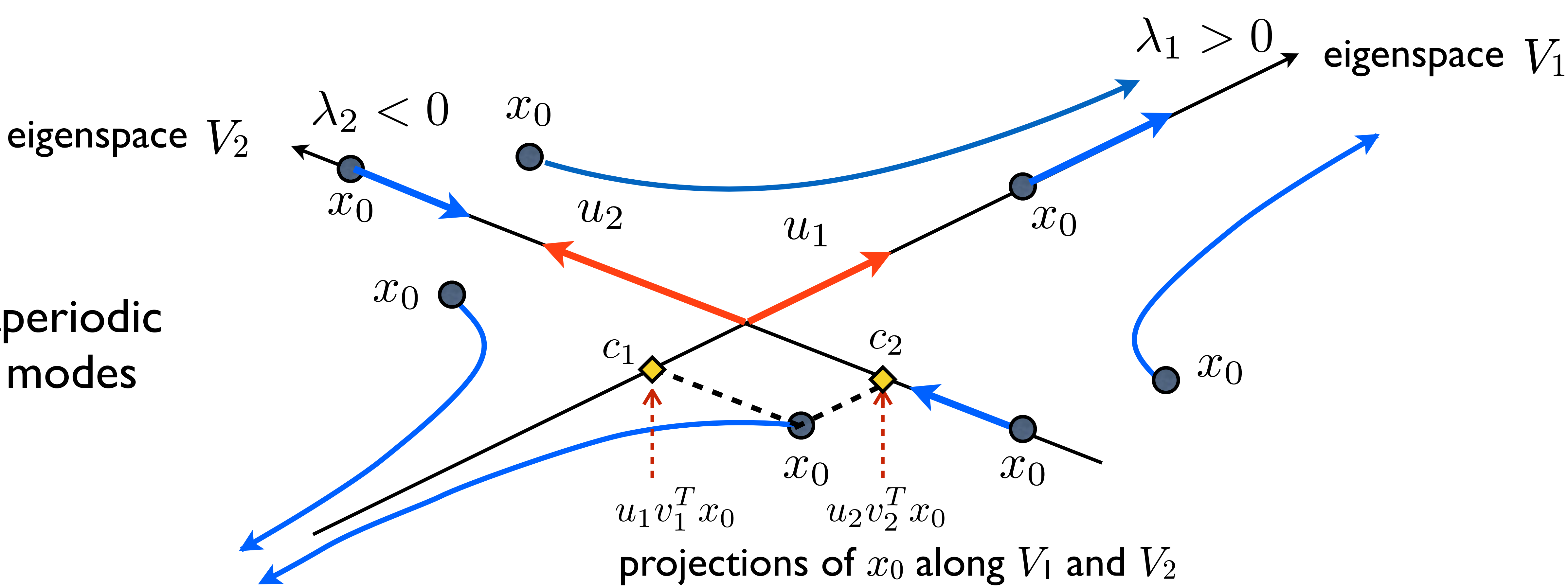
for $n = 2$

$$x_{ZIR}(t) = \sum_{i=1}^{n} e^{\lambda_i t} u_i v_i^T x_0 \rightarrow x_{ZIR}(t) = e^{\lambda_1 t} \boxed{u_1 v_1^T} x_0 + e^{\lambda_2 t} \boxed{u_2 v_2^T} x_0$$

$$= e^{\lambda_1 t} u_1 c_1 + e^{\lambda_2 t} u_2 c_2$$

projection matrices ($u_1 v_1^T$, $u_2 v_2^T$)

scalars (c_1, c_2)



examples

- $\dot{v} = \frac{1}{m} F$ eigenvalue $\lambda_1 = 0$ natural mode $e^{0t} = 1$
- $A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ eigenvalues $\lambda_1 =$ -2 $\lambda_2 = 1$ natural modes e^{-2t} and e^t

- **Mass-Spring-Damper (MSD)**

from the second order ODE we found the state space model with dynamic matrix A

$$m\ddot{s} + \mu\dot{s} + ks = u \rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{bmatrix} \rightarrow \lambda_{1/2} = \frac{-\mu \pm \sqrt{\mu^2 - 4km}}{2m}$$

the eigenvalues are:

- real when we have high or critical damping $\mu \geq 2\sqrt{k\,m}$
- complex conjugate when we have low damping $\mu < 2\sqrt{km}$

- compute eigenvalues and check, for the real case, the sign
- discuss how the eigenvalues and therefore the ZIR varies with μ in the real eigenvalues case

example: chemical reaction

consider first order and reversible chemical reactions between the two components A and B with reaction rates k_d and k_i

reversible reaction

$$A \xrightarrow{k_d} B \qquad B \xrightarrow{k_i} A$$

C_A is the concentration of the component A

C_B is the concentration of the component B

the reaction dynamics are described by the following differential equations

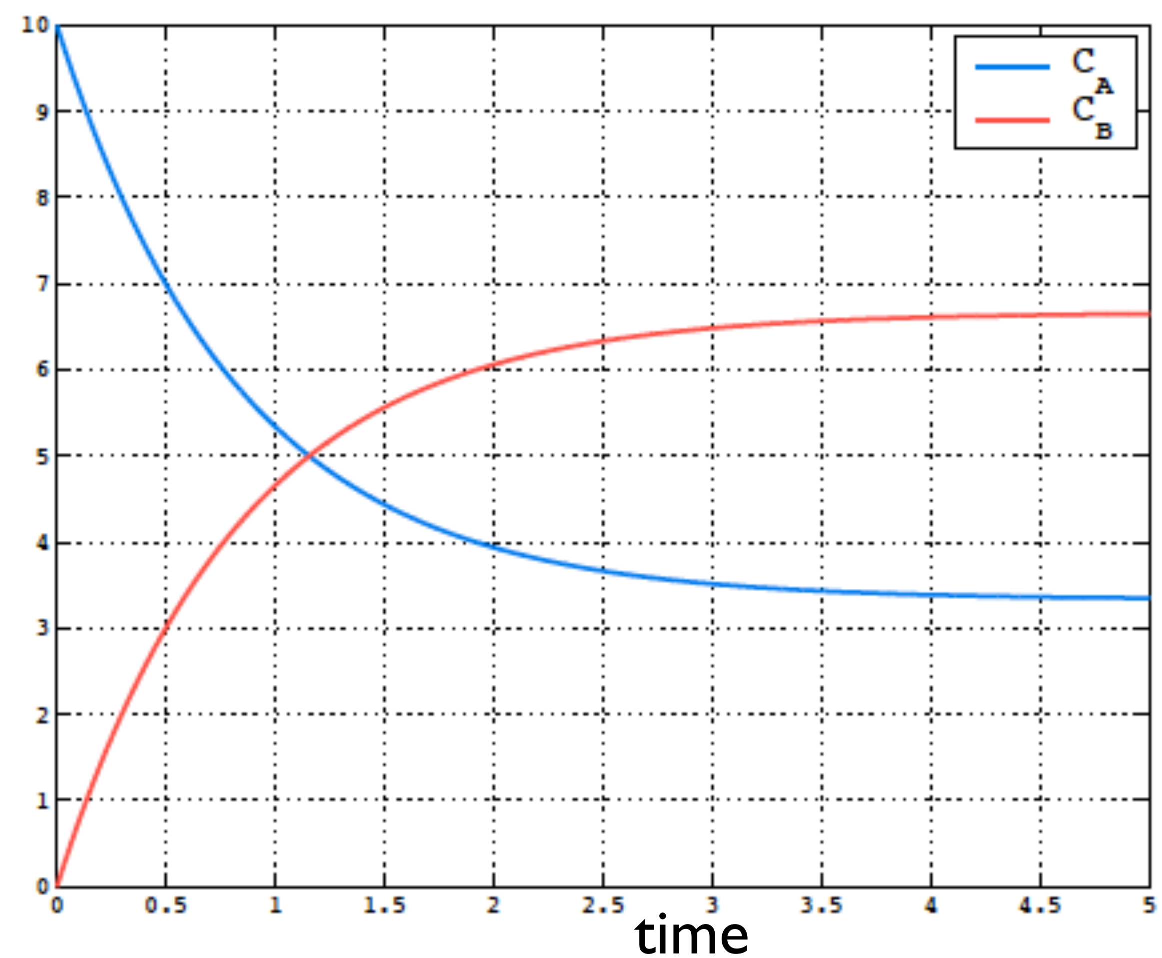
$$\frac{d\,C_A}{dt} = -k_d\,C_A + k_i\,C_B$$
$$\frac{d\,C_B}{dt} = k_d\,C_A - k_i\,C_B$$

possible exercise:

- find eigenvalues and interpret
- find diagonalizing change of coordinates
- draw the phase plane trajectories

$$\dot{C}_A + \dot{C}_B = 0 \qquad \text{mass conservation}$$

$$C_A(t) + C_B(t) = C_A(0) + C_B(0)$$



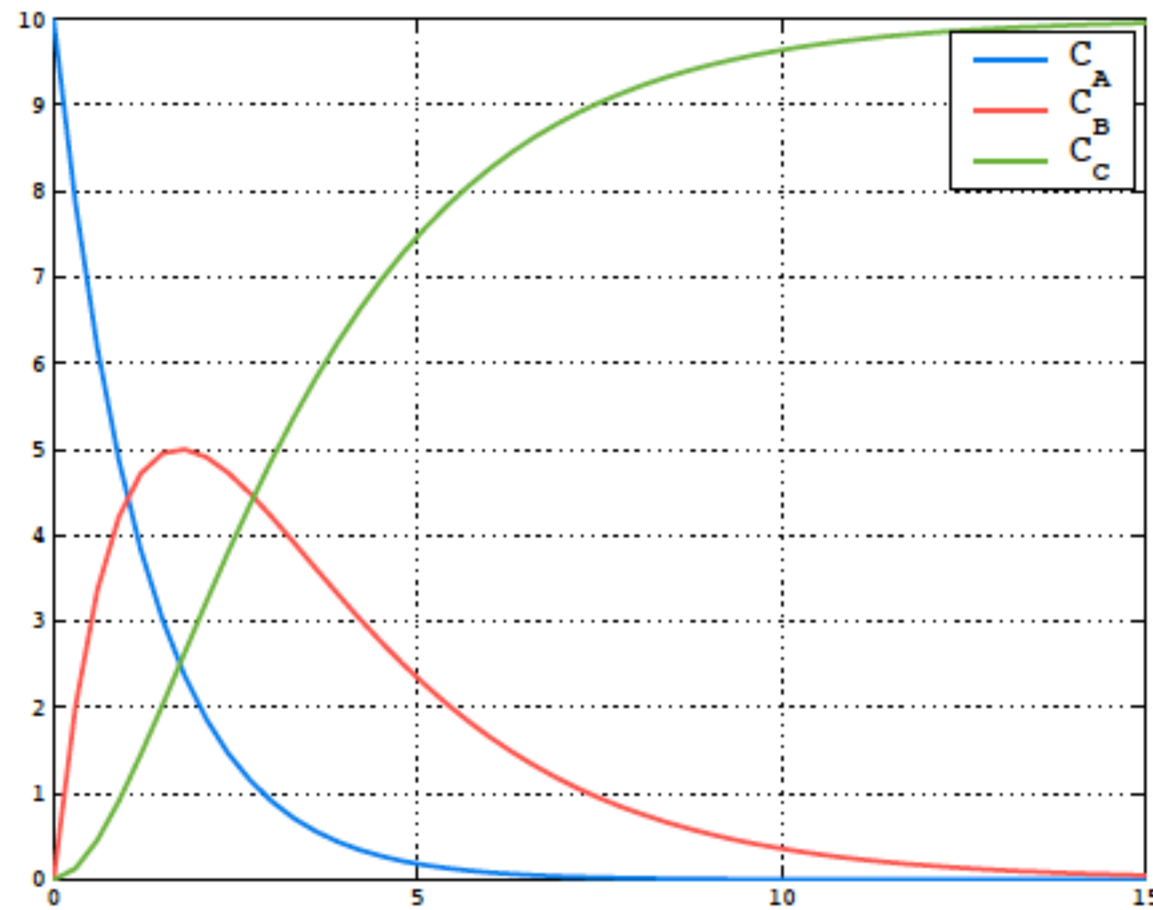
system evolution from the initial conditions $C_A(0) = 10$ and $C_B(0) = 0$

example: chemical reaction

chemical chain reactions between the three components A, B and C $\quad A \xrightarrow{k_1} B \xrightarrow{k_2} C$

the three concentrations satisfy the following differential equations

$$
\begin{aligned}
\frac{d\,C_A}{dt} &= -k_1\,C_A \\
\frac{d\,C_B}{dt} &= k_1\,C_A - k_2\,C_B \\
\frac{d\,C_C}{dt} &= k_2\,C_B
\end{aligned}
$$



possible exercise:

- find eigenvalues and interpret
- find diagonalizing change of coordinates

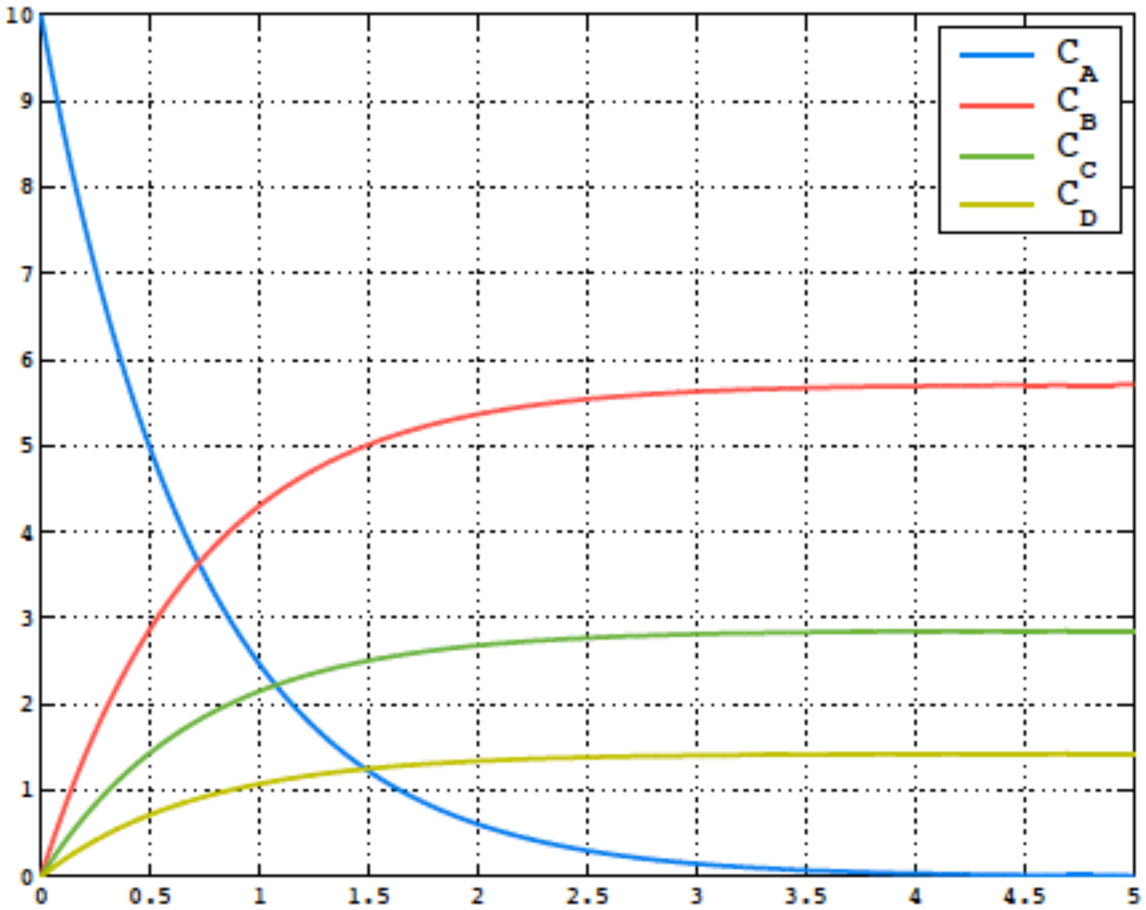
$$C_A(t) + C_B(t) + C_C(t) = \text{initial value}$$

example: chemical reaction

chemical parallel reaction of three components A, B and C

$$A \xrightarrow{k_1} B$$
$$A \xrightarrow{k_2} C$$
$$A \xrightarrow{k_3} D$$

$$\frac{d\,C_A}{dt} = -(k_1 + k_2 + k_3)\,C_A$$
$$\frac{d\,C_B}{dt} = k_1\,C_A$$
$$\frac{d\,C_C}{dt} = k_2\,C_A$$
$$\frac{d\,C_D}{dt} = k_3\,C_A$$



A diagonalizable - complex eigenvalues

Let us now consider the case of a pair of complex and conjugate eigenvalues (λ_i, λ_i^*) with $\lambda_i = \alpha_i + j\omega_i$

if A (real) diagonalizable $\quad \exists \; T_R$ such that $\; T_R A T_R^{-1} = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$ (real form)

the free state response or ZIR is

$$x_{ZIR}(t) = e^{At} x_o = T_R^{-1} e^{\begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} t} T_R x_0$$

we need to:

- compute

$$e^{\begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} t}$$

- use the change of coordinates T_R that puts a generic $(2$ x $2)$ matrix A with complex eigenvalues into the real form

$$T_R A T_R^{-1} = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$$

A diagonalizable - complex eigenvalues

- 1st step: (λ_i, λ_i^*) with $\lambda_i = \alpha_i + j\omega_i$

$$e^{\begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} t} = e^{\left(\begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i \end{bmatrix} t + \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} t\right)}$$

$$= e^{\begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i \end{bmatrix} t} \cdot e^{\begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} t}$$

since matrices commute

$$= e^{\alpha_i t} I \cdot \left(\sum_{k=0}^{\infty} \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}^k \frac{t^k}{k!} \right)$$

definition of exponential

$$= \ldots = e^{\alpha_i t} \begin{bmatrix} \cos \omega_i t & \sin \omega_i t \\ -\sin \omega_i t & \cos \omega_i t \end{bmatrix}$$

recognize known series

we obtain

$$e^{\begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix} t} = e^{\alpha_i t} \begin{bmatrix} \cos \omega_i t & \sin \omega_i t \\ -\sin \omega_i t & \cos \omega_i t \end{bmatrix}$$

A diagonalizable - complex eigenvalues

- 2nd step:
- change of coordinates for the real system representation if complex eigenvalues

eigenvector

$$\lambda_i = \alpha_i + j\omega_i \longrightarrow u_i = u_{\text{re}} + j\, u_{\text{im}} \longrightarrow T_R^{-1} = \begin{bmatrix} u_{\text{re}} & u_{\text{im}} \end{bmatrix}$$

non-singular

complex vector

real vectors → also linearly independent if A is real

- write the initial condition as

$$x_0 = c_a u_{\text{re}} + c_b u_{\text{im}} = \begin{bmatrix} u_{\text{re}} & u_{\text{im}} \end{bmatrix} \begin{bmatrix} c_a \\ c_b \end{bmatrix} = T_R^{-1} \begin{bmatrix} c_a \\ c_b \end{bmatrix} \longrightarrow T_R\, x_0 = \begin{bmatrix} c_a \\ c_b \end{bmatrix}$$

- define the quantities m_R and φ_R as

$$m_R = \sqrt{c_a^2 + c_b^2}$$

$$\sin\varphi_R = \frac{c_a}{\sqrt{c_a^2 + c_b^2}} \qquad \cos\varphi_R = \frac{c_b}{\sqrt{c_a^2 + c_b^2}} \longrightarrow \begin{aligned} c_a &= m_R \sin\varphi_R \\ c_b &= m_R \cos\varphi_R \end{aligned}$$

A diagonalizable - complex eigenvalues

combining all the previous results we have

$$
\begin{aligned}
e^{At}x_0 &= T^{-1} e^{\begin{bmatrix}\alpha_i & \omega_i \\ -\omega_i & \alpha_i\end{bmatrix} t} T x_0 \\
&= \begin{bmatrix} u_{\mathrm{re}} & u_{\mathrm{im}} \end{bmatrix} e^{\alpha_i t} \begin{bmatrix}\cos\omega_i t & \sin\omega_i t \\ -\sin\omega_i t & \cos\omega_i t\end{bmatrix} \begin{bmatrix} m_R \sin\varphi_R \\ m_R \cos\varphi_R \end{bmatrix}
\end{aligned}
$$

depend upon the initial condition

ZIR

$$
e^{At}x_0 = m_R\, e^{\alpha_i t} \left[\sin(\omega_i t + \varphi_R) u_{\mathrm{re}} + \cos(\omega_i t + \varphi_R) u_{\mathrm{im}}\right]
$$

exponential x periodic

vectors

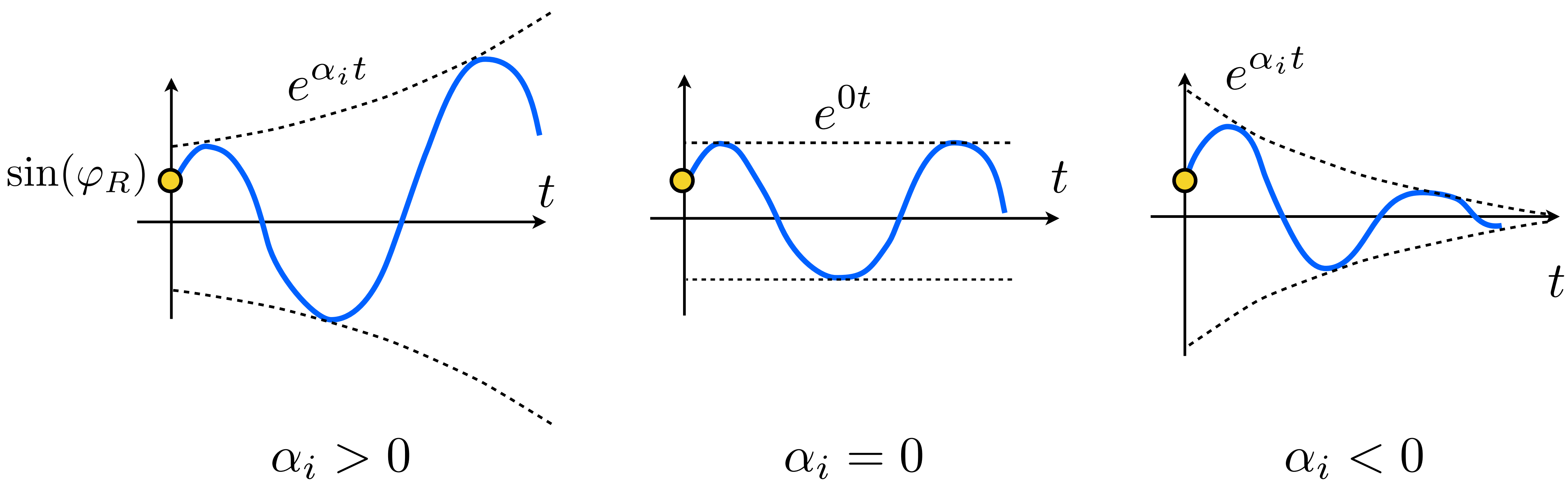
the zero input response will have components along the real and imaginary part of the eigenvector with damped (or diverging or constant) oscillating amplitudes which are scaled and shifted by quantities depending upon the initial conditions

natural modes (complex eigenvalues - A diagonalizable)

In the presence of complex eigenvalues (λ_i, λ_i^*), the corresponding natural mode is called **pseudoperiodic natural mode**

time functions of the form $e^{\alpha_i t}\sin(\omega_i t)$ $e^{\alpha_i t}\cos(\omega_i t)$

from α and ω we have a qualitative behaviour of the ZIR



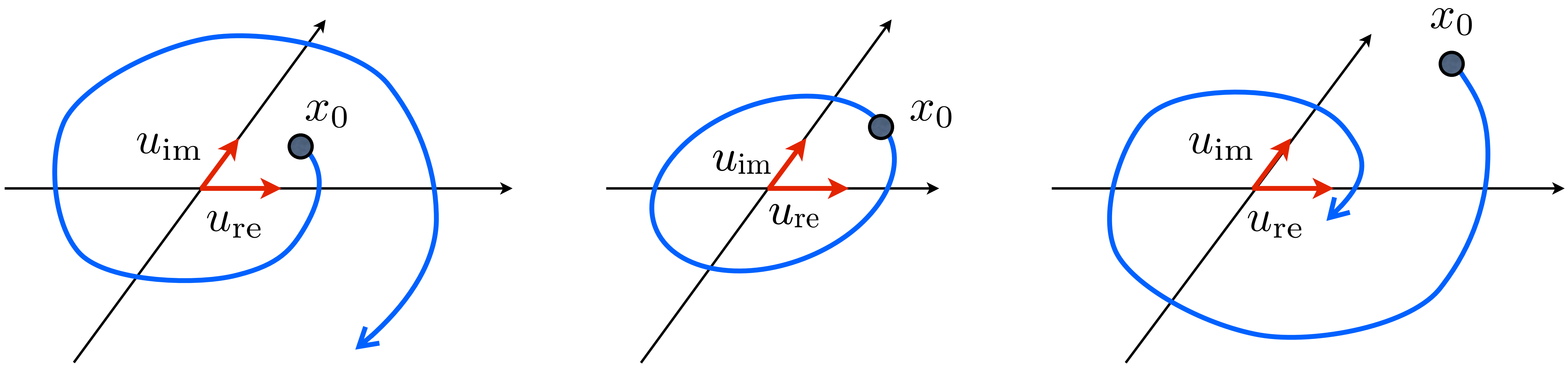
The ZIR is a linear combination (plus a time shift) of such natural modes

$$e^{At}x_0 = m_R\, e^{\alpha_i t}\left[\sin(\omega_i t + \varphi_R)u_{\text{re}} + \cos(\omega_i t + \varphi_R)u_{\text{im}}\right]$$

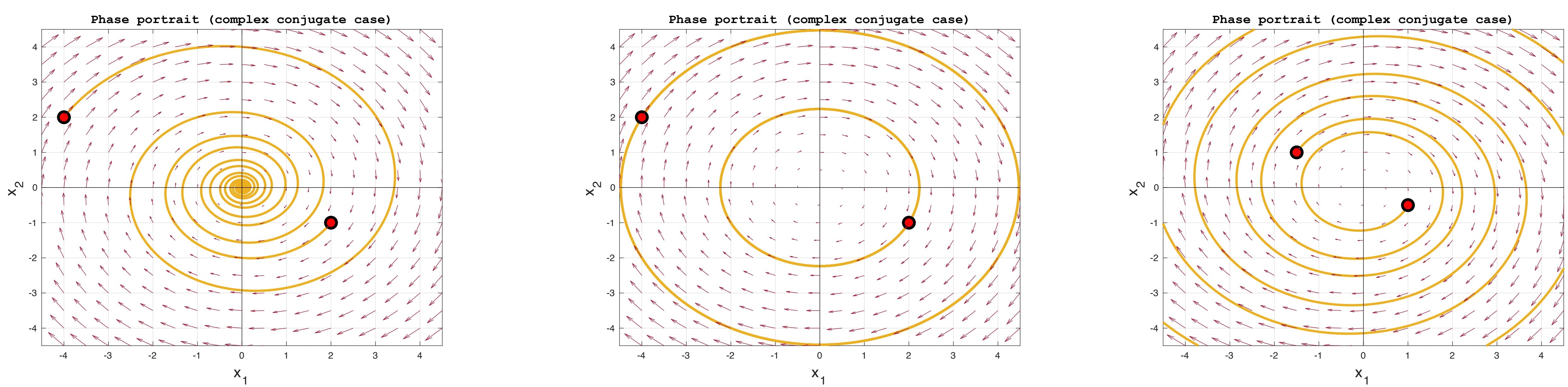
natural modes (complex - A diagonalizable)

complex and conjugate eigenvalues (λ_i, λ_i^*) $(n = 2)$ the 2D plots in the $(x_1,\ x_2)$ plane display, starting from a generic initial condition, the different behavior of the ZIR depending upon the sign of the real part of the eigenvalue $\alpha_i = \mathrm{Re}[\lambda_i]$

pseudoperiodic natural mode (prove how it turns)



$\alpha_i > 0$ $\alpha_i = 0$ $\alpha_i < 0$

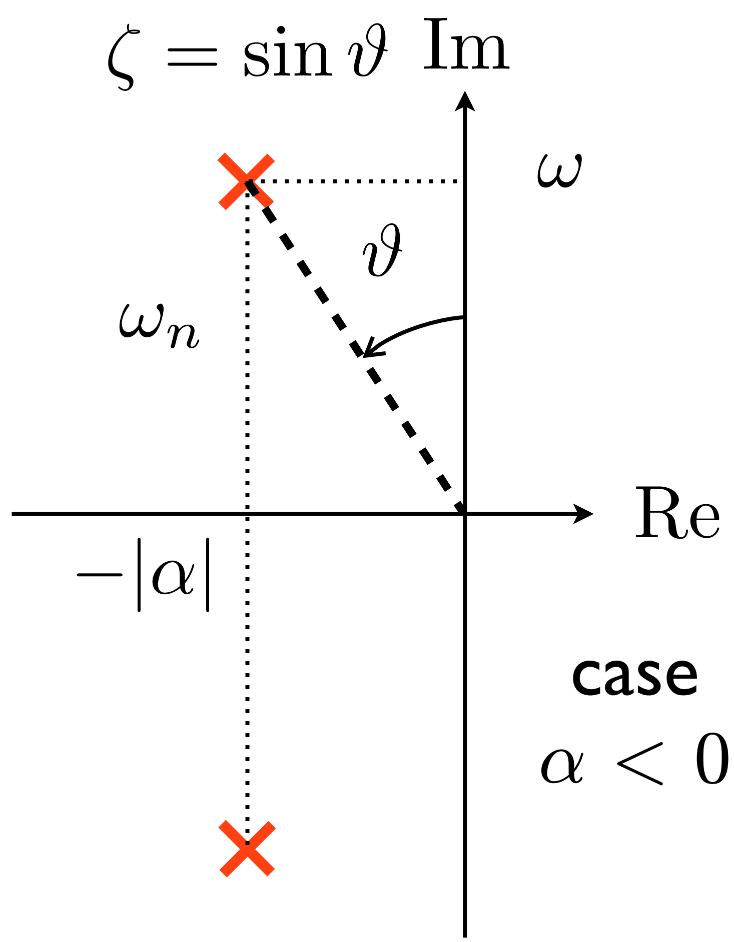


natural modes (complex - A diagonalizable)

Note that $\lambda_i = \alpha + j\omega$ can be represented by its real and imaginary part (α, ω) or by (ω_n, ζ) defined as

$$\omega_n = \sqrt{\alpha^2 + \omega^2} \qquad \zeta = \frac{-\alpha}{\sqrt{\alpha^2+\omega^2}}$$

natural frequency **damping coefficient**



(λ_i, λ_i^*) in the characteristic polynomial gives

$$(\lambda - \lambda_i)(\lambda - \lambda_i^*) = \lambda^2 - (\lambda_i + \lambda_i^*) + \lambda_i\lambda_i^*$$
$$= \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

we can express the pseudoperiodic natural modes as

$$e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t) \quad \text{and} \quad e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t)$$

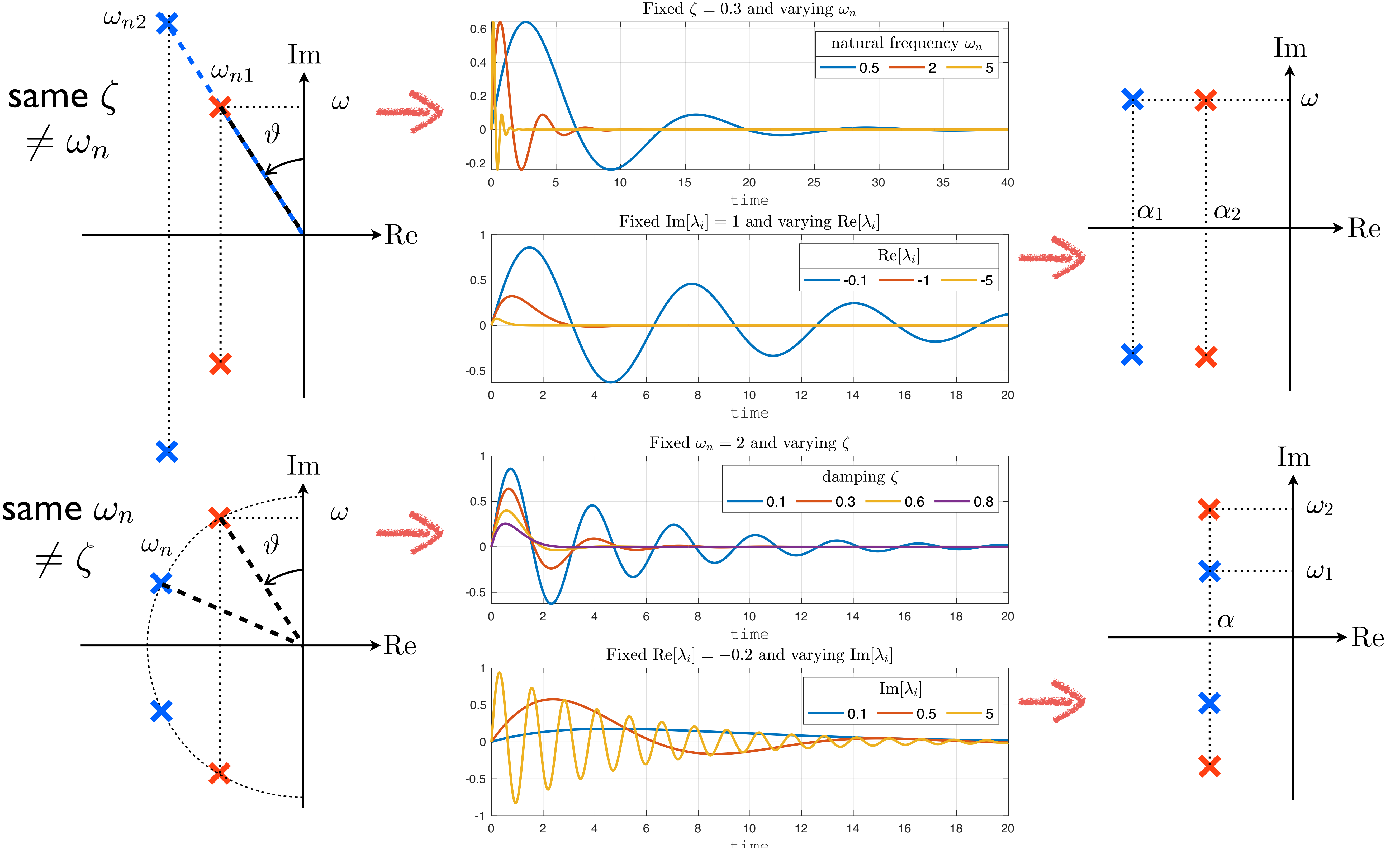
since we have the inverse relations

$$\alpha = -\zeta\omega_n \qquad \omega = \omega_n\sqrt{1-\zeta^2}$$

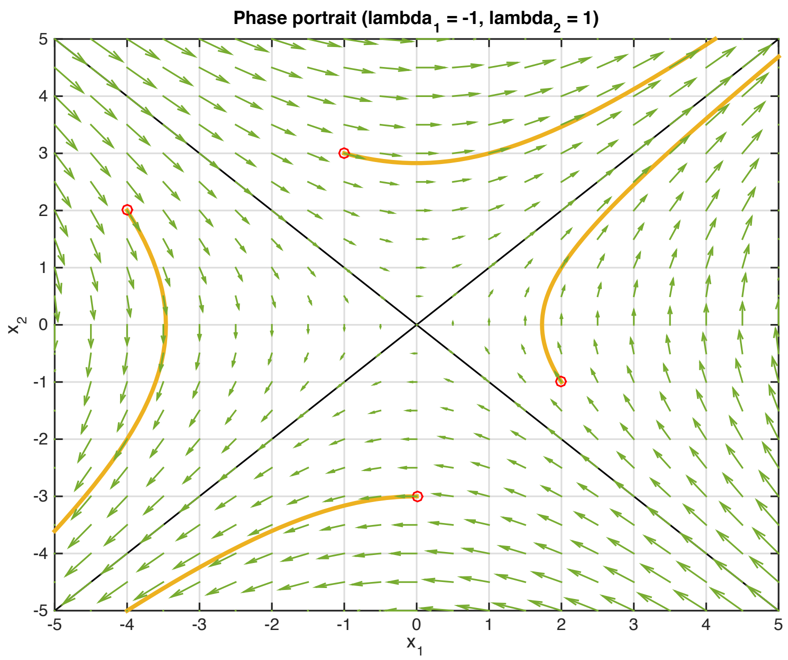
so $\quad \lambda_{1/2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = \omega_n\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)$

natural modes (complex - A diagonalizable)

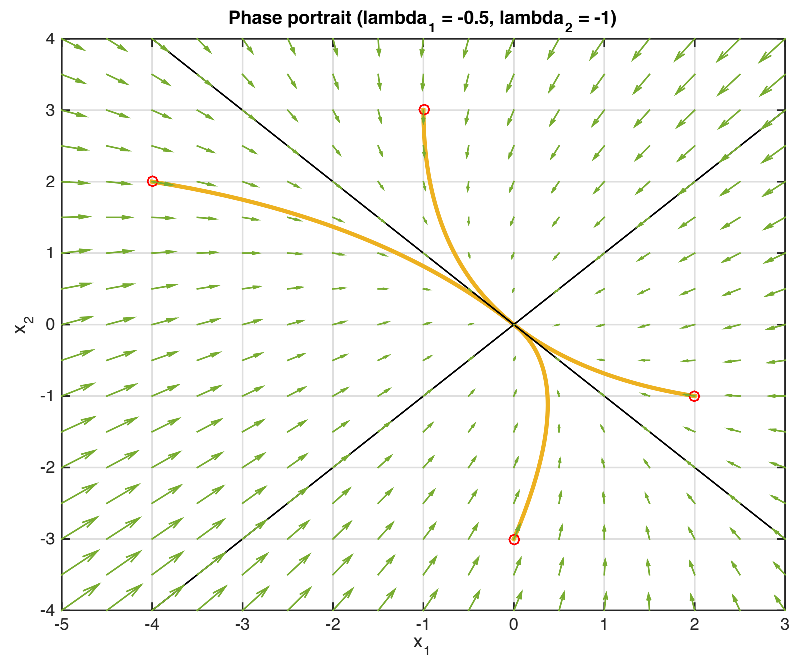
influence of the parameters (α, ω) or (ω_n, ζ) on the pseudoperiodic natural mode



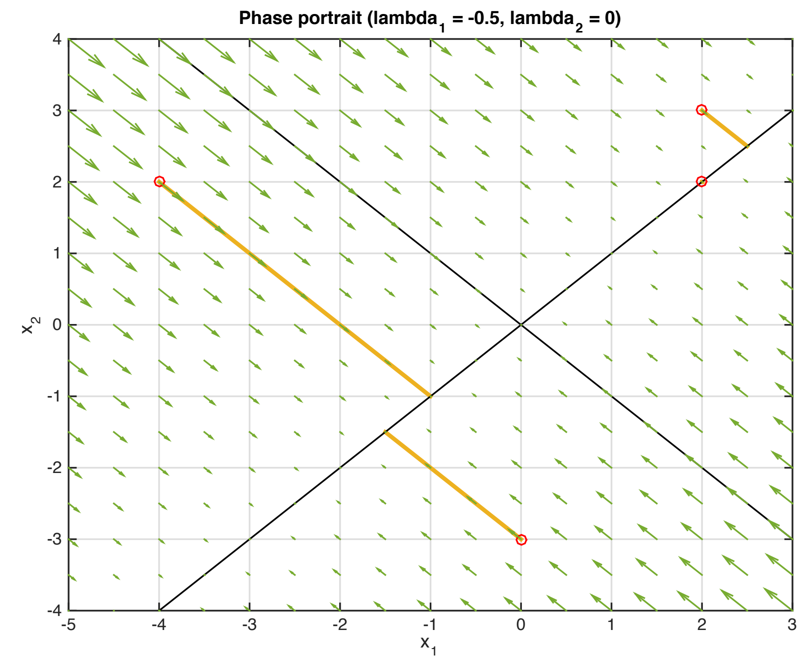
phase plane examples



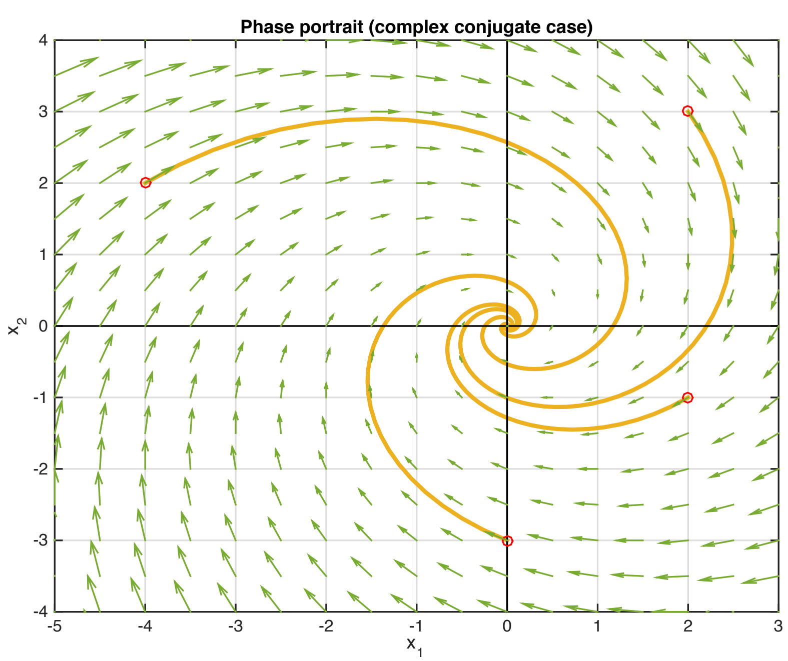
$\lambda_1 = -1 \quad \lambda_2 = 1$



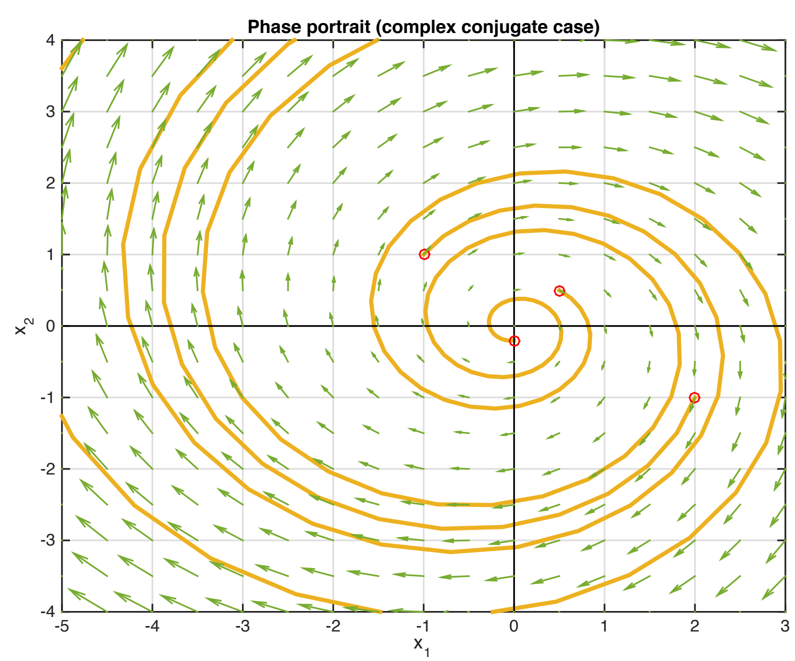
$\lambda_1 = -0.5 \quad \lambda_2 = -1$



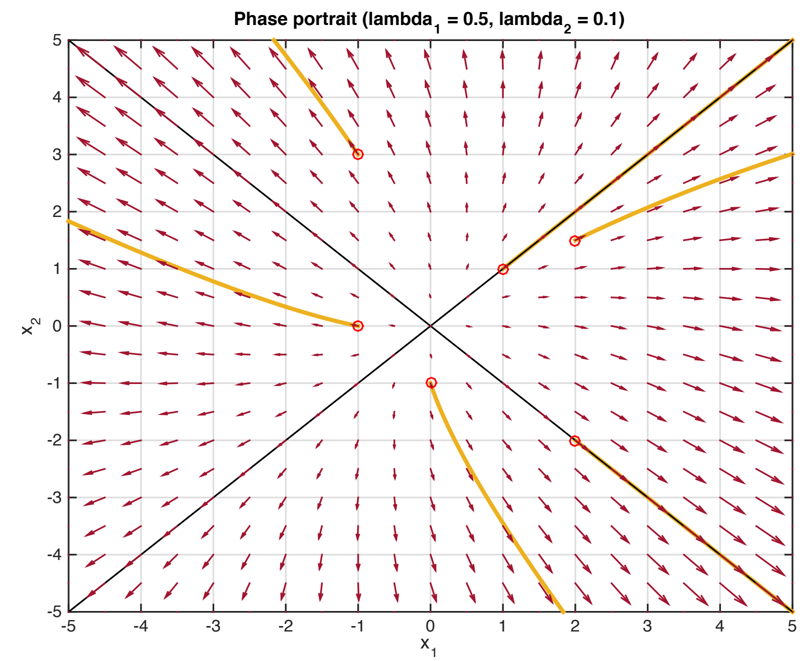
$\lambda_1 = -0.5 \quad \lambda_2 = 0$



$\lambda_{1/2} = -0.5 \pm j$



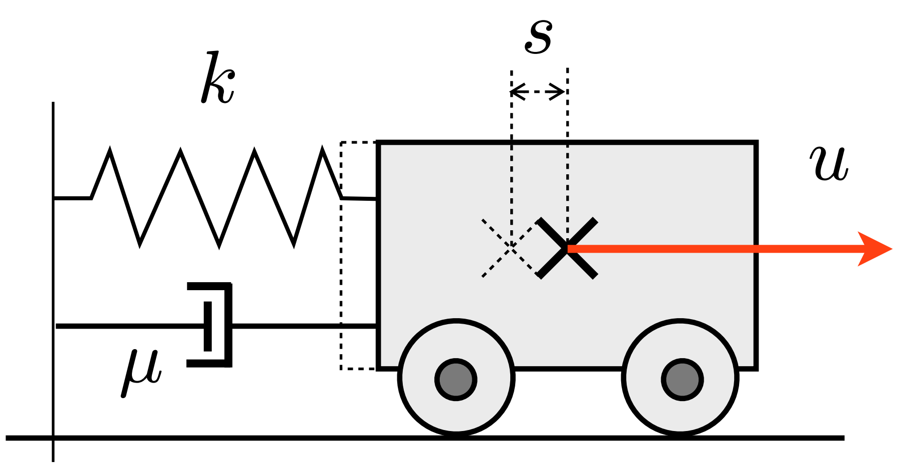
$\lambda_{1/2} = 0.2 \pm j$



$\lambda_1 = 0.5 \quad \lambda_2 = 0.1$

natural modes (Mass - Spring - Damper)

we can now study the natural modes of the Mass-Spring-Damper system



from the second order ODE $m\ddot{s} + \mu\dot{s} + ks = u$

we derived our state space model with dynamic matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{bmatrix}$$

to obtain the natural modes we first compute the eigenvalues (see previous slides)

- real eigenvalues when high damping $\mu \geq 2\sqrt{k\,m}$ — if $>$, over damping; if $=$, critical damping
- complex eigenvalues when low damping $\mu < 2\sqrt{k\,m}$ — if $<$, under damping

computing the natural frequency ω_n and damping coefficient ζ we note that the natural frequency corresponds to the mechanical frequency when there is no friction and the damping coefficient is proportional to the mechanical damping μ

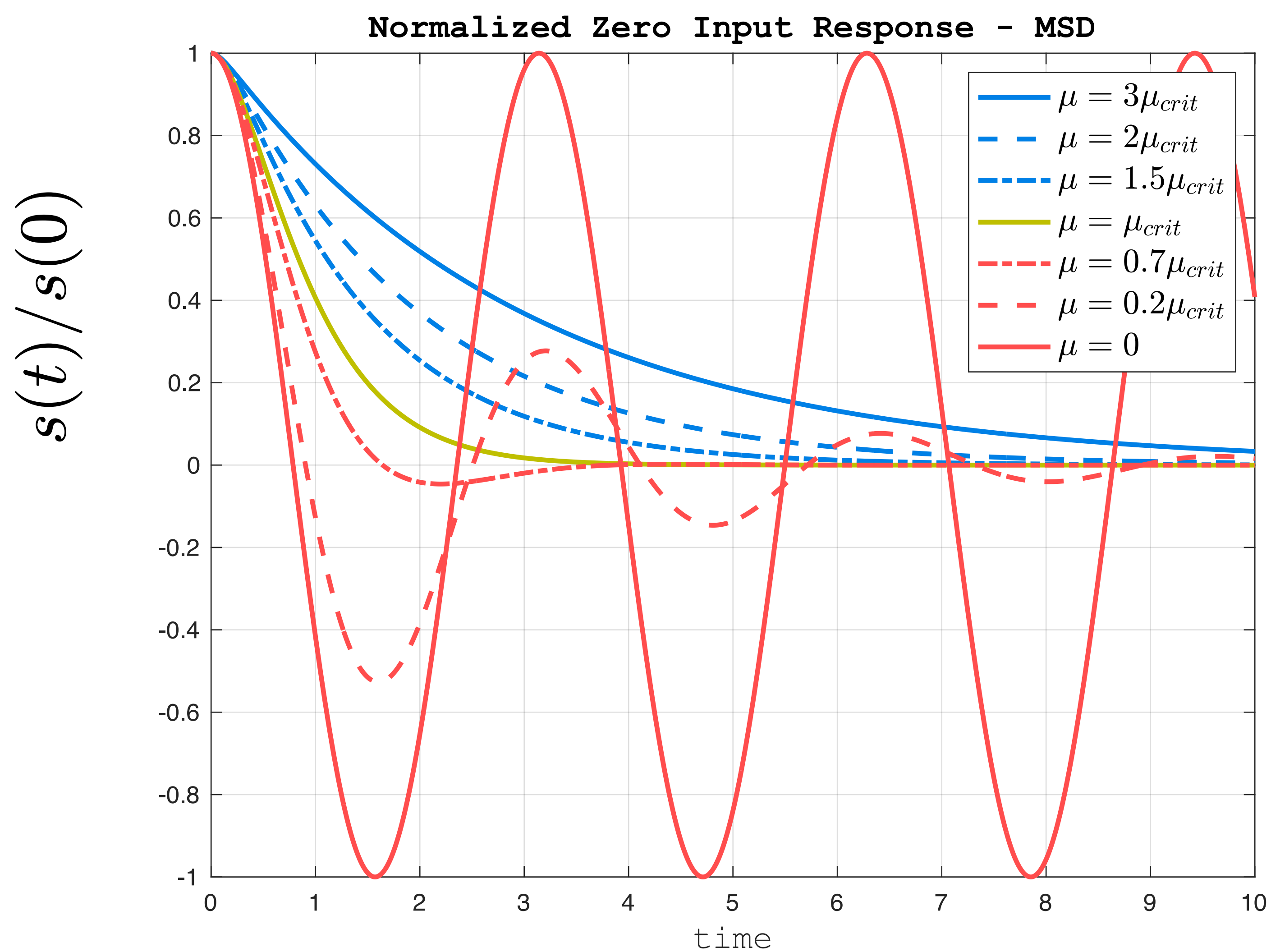
natural frequency $\omega_n = \sqrt{\frac{k}{m}}$

damping coefficient $\zeta = \frac{1}{2}\frac{\mu}{\sqrt{km}}$ ← mechanical friction coefficient

natural modes (Mass - Spring - Damper)

$m\ddot{s} + \mu\dot{s} + ks = u$ three different types of natural modes depending on μ

recall that we have pseudoperiodic natural modes only for $\mu < 2\sqrt{k\,m}$



$\mu_{crit} = 2\sqrt{km}$

normalized $(s(t)/s(0))$ plot of the ZIR

under-damped

critically damped

over-damped

natural modes (diagonalizable case)

example: consider the system characterized by a real and a pair of complex conjugate eigenvalues. The ZIR is a linear combination of an aperiodic and a pseudoperiodic natural mode. The two shown ZIR are for the same system but with similar (through a change of coordinates) dynamic matrix.

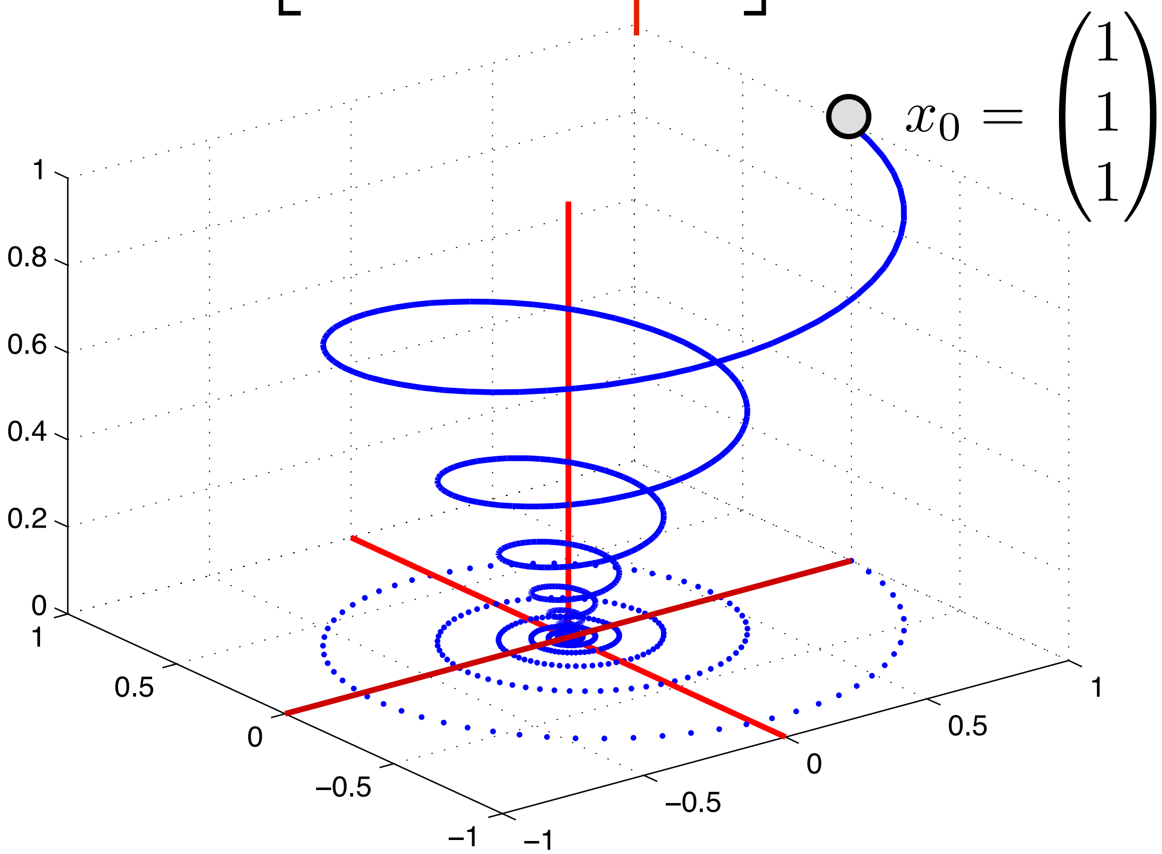
block-diagonal

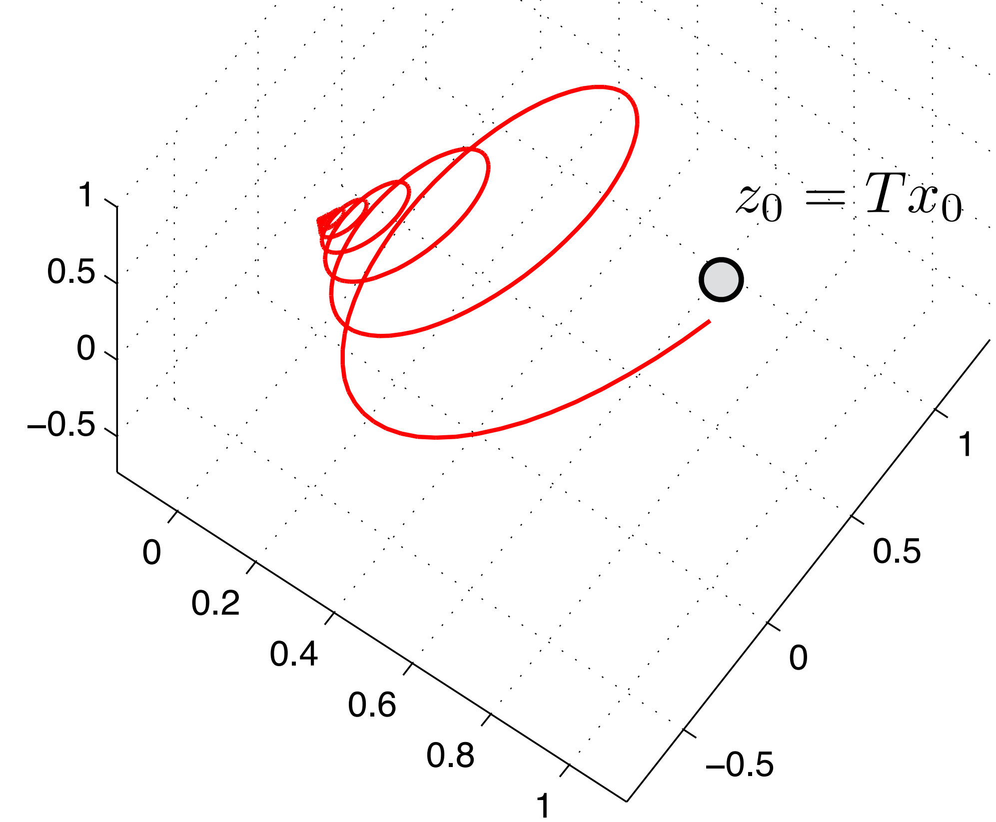
$$\left[\begin{array}{cc|c} -1 & 10 & 0 \\ -10 & -1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right]$$

$$\lambda_{1/2} = -1 \pm 10j$$
$$\lambda_3 = -1$$

same eigenvalues

$$\begin{bmatrix} -6 & 5 & 4 \\ 5 & -6 & -16 \\ -10 & 10 & 9 \end{bmatrix}$$





natural modes (non-diagonalizable case)

example: $n = 3, ma(\lambda_i) = 3,\ mg(\lambda_i) = 1$ thus one Jordan block of dimension 3

$$J = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} = \overbrace{\begin{bmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & \lambda_i \end{bmatrix}}^{J_1} + \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}^{J_2}$$

single Jordan block case

using the definition of matrix exponential one obtains:

- since J_1 and J_2 commute in the product $e^{(J_1+J_2)t} = e^{J_1 t} e^{J_2 t}$
- since J_2 is nilpotent $(J_2^3 = 0)$ the infinite sum in $e^{J_2 t}$ becomes finite

$$e^{Jt} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{1}{2}t^2 e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & 0 & e^{\lambda_i t} \end{bmatrix}$$

(proof)

the natural modes are $e^{\lambda_i t}, te^{\lambda_i t}, \frac{t^2}{2} e^{\lambda_i t}$

new time functions

maximum exponent depends on the dimension of the Jordan block

natural modes (non-diagonalizable case)

general case:

assume that the matrix A (n x n) has only one eigenvalue thus $ma(\lambda_i) = n$.
Moreover the geometric multiplicity is $mg(\lambda_i) < ma(\lambda_i) = n$ (non diagonalizable case), and thus we have $mg(\lambda_i)$ Jordan blocks J_i

In the proper coordinates, tha matrix will display its Jordan blocks

$$e^{At} = T^{-1} \operatorname{diag}\left\{e^{J_i t}\right\} T$$

with **index** (dimension of the largest Jordan block associated to λ_i) n_k

Since in general we will obtain $mg(\lambda_i)$ Jordan blocks relative to the eigenvalue λ_i, the maximum exponent of t that will appear in the natural modes will depend on the largest Jordan block, that is on the index n_k of λ_i .

New time functions appear as **natural modes**

$$e^{\lambda_i t}, te^{\lambda_i t}, \ldots, \frac{t^{n_k - 1}}{(n_k - 1)!} e^{\lambda_i t}$$

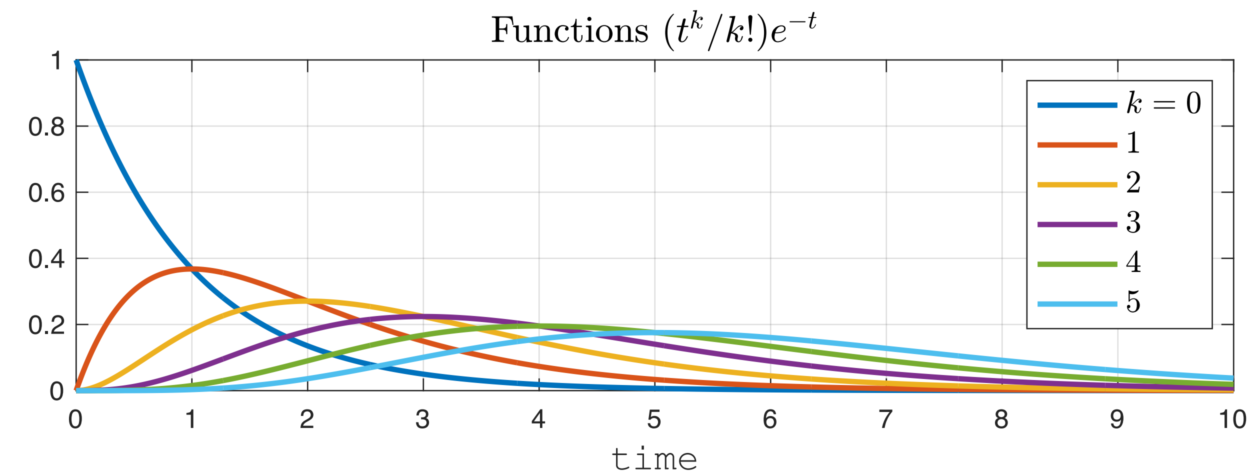
depends on n_k (the index of λ_i)

natural modes (non-diagonalizable case)

What contribution in time these new terms give?

Is asymptotic convergence to 0 affected?

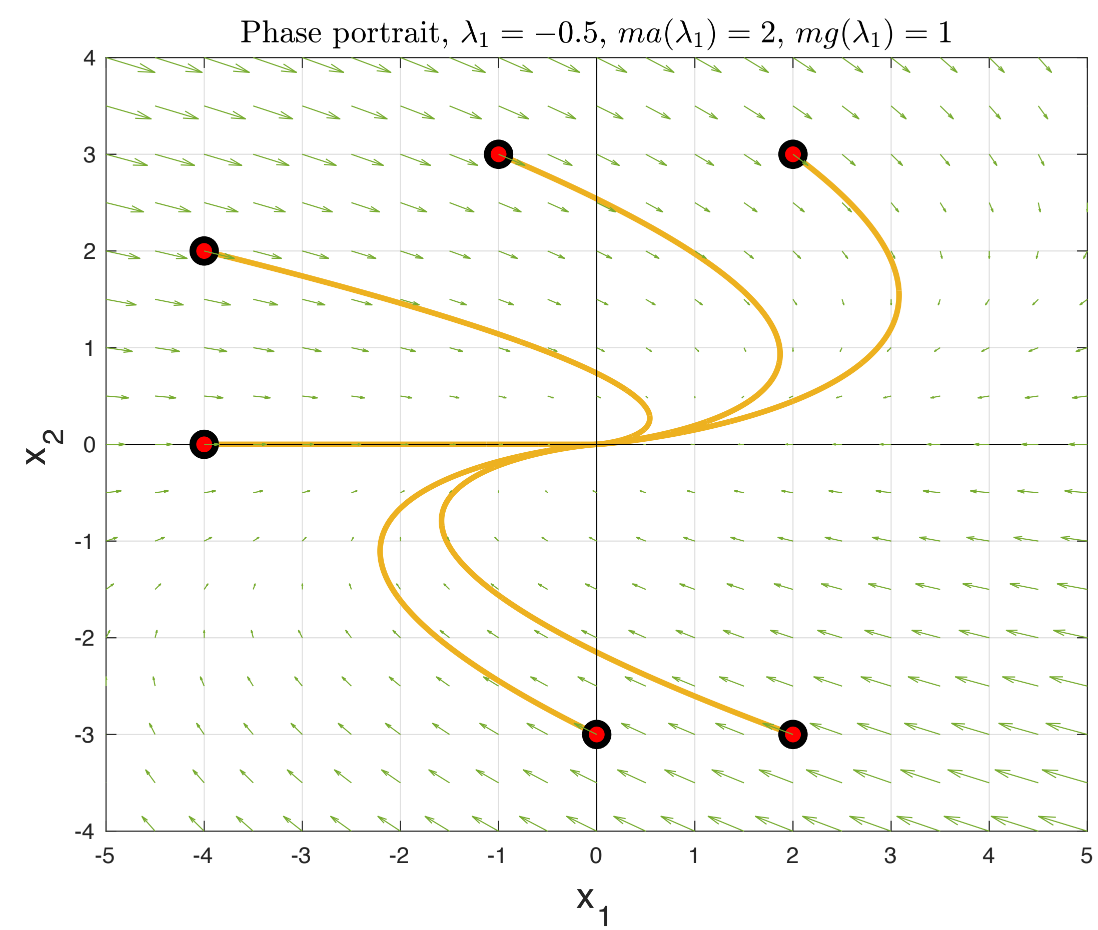
$$\frac{t^k}{k!}e^{\lambda_i t}$$



- if λ_i is real negative, exponential wins and it converges to 0 as $t \to \infty$
- if λ_i is real positive, it diverges
- if $\lambda_i = 0$, it diverges when $k \geq 0$

example

$$A = \begin{pmatrix} -0.5 & 1 \\ 0 & -0.5 \end{pmatrix}$$

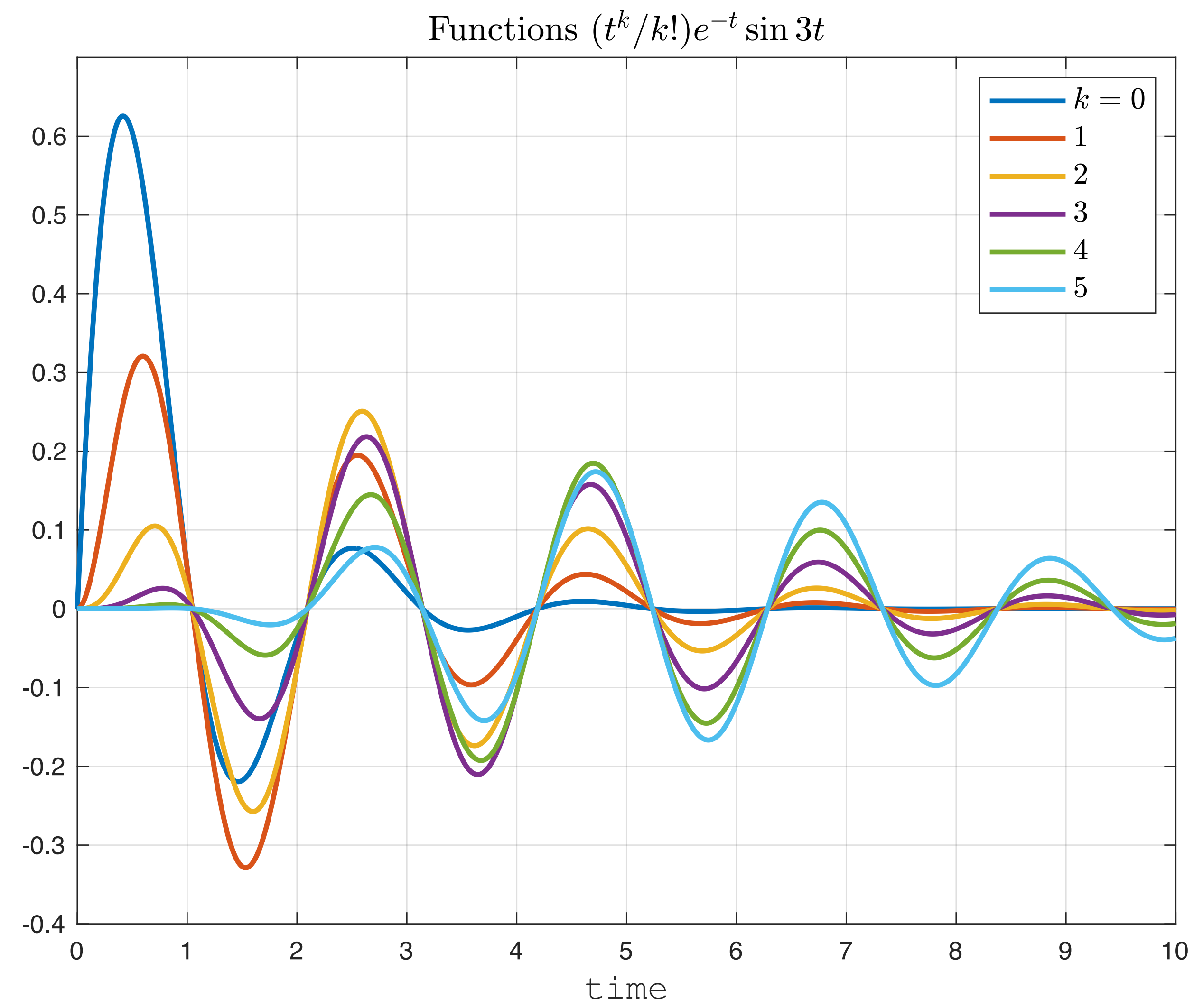


natural modes (non-diagonalizable case)

If we have complex eigenvalues (λ_i, λ_i^*) with $\lambda_i = \alpha_i + j\omega_i$ and index n_k greater than 1 then the following time functions will also appear

$$e^{\alpha_i t} \sin \omega_i t, \;\; t e^{\alpha_i t} \sin \omega_i t, \ldots, \frac{t^{n_k - 1}}{(n_k - 1)!} e^{\alpha_i t} \sin \omega_i t$$

case with $\mathrm{Re}[\lambda_i] = \alpha_i < 0$



natural modes (non-diagonalizable case)

When $\mathrm{Re}[\lambda_i] = 0$ the geometric multiplicity plays an important role in determining if the corresponding natural mode is diverging or not.

When $mg(\lambda_i) < ma(\lambda_i)$ (and $\mathrm{Re}[\lambda_i] = 0$) the corresponding natural mode will diverge asymptotically.

- $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad \lambda_i = 0 \qquad$ natural modes $\quad 1, \; t, \; t^2 \quad \longrightarrow$ **diverging**

- $\left[\begin{array}{cc:cc} j\omega_i & 1 & 0 & 0 \\ 0 & j\omega_i & 0 & 0 \\ \hdashline 0 & 0 & -j\omega_i & 1 \\ 0 & 0 & 0 & -j\omega_i \end{array}\right]$ $\qquad (\lambda_i, \lambda_i^*) = \pm j\omega_i \qquad$ or real form $\qquad \left[\begin{array}{cc:cc} 0 & \omega_i & 1 & 0 \\ -\omega_i & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & \omega_i \\ 0 & 0 & -\omega_i & 0 \end{array}\right]$

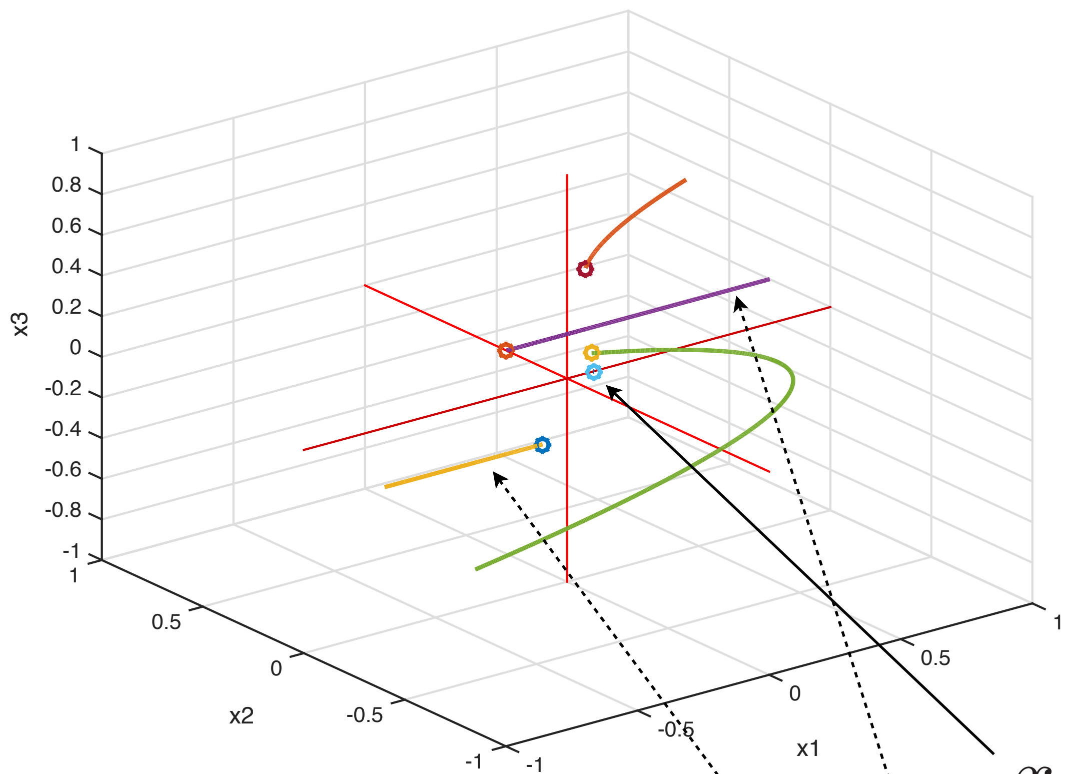
natural modes $\quad \sin(\omega_i t) \quad t\,\sin(\omega_i t) \quad \longrightarrow$ **diverging**

natural modes (non-diagonalizable case)

example $\quad \mathrm{Re}(\lambda_i) = 0 \quad \lambda_i = 0$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad mg(\lambda_i) = 1 \qquad ma(\lambda_i) = 3$$

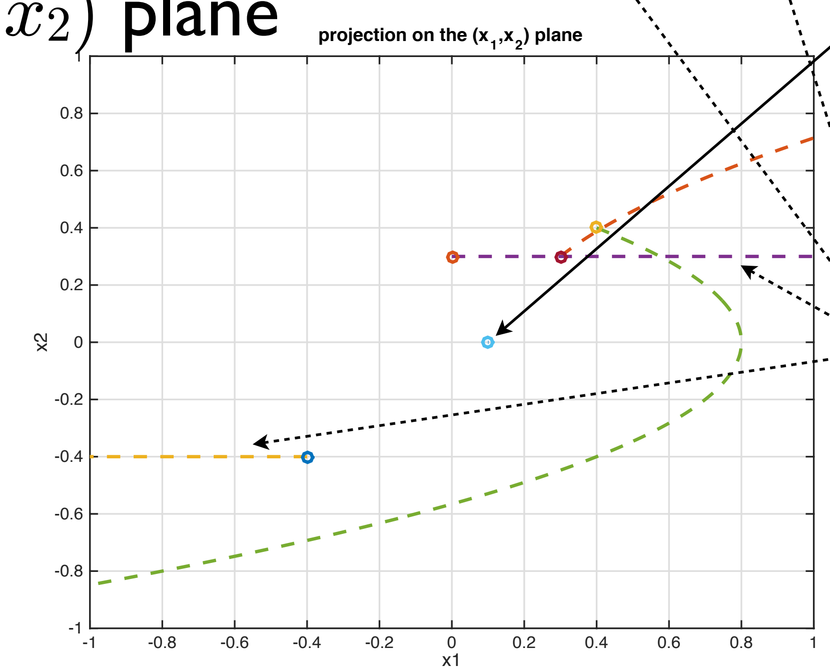
$$e^{At}x_0 = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} x_0$$



depending upon the initial condition, different natural modes are excited

$$x_0 = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$ only constant mode is selected $$x(t) = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$

$(x_1,\ x_2)$ plane



$$x_0 = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$$ constant and t mode are selected $$x(t) = \begin{pmatrix} \alpha + \beta t \\ \beta \\ 0 \end{pmatrix}$$

summary

A real diagonalizable $mg(\lambda_i) = ma(\lambda_i)$ for all i	real λ_i	aperiodic mode $e^{\lambda_i t}$
	complex $\lambda_i = \alpha_i + j\omega_i$	pseudoperiodic mode $e^{\alpha_i t}\left[\sin(\omega_i t + \varphi_R)u_{\text{re}} + \cos(\omega_i t + \varphi_R)u_{\text{im}}\right]$
	$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} u_i v_i^T$	spectral form
A real non-diagonalizable $mg(\lambda_i) < ma(\lambda_i)$ index$(\lambda_i) = n_k$	real λ_i	$\dots, \dfrac{t^{n_k-1}}{(n_k-1)!}e^{\lambda_i t}$
	complex $\lambda_i = \alpha_i + j\omega_i$	$\dots, \dfrac{t^{n_k-1}}{(n_k-1)!}e^{\alpha_i t}\sin\omega_i t$

vocabulary

English	Italiano
natural mode	modo naturale
aperiodic/pseudoperiodic natural mode	modo naturale aperiodico/ pseudoperiodico
natural frequency	pulsazione naturale
damping coefficient	coefficiente di smorzamento
spectral form	forma spettrale