Control Systems

Linear Algebra topics

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outline

- basic facts about matrices
- eigenvalues - eigenvectors - characteristic polynomial - algebraic multiplicity
- eigenvalues invariance under similarity transformation
- invariance of the eigenspace
- geometric multiplicity
- diagonalizable matrix: necessary & sufficient condition
- diagonalizing similarity transformation
- a more convenient similarity transformation for complex eigenvalues
- spectral decomposition
- not diagonalizable (A defective) case: Jordan blocks

matrices - notations and terminology

p x q matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1q} \\ m_{21} & m_{22} & \cdots & m_{2q} \\ \vdots & \vdots & . & \vdots \\ m_{p1} & \cdots & m_{p,q-1} & m_{pq} \end{pmatrix}$$

m_{ij} are the
matrix elements
(or entries)

$$M = \{m_{ij} : i = 1, \ldots, p \quad \& \quad j = 1, \ldots, q\}$$

j-th column

i-th row

transpose	$M^T = \{m'_{ij} : m'_{ij} = m_{ji}\} \longrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
vector (column)	$v : n \times 1$
row vector	$v^T : 1 \times n$
scalar	1×1
square matrix	$M : p \times p$
rectangular matrix	$M : p \times q \qquad (p \neq q)$

particular square matrices

diagonal matrix

$$\begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{pmatrix}$$

compact notation:

$$M = \operatorname{diag}\{m_i\}, \quad i = 1, \dots, p$$

block diagonal matrix

$$\begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

compact notation:

$$M = \operatorname{diag}\{M_i\}, \quad i = 1, \dots, k$$

with M_i square matrix

upper triangular matrix

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \ddots & m_{2n} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m_{nn} \end{pmatrix}$$

similarly:

- lower triangular
- strictly lower triangular

upper block triangular matrix

$$\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

similarly:

- lower block triangular
- strictly lower block triangular

with M_{ii} square matrix

square matrices

- for block triangular or block diagonal the computation of the **determinant** is simplified

$$\det\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} = \det(M_{11})\cdot\det(M_{22})$$

$$\det\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \det(M_1)\cdot\det(M_2)$$

special case

- determinant of a diagonal matrix = product of the diagonal terms

special case

- examples

$$\det\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix} = m_{11}\, m_{22}\, m_{33} \qquad \det\begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} = m_{11}\, m_{22}\, m_{33}$$

- these particular cases will be useful for the eigenvalue computation

matrix-vector multiplication

- right multiplication of a matrix M by a column vector $\boldsymbol{v}$ is equivalent to making a linear combination of the columns $\boldsymbol{m}_i$ of M with coefficients the elements v_i of $\boldsymbol{v}$

$$M\boldsymbol{v} = \begin{pmatrix}\boldsymbol{m}_1 & \boldsymbol{m}_2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = v_1\,\boldsymbol{m}_1 + v_2\,\boldsymbol{m}_2$$

example

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5\\ 6\end{pmatrix} = 5\begin{pmatrix}1\\ 3\end{pmatrix} + 6\begin{pmatrix}2\\ 4\end{pmatrix} = \begin{pmatrix}17\\ 39\end{pmatrix}$$

- left multiplication of a matrix M by a row vector $\boldsymbol{v}^T$ is equivalent to making a linear combination of the rows $\boldsymbol{m}_i{}^T$ of M with coefficients the elements v_i of $\boldsymbol{v}^T$

$$\boldsymbol{v}^T M = \begin{pmatrix}v_1 & v_2\end{pmatrix}\begin{pmatrix}\boldsymbol{m}_1^T\\ \boldsymbol{m}_2^T\end{pmatrix} = v_1\,\boldsymbol{m}_1^T + v_2\,\boldsymbol{m}_2^T$$

example

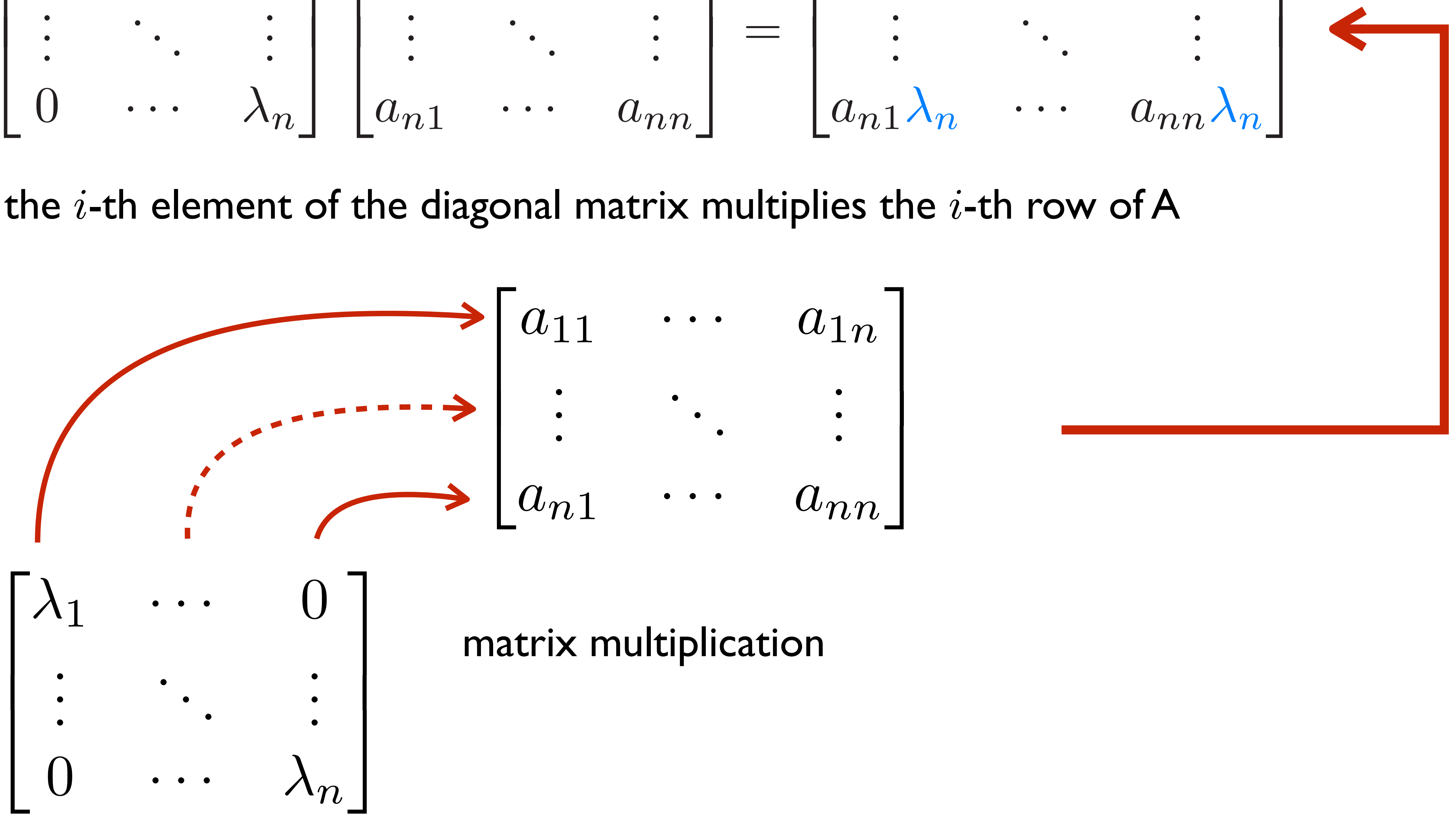
$$\begin{pmatrix}5 & 6\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = 5\begin{pmatrix}1 & 2\end{pmatrix} + 6\begin{pmatrix}3 & 4\end{pmatrix} = \begin{pmatrix}23\\ 34\end{pmatrix}$$

square matrices

- left multiplication by a diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_i\}$

$$\Lambda\, A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & \cdots & a_{1n}\lambda_1 \\ \vdots & \ddots & \vdots \\ a_{n1}\lambda_n & \cdots & a_{nn}\lambda_n \end{bmatrix}$$

the i-th element of the diagonal matrix multiplies the i-th row of A



$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

matrix multiplication

square matrices

- left multiplication by a diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_i\}$

$$\Lambda\, A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & \cdots & a_{1n}\lambda_1 \\ \vdots & \ddots & \vdots \\ a_{n1}\lambda_n & \cdots & a_{nn}\lambda_n \end{bmatrix}$$

the i-th element of the diagonal matrix Λ multiplies the i-th row of A

similarly, right multiplication by Λ corresponds to operations on columns of A

$$A\, \Lambda = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & \cdots & a_{1n}\lambda_n \\ \vdots & \ddots & \vdots \\ a_{n1}\lambda_1 & \cdots & a_{nn}\lambda_n \end{bmatrix}$$

the i-th element of the diagonal matrix Λ multiplies the i-th column of A

some matrix properties

remember that in general (except some special cases) the product of matrices is not commutative, that is $M\,N \neq N\,M$

for non-square matrices it is obvious, for example if M is 2 x 3 and N is 3 x 2, MN will be 2 x 2 while NM 3 x 3.

a square matrix M is invertible (or non-singular) $\Leftrightarrow \det(M) \neq 0$

for square matrices

$$\left(A^{-1}\right)^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$M = \mathrm{diag}\{m_i\} \quad \Longrightarrow \quad M^{-1} = \mathrm{diag}\{\frac{1}{m_i}\}$$

$$A^{-1}A = AA^{-1} = I$$

$$A^0 = I$$

$\det\left(A\right) = \det\left(A^T\right)$ $\det(\alpha A) = \alpha^n \det(A)$ and **not** $\alpha\,\xcancel{\det}(A)$

$\det(MN) = \det(M).\det(N)$ for square matrices M and N with same size

nullspace and image of M

consider the linear map from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ represented by the square matrix M

$$M: \quad \boldsymbol{R}^n \xrightarrow{\qquad\qquad} \boldsymbol{R}^n$$

- the kernel or nullspace $\mathrm{Ker}(M)$ of M is the linear subspace defined as

$$\boxed{\mathrm{Ker}(M) = \{v \in \mathbf{R}^n \,|\, Mv = 0\}}$$

since it is a linear space, it is characterized by a base; each vector of the base can be seen as a particular linear combination of the columns of M (since v multiplies on the right) that give the null vector

- the image or range $\mathrm{Im}(M)$ of M is the linear subspace defined as

$$\boxed{\mathrm{Im}(M) = \{v \in \mathbf{R}^n \,|\, Mu = v \text{ for } u \in \mathbf{R}^n\}}$$

also the image is a linear space, its base can be found choosing the set of linearly independent columns of M

- for $M: \boldsymbol{R}^n \xrightarrow{\qquad\qquad} \boldsymbol{R}^n$

Rank-nullity theorem

$$\boxed{\dim(\mathrm{Ker}(M)) + \dim(\mathrm{Im}(M)) = n}$$

nullspace and image of M

examples

$$M = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \qquad \mathrm{Ker}(M) = \mathrm{gen} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

3 times the first column of M plus the second column of M gives the null vector

$$\mathrm{Im}(M) = \mathrm{gen} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

the two columns of M are clearly linear dependent (the second one is -3 times the first one)

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \qquad \mathrm{Ker}(M) = \mathrm{gen} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

or equivalently

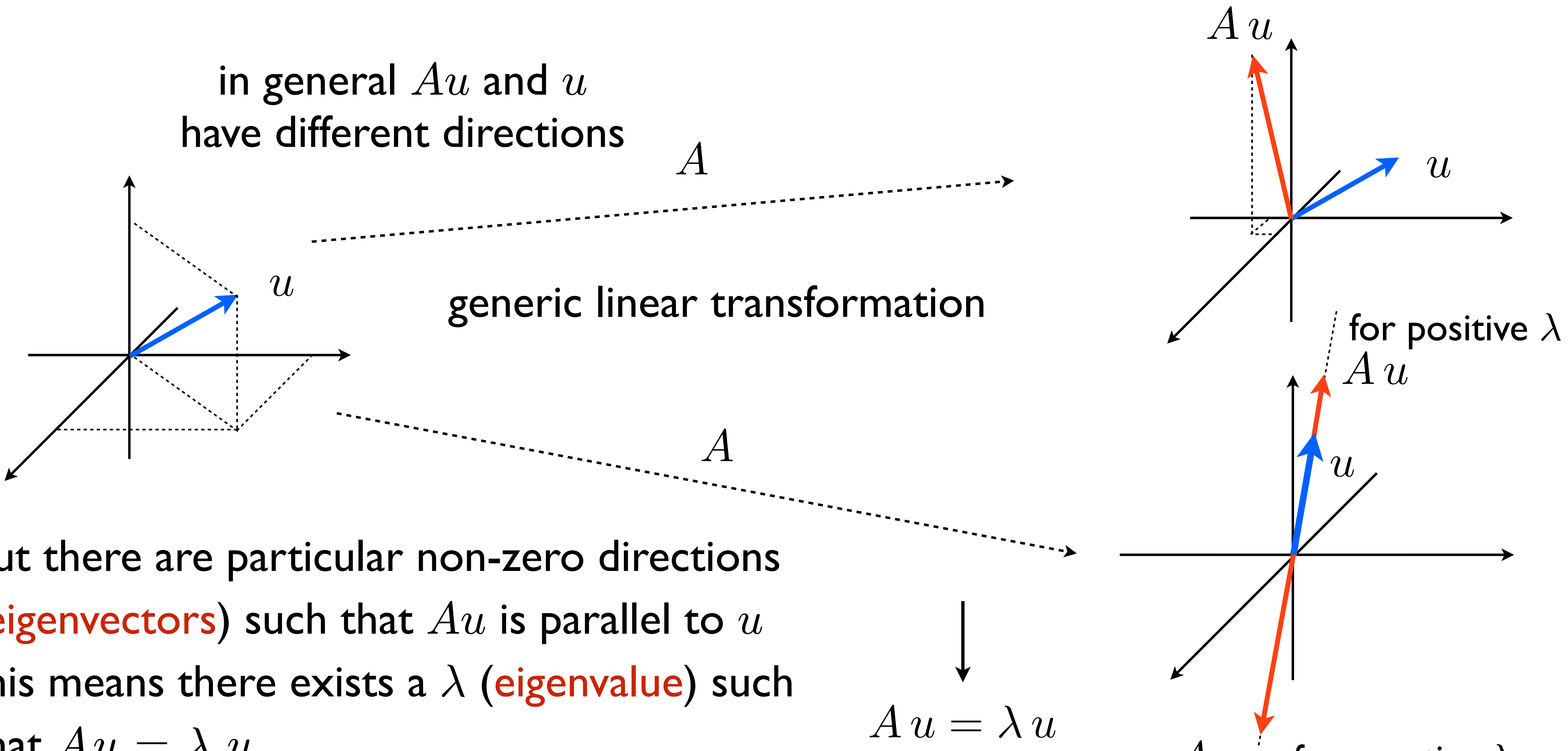
$$\mathrm{Ker}(M) = \mathrm{gen} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

these two basis generate the same linear subspace: any vector of the second base can be obtained from a linear combination of the first basis

$$\mathrm{Im}(M) = \mathrm{gen} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

eigenvalues & eigenvectors of a square matrix A

A transforms a vector u into the vector Au (linear mapping)



in general Au and u
have different directions

generic linear transformation

for positive λ

or $A\,u$ for negative λ

but there are particular non-zero directions (eigenvectors) such that Au is parallel to u

this means there exists a λ (eigenvalue) such that $Au = \lambda\, u$

$$A\,u = \lambda\,u$$

scalar
(scaling factor)

eigenvalues & eigenvectors

example

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

eigenvalues

$$\lambda_1 = -1$$

$$\lambda_2 = 2$$

no special relation between the vector and its image

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

no special relation between the vector and its image

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

the vector and its image are parallel

$$A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

scaling factor = eigenvalue

the vector and its image are parallel

$$A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

eigenvalues & eigenvectors

Definition: given the matrix A $(n$ x $n)$, if for a scalar λ_i there exists a non-zero vector $u_i \neq 0$ such that $A\, u_i = \lambda_i\, u_i$ then λ_i is an **eigenvalue** of A and u_i is an associated **eigenvector**

note that $A\, u_i = \lambda_i\, u_i \Longleftrightarrow (A - \lambda_i\, I\,)u_i = 0$

u_i belongs to the nullspace or kernel of $(A - \lambda_i I)$

non-trivial $u_i \neq 0$ solution exists iff $\det(A - \lambda_i I) = 0$

→ the eigenvalues λ_i are the roots of the n-th order **characteristic polynomial**

$$\boxed{p_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0}$$

the eigenvalues λ_i are the solutions of $p_A(\lambda) = \det(\lambda I - A) = 0$

- same solutions of $\det\,(A - \lambda I) = 0$ since $\det\,(A - \lambda I) = (\text{-}1)^n \det\,(\lambda I - A)$
- kernel or **nullspace** of M is the subspace of all vectors v s.t. $Mv = 0$

- the eigenvectors u_i associated to the eigenvalue λ_i generate a linear subspace: the **eigenspace** V_i
- if u_i is am eigenvector associated to the eigenvalue λ_i then also $\alpha\, u_i$ is an eigenvector associated to the same eigenvalue

eigenvalues & eigenvectors

ex. $$\boxed{A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}}$$

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda + 2 & 2 \\ -2 & \lambda - 3 \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$$

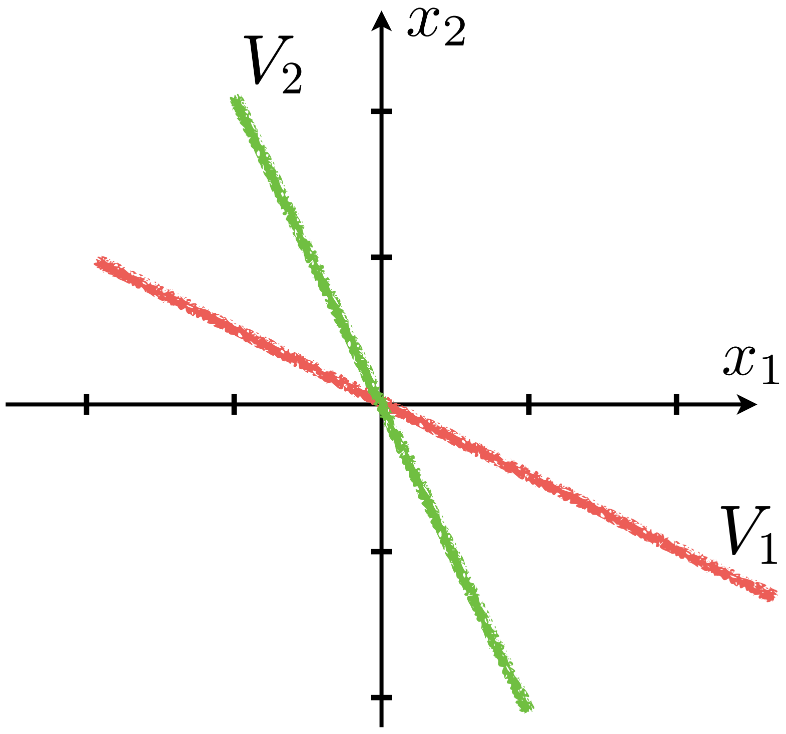
→ therefore the eigenvalues are $\lambda_1 =$ -1 and $\lambda_2 = 2$

- for $\lambda_1 =$ -1 the eigenspace V_1 is determined by

$$V_1 = \ker(A - \lambda_1 I) = \ker \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} = \text{gen} \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

- for $\lambda_2 = 2$ the eigenspace V_2 is determined by

$$V_2 = \ker(A - \lambda_2 I) = \ker \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix} = \text{gen} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$



eigenvalues & eigenvectors

hyp: A square matrix n x n with real elements

$$p_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$$

- $p_A(\lambda)$ is a polynomial of order n with **real** coefficients a_i if A is real
- set of the n solutions of $p_A(\lambda) = 0$ defined as the **spectrum** of A, symbol: $\sigma(A)$
- since the coefficients of $p_A(\lambda)$ are real its roots can be real and/or complex

→ generic solution λ_i (eigenvalue λ_i) $\left\{ \begin{array}{l} \lambda_i \in \mathbb{R} \\ \text{or} \\ \lambda_i \in \mathbb{C} \;\; \text{then also} \;\; \lambda_i^* \;\; \text{is a solution} \longrightarrow (\lambda_i, \lambda_i^*) \end{array} \right.$ **pairs**

therefore if $\left\{ \begin{array}{l} \lambda_i \in \mathbb{R} \\ \lambda_i \in \mathbb{C} \end{array} \right.$ then $\begin{array}{l} u_i \;\; \text{real components} \\ u_i \;\; \text{complex components and} \;\; \lambda_i^* \longrightarrow u_i^* \end{array}$

- define $ma(\lambda_i)$ = **algebraic multiplicity** of eigenvalue λ_i as the multiplicity of the solution $\lambda = \lambda_i$ in $p_A(\lambda) = 0$

eigenvalues & eigenvectors

special cases

diagonal matrix

$$\begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{pmatrix}$$

→ eigenvalues = $\{m_i\}$

triangular matrix (upper or lower)

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \ddots & m_{2n} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m_{nn} \end{pmatrix}$$

→ eigenvalues = $\{m_{ii}\}$

in these situations

eigenvalues = elements on the main diagonal

eigenvalues - invariance

- similarity transformation: $A \xrightarrow[\det(T) \neq 0]{T} TAT^{-1}$ has the same eigenvalues as A

→ eigenvalues are **invariant** under similarity transformations (proof)

- right and left eigenvectors:

$$Au_i = \lambda_i u_i \quad \rightarrow \quad u_i$$ **right eigenvector** (column)

$$v_i^T A = v_i^T \lambda_i = \lambda_i v_i^T \quad \rightarrow \quad v_i^T$$ **left eigenvector** (row)

we will often choose the left eigenvectors such that $v_i^T u_j = \delta_{ij}$

where $\delta_{ij} \begin{cases} = 0 & \text{if} \quad i \neq j \\ = 1 & \text{if} \quad i = j \end{cases}$ Kronecker delta

- $A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$ if $u_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ we will choose $v_1^T = \frac{1}{3} \begin{pmatrix} 2 & 1 \end{pmatrix}$

if $u_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ we will choose $v_2^T = \frac{1}{3} \begin{pmatrix} -1 & -2 \end{pmatrix}$

eigenspace

the **eigenspace** V_i corresponding to the eigenvalue λ_i of A is the **vector space**

$V_i = \{u \in \mathbb{R}^n | Au = \lambda_i u\}$ or equivalently $V_i = \mathrm{Ker}(A - \lambda_i I)$

reminder:

- basis of $\mathrm{Ker}(M)$ is a set of linearly independent vectors which spans the whole subspace $\mathrm{Ker}(M)$
- $\mathrm{span}\{v_1, v_2, \dots, v_k\}$ = vector space generated by all possible linear combinations of the vectors $v_1, v_2, \dots, v_k$

→ as a consequence, the **eigenvector** u_i associated to λ_i is **not unique**

example: $A(\alpha u_i) = \alpha A u_i = \alpha \lambda_i u_i = \lambda_i (\alpha u_i)$

→ all belong to the same linear subspace

eigenspace

example:

- $A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$ $\qquad p_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)^2(\lambda + 1)$

$\lambda_1 = 1 \qquad ma(\lambda_1) = 2$

$\lambda_2 = -1 \qquad ma(\lambda_2) = 1$

$\lambda_1 = 1$ has the eigenspace $\quad V_1 = \ker(A - I) = \ker \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} = \text{gen}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

since any linearly independent vectors generated by the chosen basis can be chosen as new basis, V_1 can also be generated by

→

$$V_1 = \text{gen}\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ \sqrt{2} \\ 2 \end{pmatrix} \right\}$$

but not as

$$V_1 \neq \text{gen}\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \right\}$$

since, although each vector belongs to V_1, they are not linearly independent

geometric multiplicity

- if λ_i eigenvalue with $ma(\lambda_i) > 1$ it will have one or more linearly independent eigenvectors u_i

ex. $$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad u_{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2 linearly independent eigenvectors

$\dim(V_1) = 2$

$$A_2 = \begin{pmatrix} \lambda_1 & \beta \\ 0 & \lambda_1 \end{pmatrix}, \quad \text{with} \quad \beta \neq 0, \quad \text{only} \quad u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1 linearly independent eigenvector

$\dim(V_1) = 1$

note that A_1 and A_2 have the same eigenvalue λ_1 both with same $ma(\lambda_1) = 2$

since they have the same characteristic polynomial $p_A(\lambda) = (\lambda - \lambda_1)^2$

Definition

the **geometric multiplicity** of λ_i is the dimension of the eigenspace associated to λ_i

$$\boxed{\dim(V_i) = mg(\lambda_i)}$$

$$mg(\lambda_i) = \dim\left[\mathrm{Ker}(A - \lambda_i I)\right] = n - \mathrm{rank}(A - \lambda_i I)$$

geometric multiplicity

useful property $\boxed{1 \leq mg(\lambda_i) \leq ma(\lambda_i) \leq n}$ → note that if $ma(\lambda_i) = 1$ then $mg(\lambda_i) = 1$

ex. 1

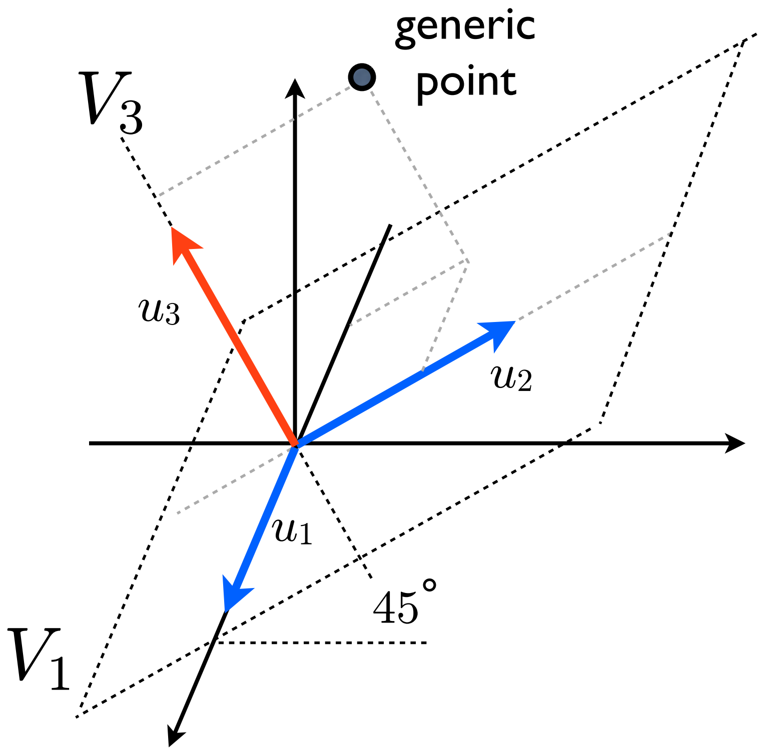
$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad (A_1 - \lambda_1 I) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad mg(\lambda_1) = 2 = ma(\lambda_1)$$

$$A_2 = \begin{pmatrix} \lambda_1 & \beta \\ 0 & \lambda_1 \end{pmatrix}, \quad (A_2 - \lambda_1 I) = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \qquad mg(\lambda_1) = 1 < ma(\lambda_1)$$

ex. 2

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

projection matrix



$$\lambda_1 = \lambda_2 = 1 \qquad u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$ma(\lambda_1) = 2 = mg(\lambda_1)$$

$$\lambda_3 = 0 \qquad u_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

diagonalization

Definition

An $(n$ x $n)$ matrix A is said to be **diagonalizable** if there exists an invertible $(n$ x $n)$ matrix T such that TAT^{-1} is a diagonal matrix

Theorem

An $(n$ x $n)$ matrix A is **diagonalizable** if and only if it has n **linearly independent eigenvectors**

since the eigenvalues are invariant under similarity transformations

if A diagonalizable $\quad TAT^{-1} = \Lambda = \text{diag}\{\lambda_i\}, i = 1, \ldots, n$

the elements on the diagonal of Λ
are the eigenvalues of A

diagonalization

Hyp: A has n linearly independent eigenvectors

(Note: this does not mean necessarily n distinct eigenvalues)

- we need to find T such that TAT^{-1} is a diagonal matrix

since by hypothesis the n eigenvectors u_i are linearly independent, the matrix having the vectors u_i as columns is necessarily non-singular

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$ non-singular nxn matrix

we rewrite the n relations defining the eigenvalues $Au_i = \lambda_i u_i, \qquad i = 1, \ldots, n$

in matrix form

$$A \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}}_{\Downarrow}$$

$$\boxed{A\mathcal{U} = \mathcal{U}\Lambda}$$

with $\Lambda = \mathrm{diag}\{\, \lambda_1, \ldots, \lambda_n \,\}$

diagonalization

from $A\mathcal{U} = \mathcal{U}\Lambda$ being $\mathcal{U}$ non-singular, we can define T such that $T^{-1} = \mathcal{U}$

$$AT^{-1} = T^{-1}\Lambda \quad \rightarrow \quad A = T^{-1}\Lambda T \quad \rightarrow \quad \Lambda = TAT^{-1}$$

therefore the **diagonalizing similarity transformation** is T s.t.

$$T^{-1} = \mathcal{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

We have an alternative necessary & sufficient condition for **diagonalizability**

> A: $(n$ x $n)$ is **diagonalizable** if and only if
>
> $mg(\lambda_i) = ma(\lambda_i)$ for every eigenvalue λ_i

rationale:
when $mg(\lambda_i) = ma(\lambda_i)$ for every eigenvalue λ_i we have $mg(\lambda_i) = ma(\lambda_i)$ linearly independent eigenvectors for every eigenvalue and therefore a total of n linearly independent eigenvectors

Warning: distinct eigenvalues (real and/or complex) $\underset{\cancel{\Longleftarrow}}{\Longrightarrow}$ A diagonalizable

diagonalization

- $$\boxed{A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}} \qquad T^{-1} = \mathcal{U} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} \qquad T = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

  $\lambda_1 =$ -1 $\qquad \lambda_2 =$ 2

  $ma(\lambda_1) = ma(\lambda_2) = 1$

  $$\frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

  note that: since the algebraic multiplicity is 1 necessarily also the geometric one is 1

- $$\boxed{A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}} \qquad V_1 = \text{gen}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\} \qquad V_2 = \text{gen}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

  $\lambda_1 =$ 1 $\qquad ma(\lambda_1) = 2 = mg(\lambda_1)$

  $\lambda_2 =$ -1 $\qquad ma(\lambda_2) = 1 = mg(\lambda_2)$

$$T^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} -1/2 & 1 & -1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \Rightarrow \Lambda = TAT^{-1} = \mathcal{U}^{-1}A\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

diagonalization

As already noted, A having **distinct eigenvalues** implies that A is **diagonalizable** (sufficient condition) but the A being diagonalizable does not necessarily imply that its eigenvalues are distinct

Sufficiency comes from the fact that distinct eigenvalues (real and/or complex) generate linearly independent eigenvectors

However the condition is **not necessary**, see for example

- $A = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix}$ diagonal but coincident eigenvalues, $ma(\lambda_i) = 2$

  however there are 2 linearly independent eigenvectors for the eigenvalue λ_i and hence the geometric multiplicity is also 2

- $A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$ diagonalizable but not distinct eigenvalues

  similarly the eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 2 has also $mg(\lambda_1) = 2$

diagonalization: complex eigenvalues case

hyp: A square matrix n x n with real elements

We know complex eigenvalues for A real come necessarily in pairs (λ_i,λ_i^*)
the complex eigenvalue λ_i will have eigennvector u_i with complex elements

consider the case $n = 2$

$$\begin{aligned} \lambda_i &= \alpha_i + j\omega_i \\ \lambda_i^* &= \alpha_i - j\omega_i \end{aligned} \quad \xrightarrow{\text{with its eigenvector}} \quad \begin{aligned} u_i &= u_{ai} + ju_{bi} \\ u_i^* &= u_{ai} - ju_{bi} \end{aligned}$$

2 choices

- **diagonalization** (since the eigenvalues are distinct, A is diagonalizable)

$$T^{-1} = \begin{bmatrix} u_i & u_i^* \end{bmatrix} \quad \rightarrow \quad D_i = TAT^{-1} = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i^* \end{bmatrix}$$

but
complex
elements

- or **real block** 2 x 2 (no diagonalization)

$$T_R^{-1} = \begin{bmatrix} u_{ai} & u_{bi} \end{bmatrix} \quad \rightarrow \quad M_i = T_R A T_R^{-1} = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$$

real
elements

linearly independent when A is real
consequence of u_i and u_i^* being linearly independent

real system representation
for complex eigenvalues

diagonalization: complex eigenvalues case

- $A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$

$$p_A(\lambda) = \lambda^2 - 2\lambda + 2 = (\lambda - 1 - j)(\lambda - 1 + j)$$

$$\lambda_1 = 1 + j \qquad u_1 = \begin{pmatrix} 2 \\ -1-j \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + j \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\lambda_1^* = 1 - j \rightarrow u_1{}^*$$

therefore $T_R^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ $T_R = \begin{bmatrix} 1/2 & 0 \\ -1/2 & -1 \end{bmatrix}$ and $T_R A T_R^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}$

- $A = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ \hline 1 & 1 & -2 & 1 \\ 2 & 1 & -2 & 0 \end{array}\right)$

$$\lambda_1 = 1 + j \rightarrow u_1 = \begin{pmatrix} -1+j \\ -2 \\ -1+j \\ -1+j \end{pmatrix} \qquad ma(\lambda_1) = 1 = mg(\lambda_1)$$

$$\lambda_2 = -1 + j \rightarrow u_2 = \begin{pmatrix} 0 \\ 0 \\ 1-j \\ 2 \end{pmatrix} \qquad ma(\lambda_2) = 1 = mg(\lambda_2)$$

$$T_R^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 2 & 0 \end{pmatrix} \qquad T_R = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ 1 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & -1 & 1/2 \end{pmatrix} \qquad T_R A T_R^{-1} = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

check with Matlab

diagonalization

- simultaneous presence of real and complex eigenvalues

If A **diagonalizable**, there exists a non-singular matrix R such that

$$RAR^{-1} = \text{diag}\left\{\Lambda_r, M_{r+1}, M_{r+3}, \dots, M_{q-1}\right\}$$

with $\Lambda_r = \text{diag}\{\lambda_1, \dots, \lambda_r\}$ for real eigenvalues

$$M_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$$

for each complex pair of eigenvalues $(\lambda_i,\lambda_i{}^*)$

and $T_R^{-1} = \begin{pmatrix} u_1 & \dots & u_r & \text{Re}[u_{r+1}] & \text{Im}[u_{r+1}] \dots & \text{Re}[u_{q-1}] & \text{Im}[u_{q-1}] \end{pmatrix}$

- $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

$$\lambda_{1,2} = -1 \pm j$$
$$\lambda_3 = -1$$

Matlab code

```
A = [0,0,-1;0,-2,1;1,-1,-1];
[V,D] = eig(A)
Tinv = [real(V(:,1)),imag(V(:,1)),V(:,3)];
inv(Tinv)*A*Tinv
```

$$\rightarrow \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

diagonalization

example

$$A = \begin{bmatrix} -6 & 5 & 4 \\ 5 & -6 & -16 \\ -10 & 10 & 9 \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = -1 \\ \lambda_{2/3} = -1 \pm 10j \end{array}$$

$$\lambda_1 = -1 \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1 + 10j \qquad u_2 = \begin{bmatrix} 0.5 + 0.1j \\ -0.5 + 1.1j \\ 1 \end{bmatrix} \qquad u_{2a} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} \qquad u_{2b} = \begin{bmatrix} 0.1 \\ 1.1 \\ 0 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} u_1 & u_{2a} & u_{2b} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 1 & -0.5 & 1.1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow TAT^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 10 \\ 0 & -10 & -1 \end{bmatrix}$$

A diagonalizable: spectral decomposition

Hyp: A diagonalizable

$$A = \mathcal{U} \Lambda \mathcal{U}^{-1}$$

columns (linearly independent)

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \longrightarrow \mathcal{U}^{-1} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

rows

these are also left eigenvectors

$$\mathcal{U}^{-1}\mathcal{U} = I \quad \Rightarrow \quad \underline{v_i^T u_j = \delta_{ij}}, \quad i, j = 1, \dots, n$$

here, the choice of the left eigenvector should be so that this condition is met

therefore computing explicitly

$$A = \mathcal{U} \Lambda \mathcal{U}^{-1} = \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

spectral form of A

$$\boxed{A = \sum_{i=1}^{n} \lambda_i \; u_i \; v_i^T}$$

or **eigen-decomposition**

A diagonalizable: spectral decomposition

$$A = \sum_{i=1}^{n} \lambda_i \; u_i \; v_i^T$$

column

row

$$\Big\} \; (n \times n) \longrightarrow \quad P_i = u_i v_i^T$$

is the **projection matrix** on the invariant subspace generated by u_i

$$\boxed{A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}$$

$$\lambda_1 = 2$$

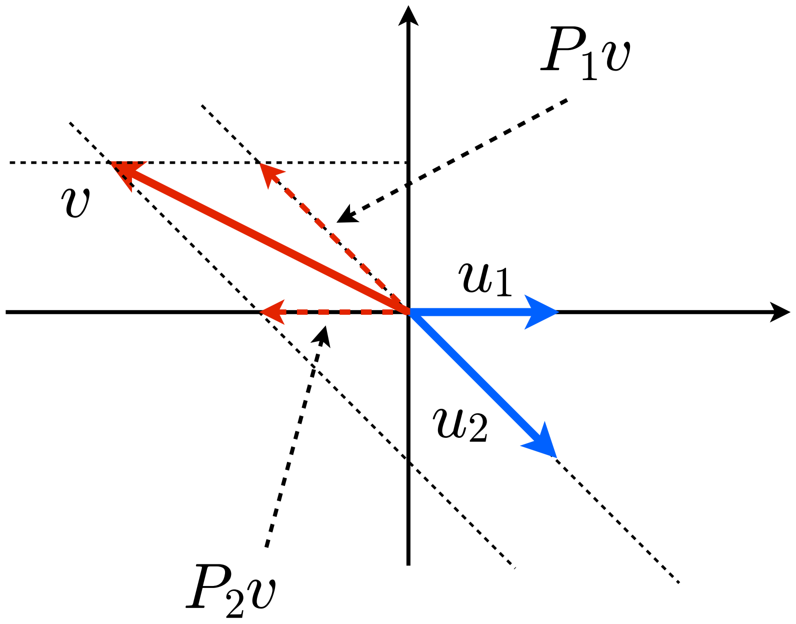
$$\lambda_2 = 1$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_1^T = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad v_2^T = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

$$P_1 v = -1 \cdot u_1 \qquad P_2 v = 1 \cdot u_2$$



$$v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

non-diagonalizable case

for illustrative purposes, we assume that λ_i is the only eigenvalue and therefore the algebraic multiplicity will be equal to n if A is n x n

since A is not diagonalizable we have $0 < mg(\lambda_i) < ma(\lambda_i) = n$

in this case A is said to be **defective**

→ then there exists a change of coordinates such that in TAT^{-1} the eigenvalue λ_i will have associated $mg(\lambda_i)$ Jordan blocks J_k with $k = 1, \ldots, mg(\lambda_i)$, each of dimension n_k (we will not explore how to determine n_k)

single
Jordan block
of dimension n_k

$$J_k = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_k \times n_k}$$

- note that the null space of J_k - $\lambda_i I$ has dimension 1
- the dimension of the largest Jordan block associated to λ_i is called **index**

non-diagonalizable case (A defective)

In this case the eigenvector is replaced by the generalized eigenvector which will form chains of n_k generalized eigenvectors (not part of this course)

The resulting matrix, after a proper change of coordinates, is the **Jordan canonical form** which is block diagonal matrix having the Jordan blocks on the main diagonal

- example:

unique eigenvalue λ_i of matrix A $(n$ x $n)$ with geometric multiplicity $mg(\lambda_i) = p$

and algebraic multiplicity $ma(\lambda_i) = n = \sum_{k=1}^{p} n_k$

then there exists a nonsingular matrix T such that $TAT^{-1} = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$

with $J_k \in \mathbb{R}^{n_k \times n_k}$ Jordan block of dim n_k for each $k = 1, \ldots, p$

non-diagonalizable case (multiple c.c. eigenvalues)

What happens if we have repeated complex conjugate eigenvalues in the non-diagonalizable case, i.e., $mg(\lambda_i) < ma(\lambda_i)$?

Let’s consider, for illustrative purposes, $n = 4$ with the repeated pair of complex conjugate eigenvalues $(\lambda_i, \lambda_i{}^*)$, $\lambda_i = \alpha_i + j\omega_i$ with $ma(\lambda_i) = 2$ and $mg(\lambda_i) = 1$

→ we know there exists a change of coordinates T which will lead to one Jordan block of dimension 2 for each eigenvalue

$$TAT^{-1} = \left(\begin{array}{cc:cc} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \\ \hdashline 0 & 0 & \lambda_i^* & 1 \\ 0 & 0 & 0 & \lambda_i^* \end{array}\right)$$

block diagonal complex

→ however we have the usual problem that the resulting matrix is complex; nevertheless as in the diagonalizable case, there exists a change of coordinates T_R which leads to a real matrix

$$T_R A T_R^{-1} = \left(\begin{array}{cc:cc} \alpha_i & \omega_i & 1 & 0 \\ -\omega_i & \alpha_i & 0 & 1 \\ \hdashline 0 & 0 & \alpha_i & \omega_i \\ 0 & 0 & -\omega_i & \alpha_i \end{array}\right)$$

block triangular real

special cases

- if $ma(\lambda_i) = 1$ then $mg(\lambda_i) = 1$

- if $mg(\lambda_i) = 1$ then only one Jordan block of dimension $ma(\lambda_i)$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \lambda_1 = -1 \qquad ma(\lambda_1) = 2 \qquad mg(\lambda_1) = 1$$

- if λ_i unique eigenvalue of matrix A and if $\mathrm{rank}(\lambda_i I$ - $A) = ma(\lambda_i)$ - 1

  then only one Jordan block of dimension $ma(\lambda_i)$

  consequence of the **rank-nullity theorem** applied to A - $\lambda_i I$

$$A - \lambda_i I : \mathbb{R}^n \to \mathbb{R}^n$$

$$\dim\left(\mathbb{R}^n\right) = \dim\left(\mathrm{Ker}(A - \lambda_i I)\right) + \dim\left(\mathrm{Im}(A - \lambda_i I)\right)$$

n nullity$(A$ - $\lambda_i I)$ rank$(A$ - $\lambda_i I)$

summary

A real diagonalizable $mg(\lambda_i) = ma(\lambda_i)$ for all i	$\exists\ T$ s.t. $TAT^{-1} = \Lambda$ $\Lambda = \text{diag}\{\lambda_i\}$	for real & complex λ_i $\Lambda_r = \text{diag}\{\lambda_1, \dots, \lambda_r\}$ alternative choice for complex $(\lambda_i,\lambda_i{}^*)$ $M_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$
	$A = \sum_{i=1}^{n} \lambda_i\ u_i\ v_i^T$	spectral form
A real not diagonalizable $mg(\lambda_i) < ma(\lambda_i)$	$\exists\ T$ s.t. $TAT^{-1} = \text{diag}\{J_k\}$ block diagonal	$mg(\lambda_i)$ Jordan blocks of the form $J_k = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_i & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_k \times n_k}$

vocabulary

English	Italiano
eigenvalue/eigenvector	autovalore/autovettore
characteristic polynomial	polinomio caratteristico
algebraic/geometric multiplicity	molteplicità algebrica/geometrica
similar matrix	matrice simile
spectral form	forma spettrale