Control Systems

Models

L. Lanari

Dipartimento di Ingegneria Informatica
Automatica e Gestionale Antonio Ruberti

Sapienza
Università di Roma

some concepts from the last lecture

- analysis & control of dynamical systems
- dynamics/motion
- models/mathematical models
- prediction/simulation
- linearity
- time-invariance
- feedback control scheme: principle and example

this lecture

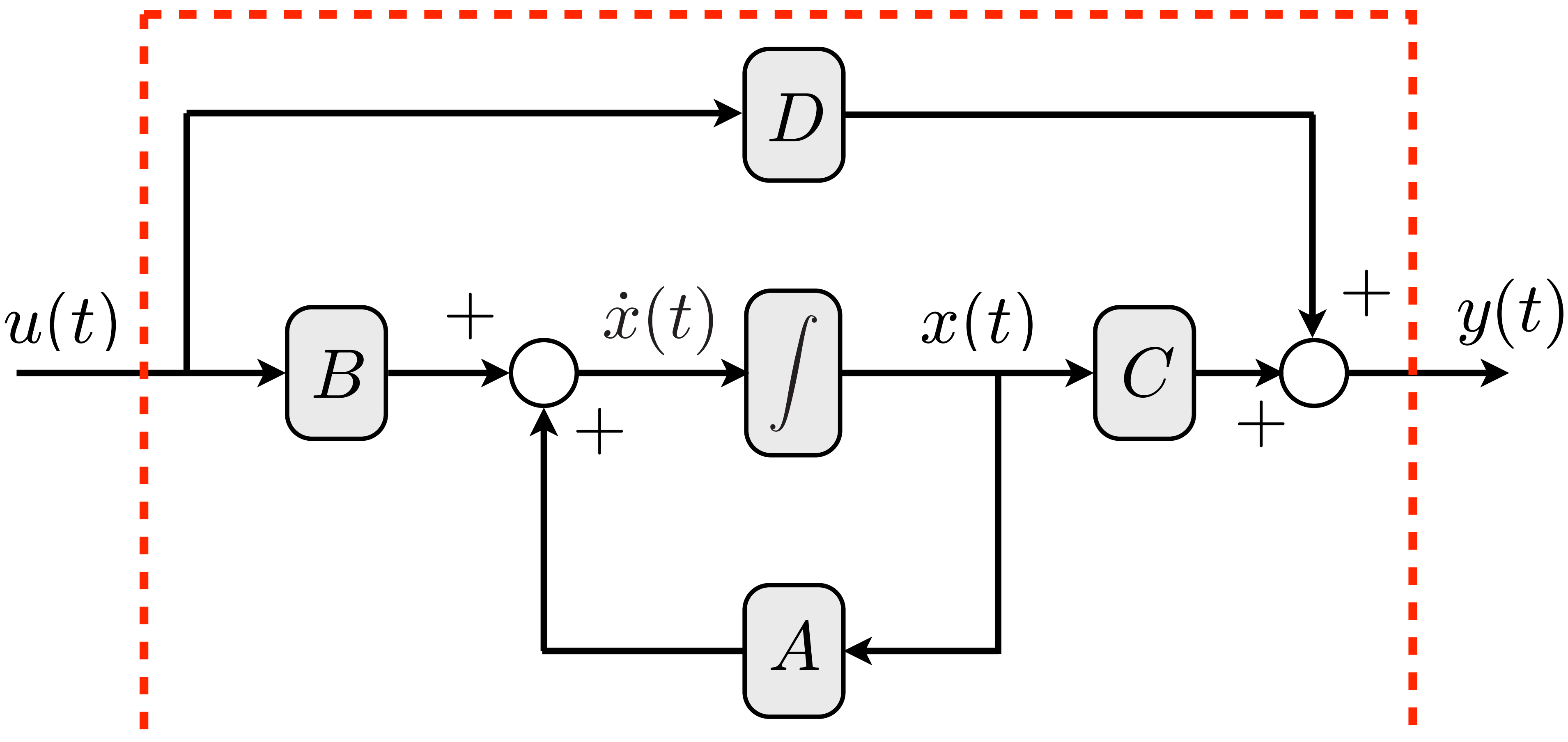
- models: from differential equations to state space representation
- input & output choice
- state
- similarity transformations

general mathematical model

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$
$$x(0) = x_0$$

state space representation (A, B, C, D) with state x
of a linear time invariant (LTI)
dynamical system (continuous time)

block diagram representation



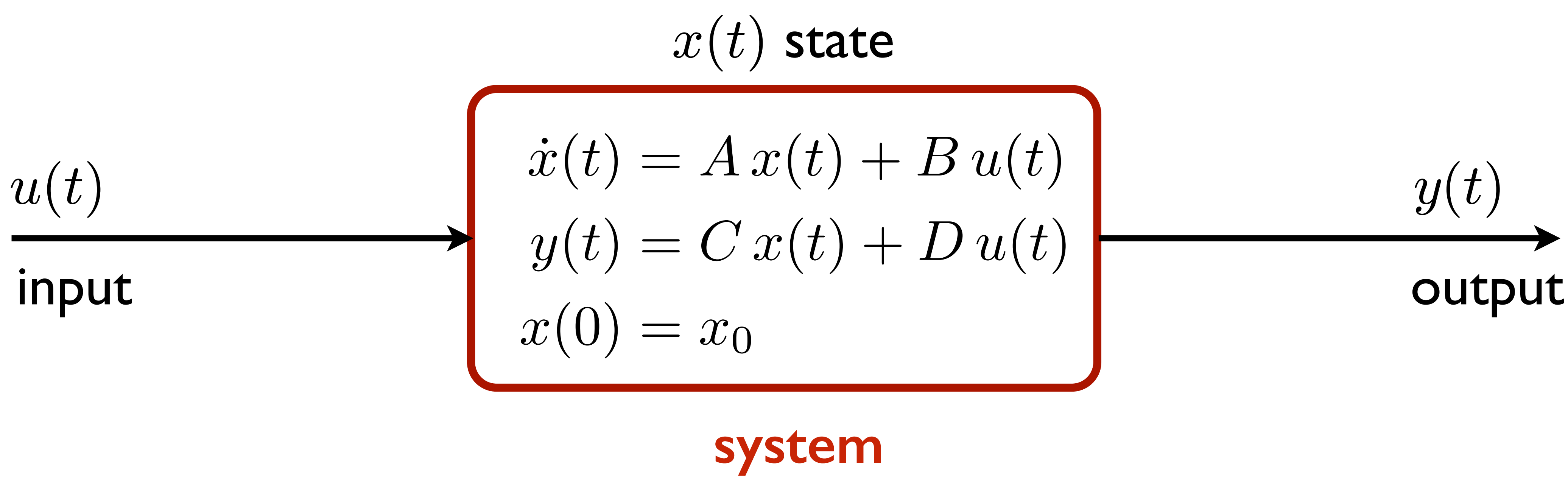
simulation model

$x(t)$ **state**
$u(t)$ **input**
$y(t)$ **output**

A: n x n
B: n x 1
C: 1 x n
D: 1 x 1

SISO

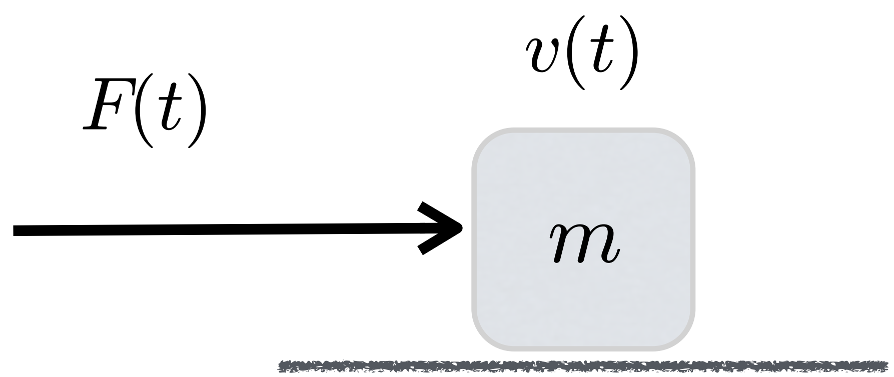
mathematical model



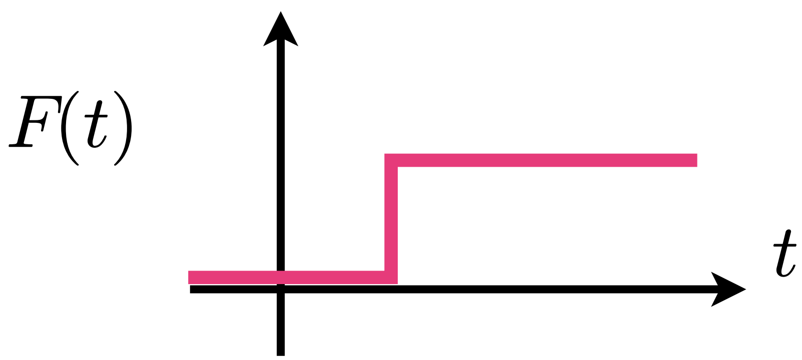
- the state evolution is influenced by the initial condition and the input
- the output displays the measurable effect of such state evolution (and potentially may also depend directly on the input when D is non-zero)

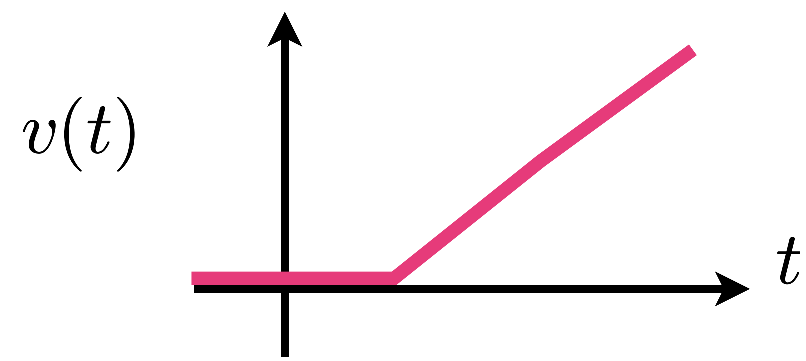
the system transforms an input signal into an output through the state evolution (and possibly the input)

system as a signal transformer (optional)



no friction case
and $v_0 = 0$



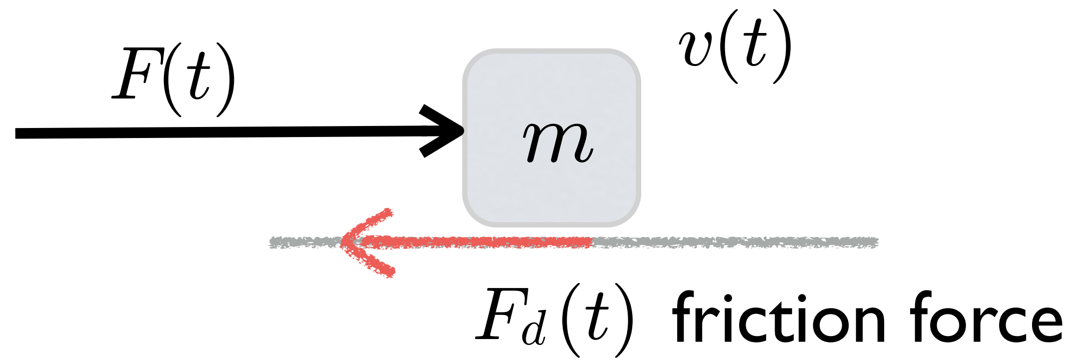


model
(system representation)

$$\dot{v}(t) = \frac{1}{m} F(t)$$

hyp: no friction

mass model with viscous friction



we add a viscous friction force $F_d(t)$, acting on the mass, which can be considered proportional to the velocity (this is a modeling hypothesis) and acts in the opposite direction w.r.t. the motion, that is $F_d(t) = \mu\, v(t)$ and with $\mu > 0$

Newton's equation gives $$m\,\dot{v}(t) = -F_d(t) + F(t) = -\mu\, v(t) + F(t)$$

or $$\dot{v}(t) = -\frac{\mu}{m}\, v(t) + \frac{1}{m} F(t)$$

the state space system representation (model) is in the form $\dot{x}(t) = A\, x(t) + B\, u(t)$

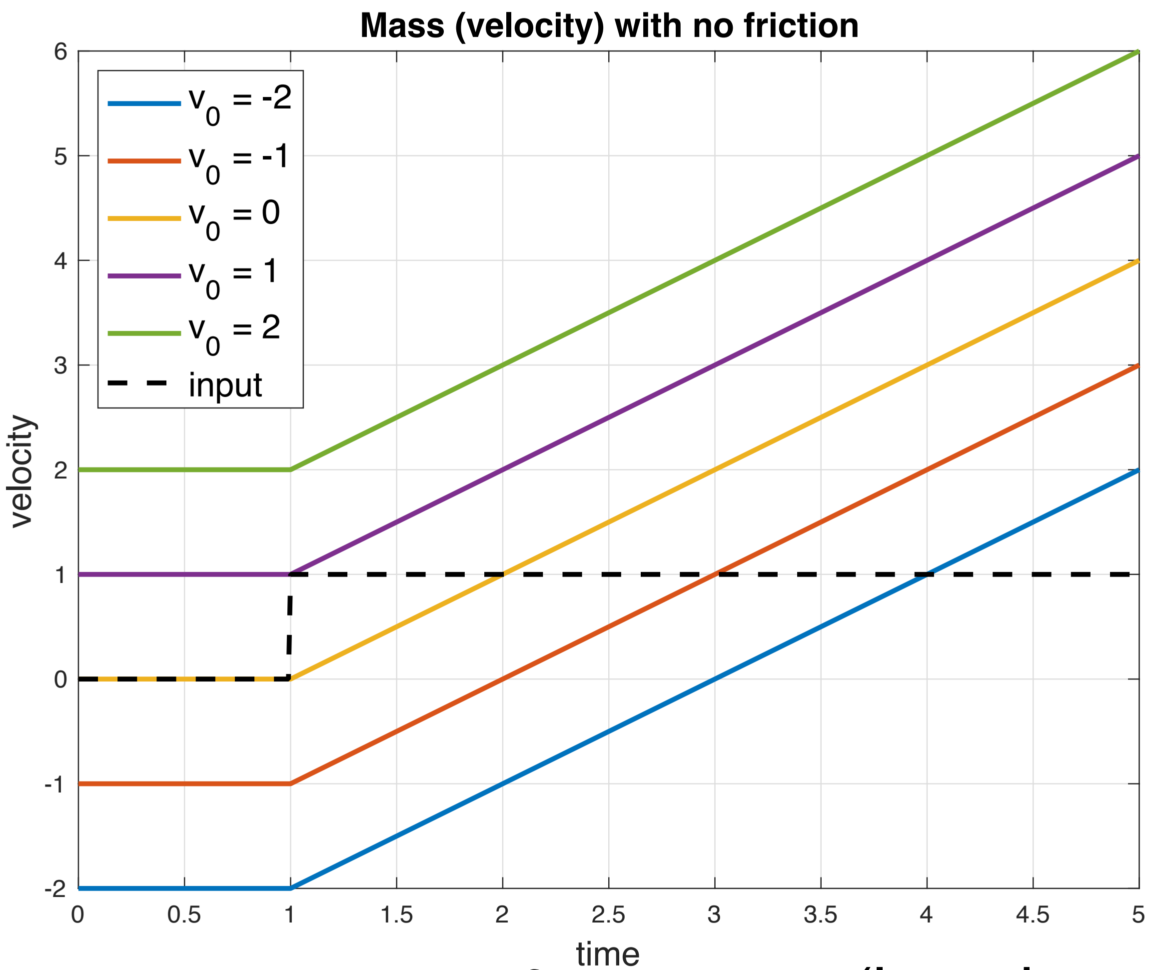
with state is $x(t) = v(t)$ and input $u(t) = F(t)$ and $A = -\dfrac{\mu}{m}$ $\quad B = \dfrac{1}{m}$

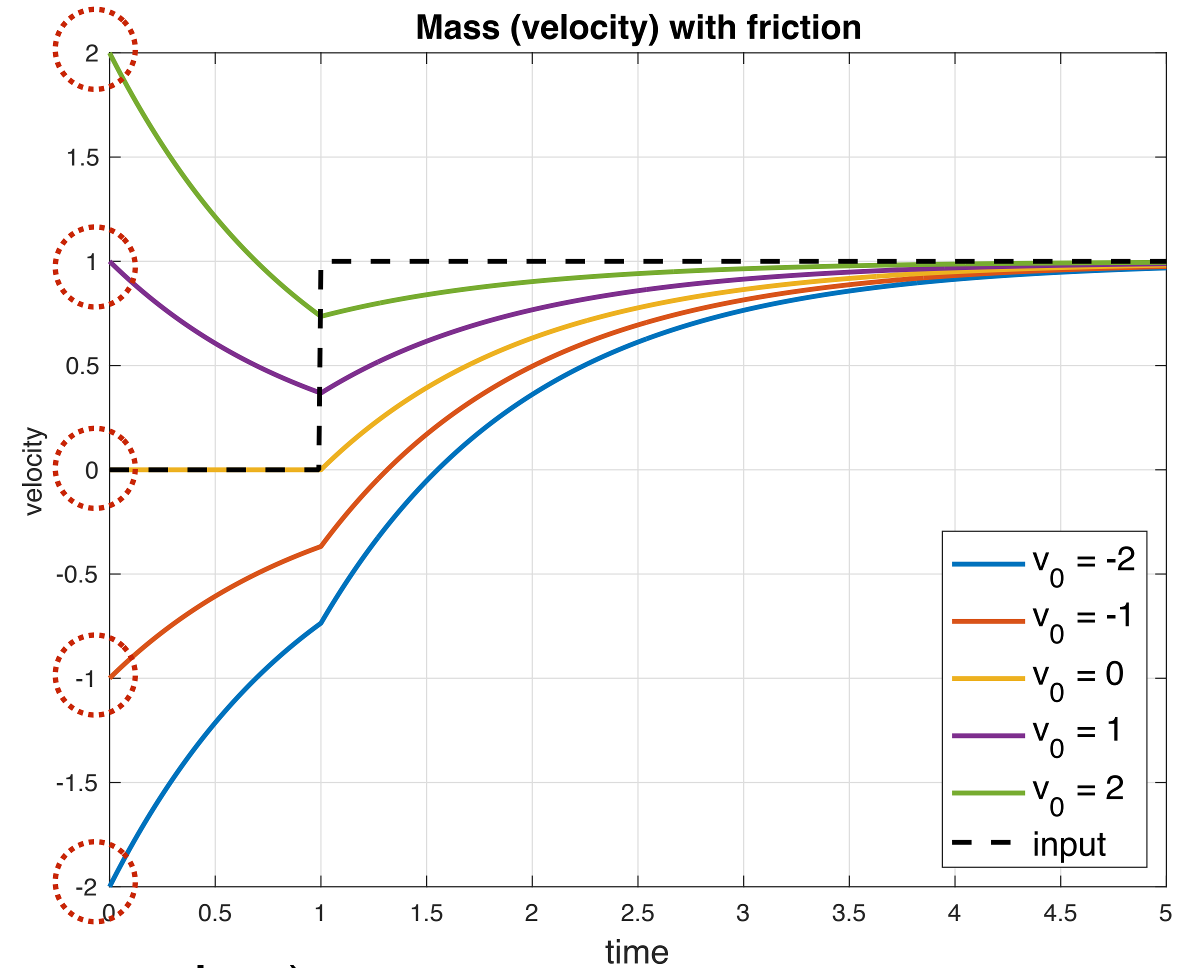
this could be a car simplified model that could be used in a speed control problem

goal $v(t) = v^{des}(t)$

mass system - simulations

initial velocities





no friction
(different initial velocities)

(learn how to interpret plots)

with friction
(different initial velocities)

velocity time evolution when a constant 1 N force is applied from $t = 1$ sec

suggested problem:
use only piece-wise constant $F(t) = \pm F$ in order to change velocity from v_1 to v_2
(you can switch value at any time)

signals & systems perspective (optional)

MIT OpenCourseware Dennis Freeman 600.3 - Fall 2011 Signals & Systems

Lecture 1: Signals and Systems

analysis & design of systems via their signal transformation properties

system transforms an input signal into an output signal

how: system description (we saw mathematical model)

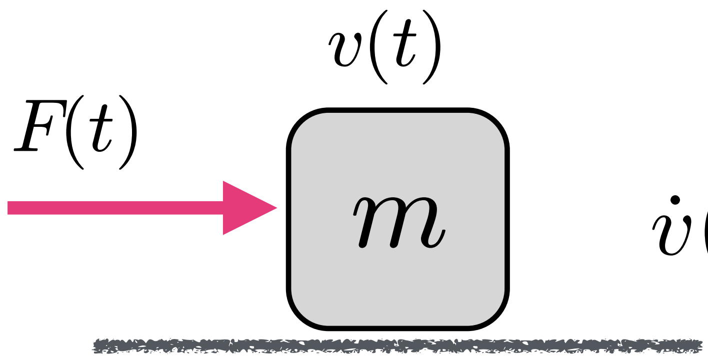
is independent from physical substrate (e.g., cell phone)

focus: flow of information

abstract, widely applicable, modular, hierarchical

different systems may have similar models

- linear motion under the action of a force



$$\dot{v}(t) = \frac{1}{m} F(t)$$

choosing

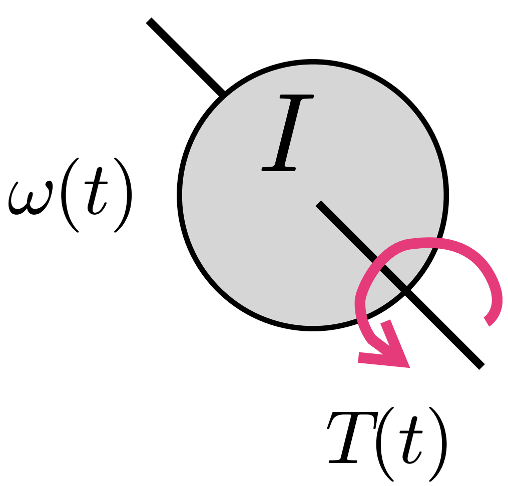
$$x(t) = v(t) \qquad b = 1/m$$
$$u(t) = F(t)$$

$$\dot{x}(t) = 0 \cdot x(t) + b \cdot u(t)$$

same model structure

$$\dot{x}(t) = b \cdot u(t)$$

- rotational motion under the action of a torque

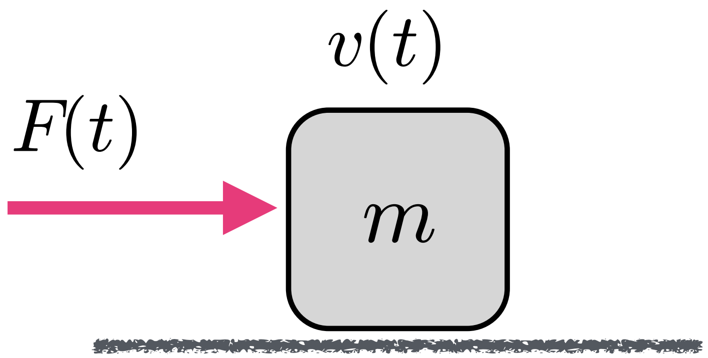


$$\dot{\omega}(t) = \frac{1}{I} T(t)$$

$$x(t) = \omega(t)$$
$$u(t) = T(t) \qquad b = 1/I$$

similar models and similar behavior

- linear motion under the action of a force



$$v(t) = v_0 + \frac{1}{m}\int_0^t F(\tau)d\tau$$

becomes

for the generic system

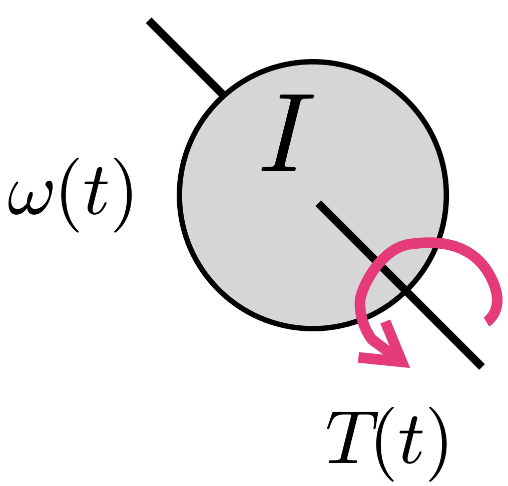
$$\dot{x}(t) = 0 \cdot x(t) + b \cdot u(t)$$

the full solution is

$$x(t) = x_0 + b\int_0^t u(\tau)d\tau$$

becomes

- rotational motion under the action of a torque



$$\omega(t) = \omega_0 + \frac{1}{I}\int_0^t T(\tau)d\tau$$

similar models and similar behavior

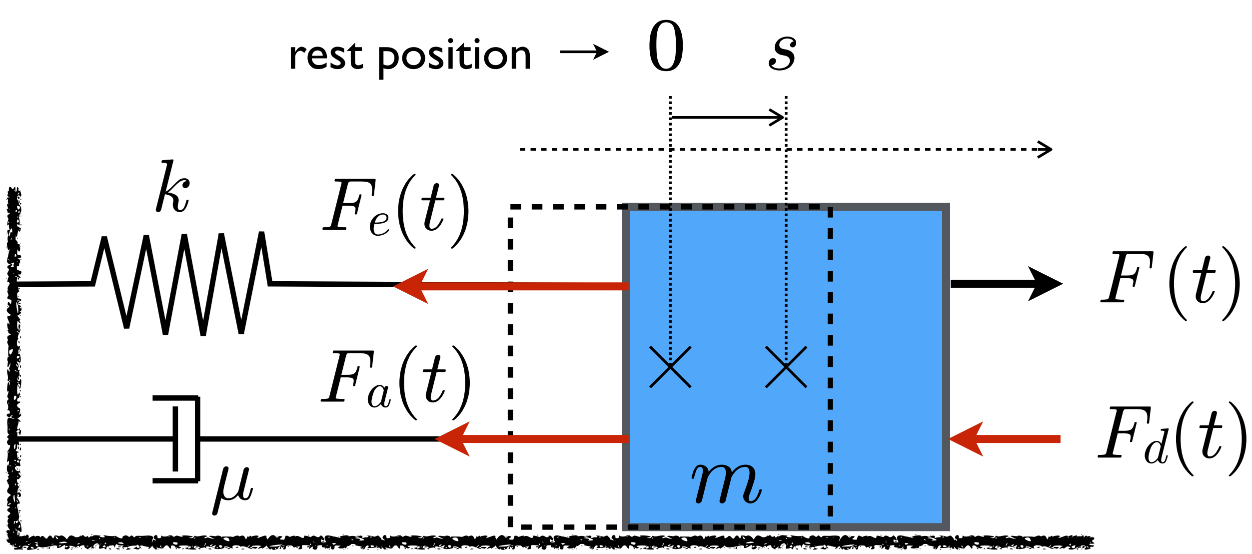
$$\dot{v}(t) = \frac{1}{m} F(t) \quad \text{linear motion} \qquad\qquad \dot{v}_C(t) = \frac{1}{C} i_C(t) \quad \text{capacitor}$$

$$\dot{\omega}(t) = \frac{1}{I} T(t) \quad \text{angular motion} \qquad\qquad \dot{i}_L(t) = \frac{1}{L} v_L(t) \quad \text{inductor}$$

the problem of making the mass move at a constant desired velocity through the action of a force is formally similar than the one of making a rigid body rotate at a given angular velocity through the action of a torque or making the voltage capacitor reach a desired value (through the current) and an inductor current converge to a reference value (through the voltage).

The inputs (respectively $F(t), T(t), i_C(t)$ and $v_L(t)$) and the system parameters (respectively m, I, C and L) have clearly different physical meanings but the structure of the differential equation is identical so the structure of the general solution will also be the same.

mass-spring-damper (MSD)



$F(t)$ control force

$F_d(t)$ disturbance force

$F_e(t)$ elastic force

$F_a(t)$ friction force

rest position: with zero velocity and no forces applied

we have added a disturbance force
(e.g. some external unknown action or if the motion is not on a horizontal plane)

Newton's second law of motion $\quad m\,a(t) = F(t) - F_e(t) - F_a(t) - F_d(t)$

$s(t) =$ deviation from the spring rest position

modeling hypothesis

$F_e(t) = k\ s(t)$ linear spring

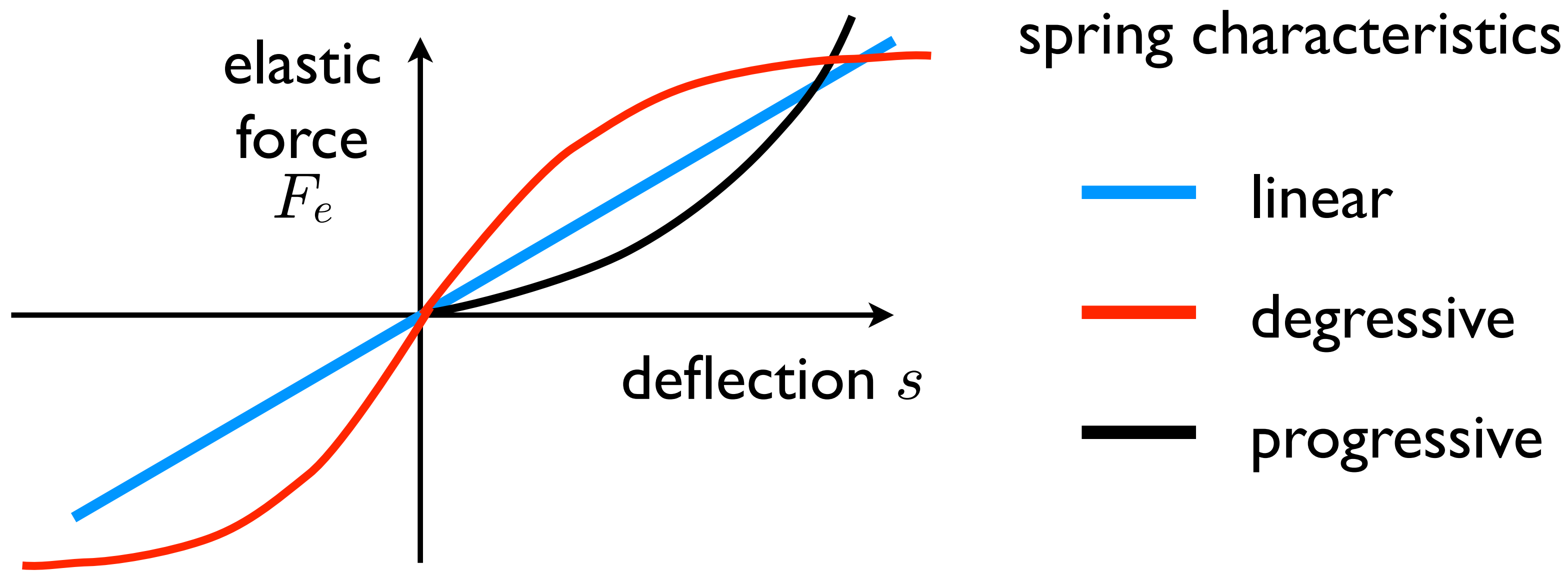
$F_a(t) = \mu\ v(t)$ linear viscous friction

$v(t) = \dot{s}(t) \qquad a(t) = \ddot{s}(t)$

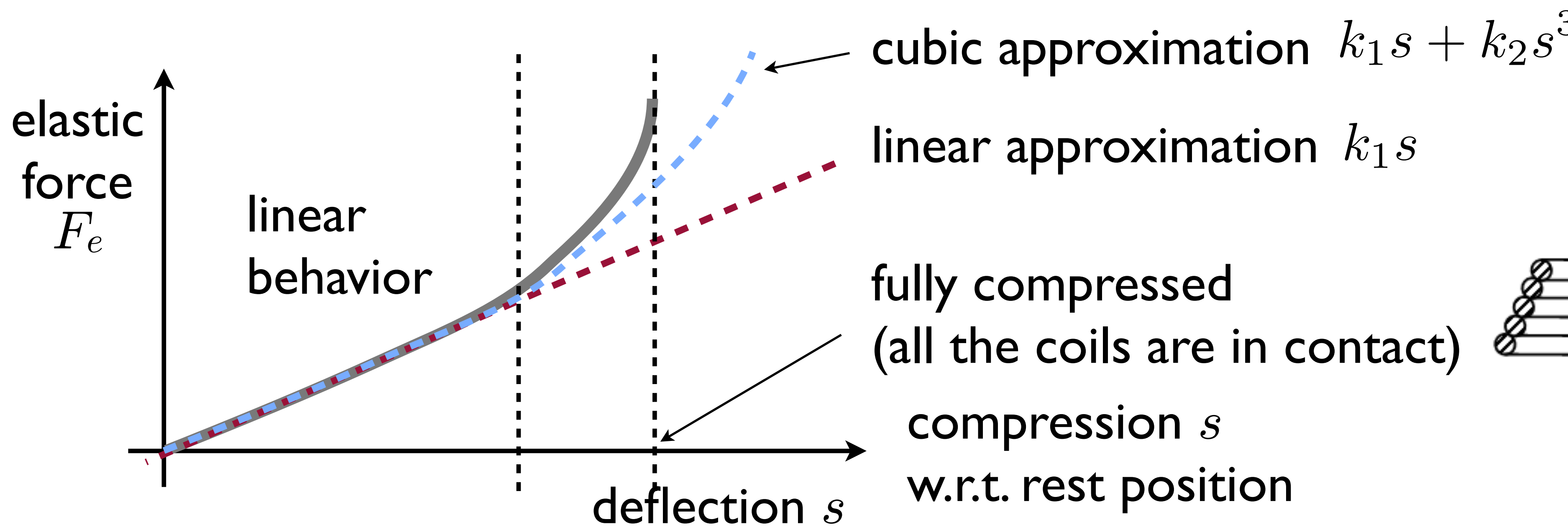
$$m\,\ddot{s} = F(t) - k\,s - \mu\,\dot{s} - F_d(t)$$

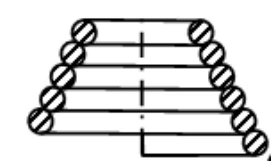
MSD - elastic force

modeling hypothesis



example:
progressive spring



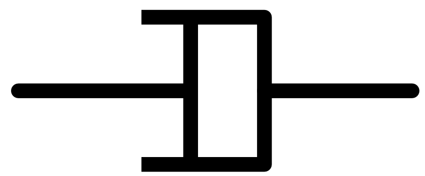


the (linear) approximation of the spring characteristic is part of the modeling phase similarly for the hypothesis on the friction force

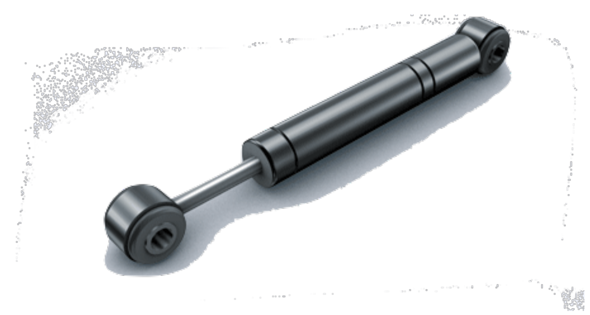
MSD - friction force

we assumed the viscous friction force $F_d\ (t)$ to be proportional to the velocity and acting in the opposite direction

we indicate schematically the presence of viscous friction with the symbol and call it a **damper**

damper 

a mechanical damper is also called **dashpot**



other models could be

- considering the friction independent from the velocity but proportional to the normal force (w.r.t. the ground)
- or static friction

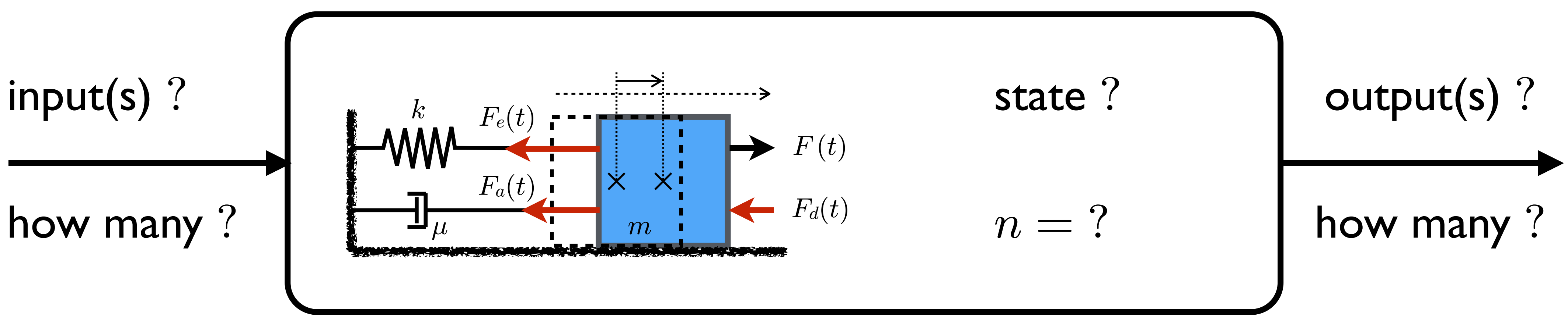
MSD - state space representation

$$m\,a(t) = F(t) - F_e(t) - F_a(t) - F_d(t) \xrightarrow[\text{hypothesis}]{\text{modeling}} \boxed{m\,\ddot{s} = F(t) - k\,s - \mu\,\dot{s} - F_d(t)}$$

how do we rewrite this linear model (second order differential equation) in the standard state space form which considers only first order differential equations?

we need to

- define the state
- define the input(s) and output(s)
- rewrite the second order differential equation in terms of the state and its derivative



can also be seen as a signal transformer

MSD - state space representation

$$m\,\ddot{s}(t) = F(t) - k\,s(t) - \mu\,\dot{s}(t) - F_d(t) \quad \Longleftrightarrow \quad \dot{x}(t) = A\,x(t) + B\,u(t)$$

if yes, how ?

input: we may have 2 choices for defining the input vector:

- single (scalar) input $u(t) = F(t) - F_d(t)$
  (if it is not necessary to distinguish between the control input $F(t)$ and the disturbance $F_d(t)$, for example in a pure analysis context)

- 2-dimensional vector: two distinctive inputs $F(t)$ and $F_d(t)$ become a unique two dimensional input vector $u(t)$

$$u(t) = \begin{pmatrix} F(t) \\ F_d(t) \end{pmatrix} \quad \Longrightarrow \quad B : 2 \text{ x } 2$$

state dimension

2 inputs

the choice depends on the problem of interest: analysis or control

MSD - state space representation

choosing as state the **position** displacement and the **velocity** of the mass

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} s(t) \\ \dot{s}(t) \end{pmatrix}$$

we can rewrite the second order differential equation as

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{k}{m} x_1(t) - \frac{\mu}{m} x_2(t) + \frac{1}{m}(F(t) - F_d(t))$$

from $\boxed{\text{1 second order differential equation}}$ to $\boxed{\text{2 first order differential equations}}$

how does this translate in matrix form ?

MSD - state space representation

we want to rewrite the system model in the form $\dot{x}(t) = Ax(t) + Bu(t)$

- single input case

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}}^{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}^{B} \overbrace{(F(t) - F_d(t))}^{u(t)}$$

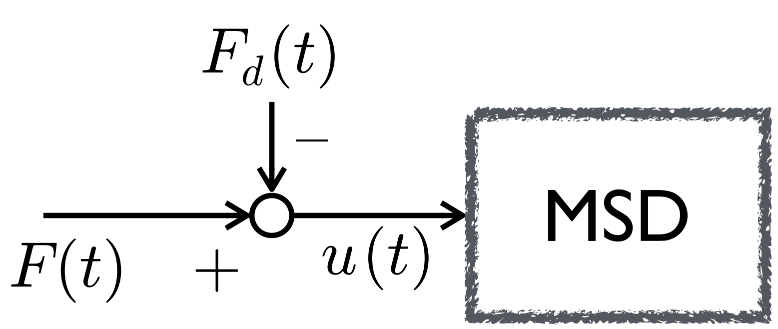
with $u(t) = F(t) - F_d(t)$

this is a special case since the two inputs enter at the same "level"

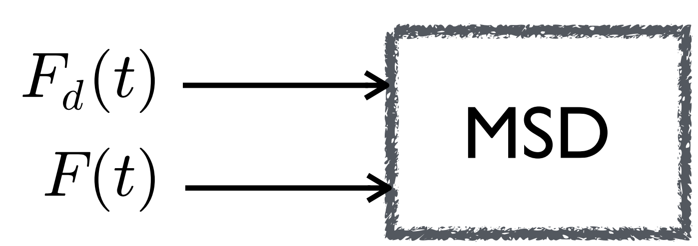
- input vector case

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}^{B_1} F(t) - \overbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}^{B_2} F_d(t)$$

with $u(t) = \begin{pmatrix} F(t) \\ F_d(t) \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & -B_2 \end{pmatrix}$



or



MSD - state space representation

matrices A and B are characteristics of the given system,
while C and D depend upon the particular chosen **output**

examples of possible choices for the MSD system:

$$y(t) = s(t) \longrightarrow \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D = 0$$

$$y(t) = \dot{s}(t) \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad D = 0$$

$$y(t) = s(t) - \pi \dot{s}(t) \qquad C = \begin{pmatrix} 1 & -\pi \end{pmatrix} \quad D = 0$$

no special physical meaning

$$y(t) = \ddot{s}(t)$$

we use $\ddot{s} = \underbrace{-\frac{k}{m} s - \frac{\mu}{m} \dot{s}} + \boxed{\frac{1}{m}} u$

$$C = \begin{pmatrix} -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}$$

feedthrough term D

high order ODE

we can generalize to a nth order differential equation

$$z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_2 z^{(2)} + a_1 z^{(1)} + a_0 z^{(0)} = bu(t)$$

with $\quad z^{(i)}(t) = \dfrac{d^i z(t)}{dt^i}$

one possible choice for the **state** is $x(t) = \begin{pmatrix} z^{(0)} \\ z^{(1)} \\ \vdots \\ z^{(n-1)} \end{pmatrix}$ n dimensional vector

then we find (A,B)

the dimension of the state vector is $n =$ number of initial conditions necessary to define a unique solution of the n-th order differential equation

other choices for the state vector are possible but all have dimension n

MSD as a special case $\qquad m\ddot{s} + \mu\dot{s} + ks = u$

high order ODE

with this choice of the state we can determine (A,B)

$$\dot{x}(t) = \begin{pmatrix} z^{(1)} \\ z^{(2)} \\ \vdots \\ z^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & 1 \\ -a_0 & -a_1 & & \ddots & -a_{n-1} \end{pmatrix} \begin{pmatrix} z^{(0)} \\ z^{(1)} \\ \vdots \\ z^{(n-1)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix} u$$

$$= \quad A \quad x(t) \quad + \quad B \quad u(t)$$

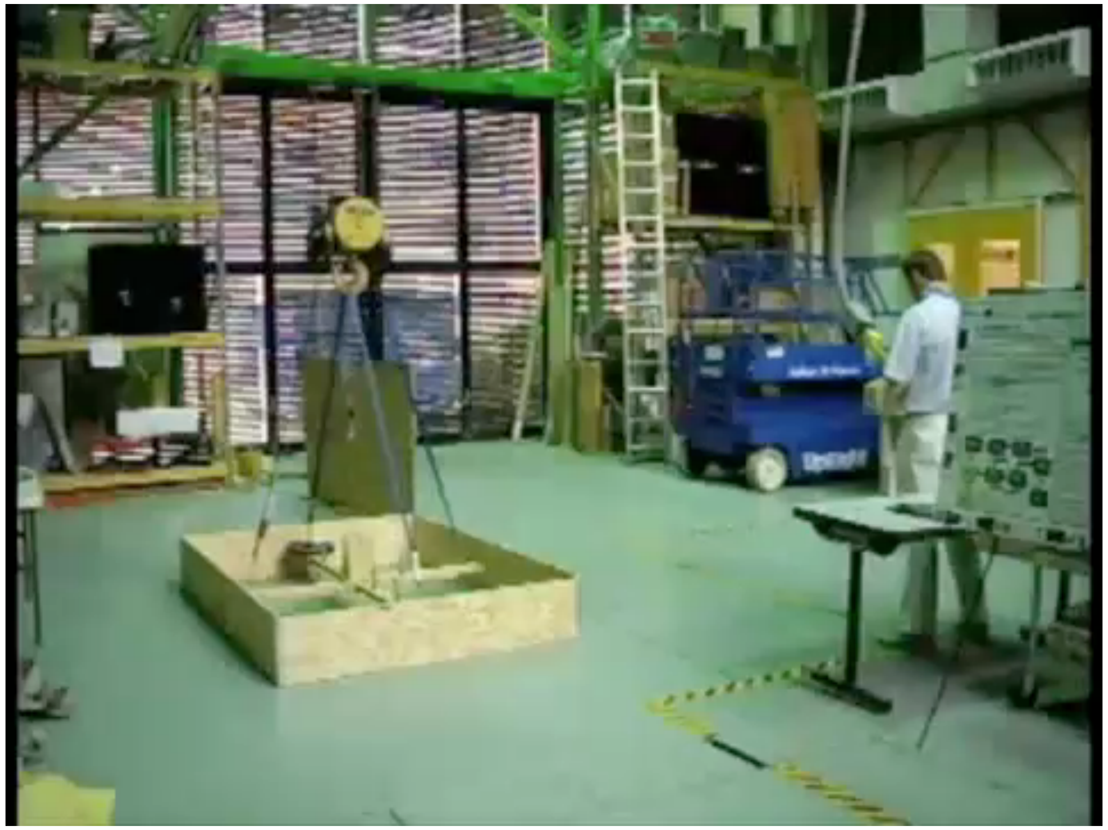
then, for example, choosing the output as $y(t) = z(t)$ we determine C and D

$$y(t) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} x(t)$$

$$= \quad C \quad x(t) \quad + \quad 0 \quad u(t) \qquad \text{i.e.} \quad D = 0$$

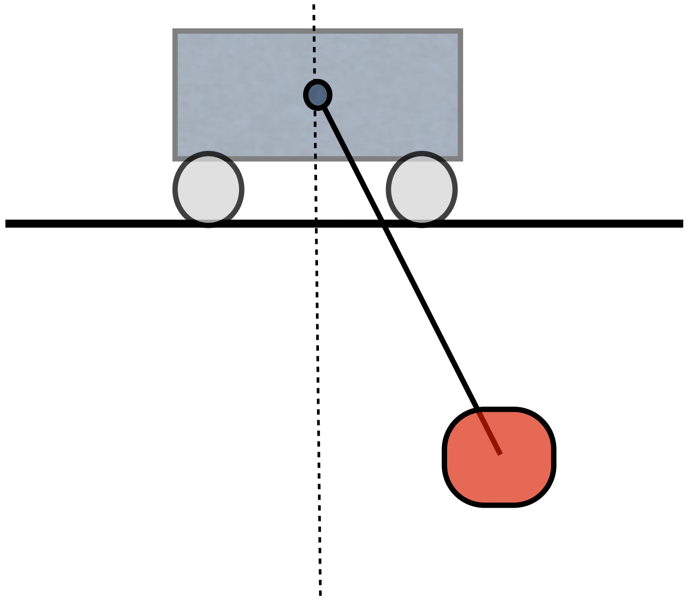
importance of dynamics

- description of the motion (e.g. satellite trajectory)
- simulation models (system behavior w.r.t. to inputs)



https://youtu.be/zlor8vGNL0c?feature=shared

simplified model



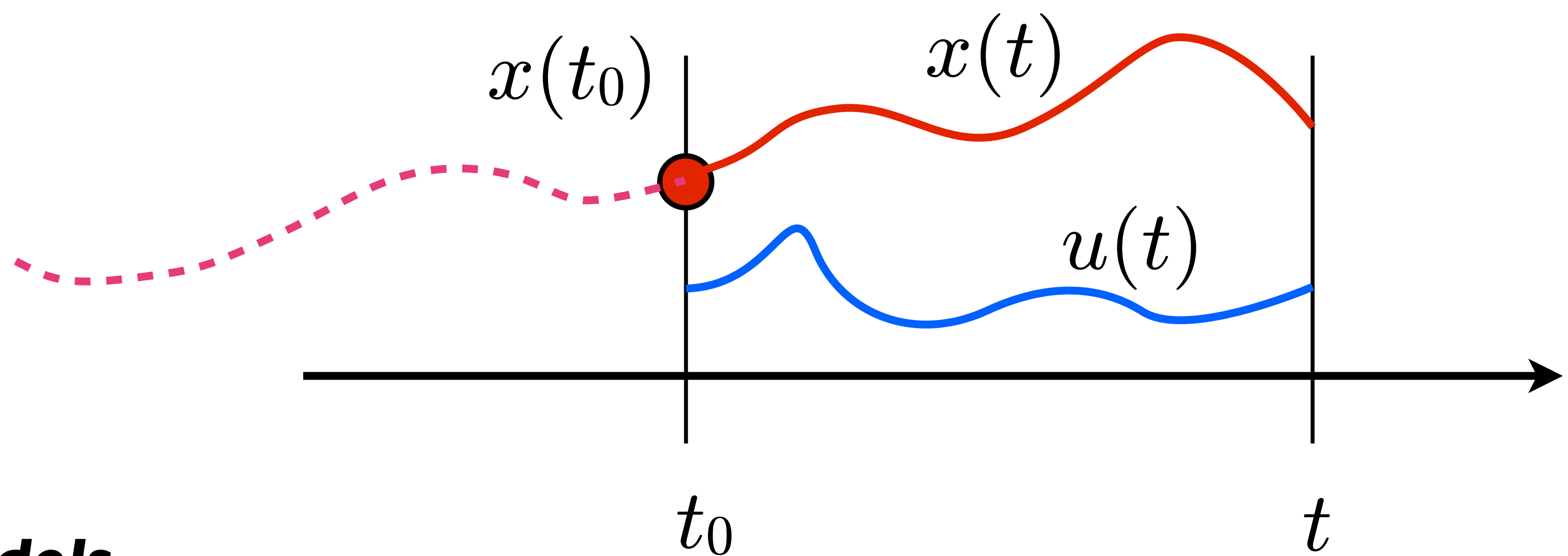
crane control: how control can help an inexpert operator who has to move a payload with a crane - here using a technique called “input shaping” (Georgia Tech)

state

The **state** of a dynamical system is a set of physical quantities (**state variables**), the specification of which (in the absence of external excitation) completely determines the evolution of the system
(B. Friedland)

Specific physical quantities that define the state are not unique, although their number (**system order**) is unique

Alternatively, the state can be seen as the minimum set of variables such that their knowledge at time t_0, together with the knowledge of the external excitations (inputs) in $[t_0,t)$, allows the complete characterization of the system evolution in $[t_0,t)$



state dimension - examples

- first order system $\quad x \in \mathbf{R} \quad \dot{v}(t) = \frac{1}{m} F(t)$
- second order system $\quad x \in \mathbf{R}^2 \quad m\ddot{s} + \mu\dot{s} + ks = u$
- n-th order system $\quad x \in \mathbf{R}^n$

$$z^{(n)} + a_{n-1} z^{(n-1)} + \cdots + a_2 z^{(2)} + a_1 z^{(1)} + a_0 z^{(0)} = bu(t)$$

- $2{+}1 = 3$rd order system

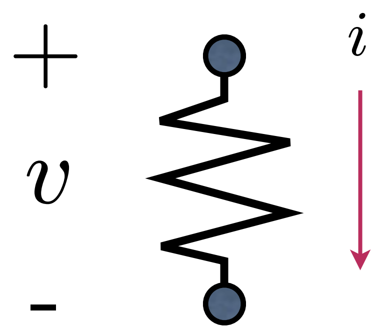
$$\begin{aligned} a_2 \ddot{x}_1 + a_1 \dot{x}_1 + a_0 x_1 &= \alpha u \\ b_1 \dot{x}_2 + b_0 x_2 &= \beta u \end{aligned} \qquad x = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^3$$

- two 2nd order equations

$$\begin{aligned} a_{11} \ddot{x}_1 + a_{12} \ddot{x}_2 + \dot{x}_1 - x_2 &= 0 \\ a_{21} \ddot{x}_1 + a_{22} \ddot{x}_2 - \dot{x}_1 - \dot{x}_2 + 3x_2 &= 0 \end{aligned} \qquad x = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} \in \mathbf{R}^4$$

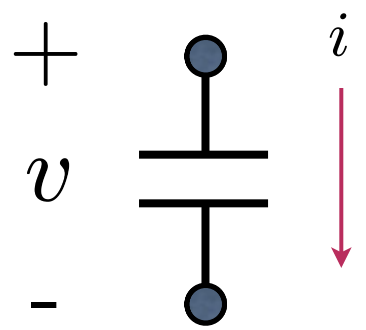
models of electrical circuits

Resistor



$$v(t) = R\, i(t)$$

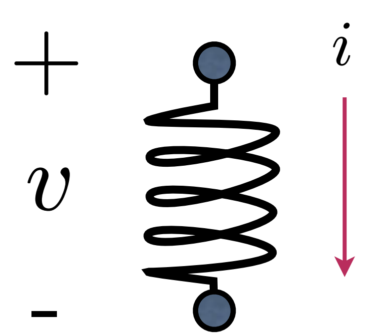
Capacitor



$$i(t) = C\,\frac{dv(t)}{dt}$$

or $$\dot v(t) = \frac{1}{C}\, i(t)$$

Inductor

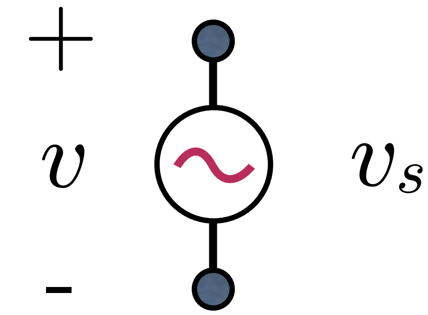


$$v(t) = L\,\frac{di(t)}{dt}$$

or $$\dot i(t) = \frac{1}{L}\, v(t)$$

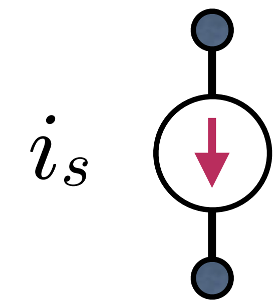
elements which store energy → **state components**

(one for each energy storing element)



voltage source

$$v(t) = v_s(t)$$



current source

$$i(t) = i_s(t)$$

possible inputs

Kirchhoff's laws

To determine the model of an electrical circuit we can use the laws of Kirchhoff (current and voltage)

KCL (**Kirchhoff current law**):
the algebraic sum of all the currents entering and leaving a node must be equal to zero

$$\sum_k i_k = 0$$

KVL (**Kirchhoff voltage law**):
the algebraic sum of all the voltages within a closed circuit loop must be equal to zero

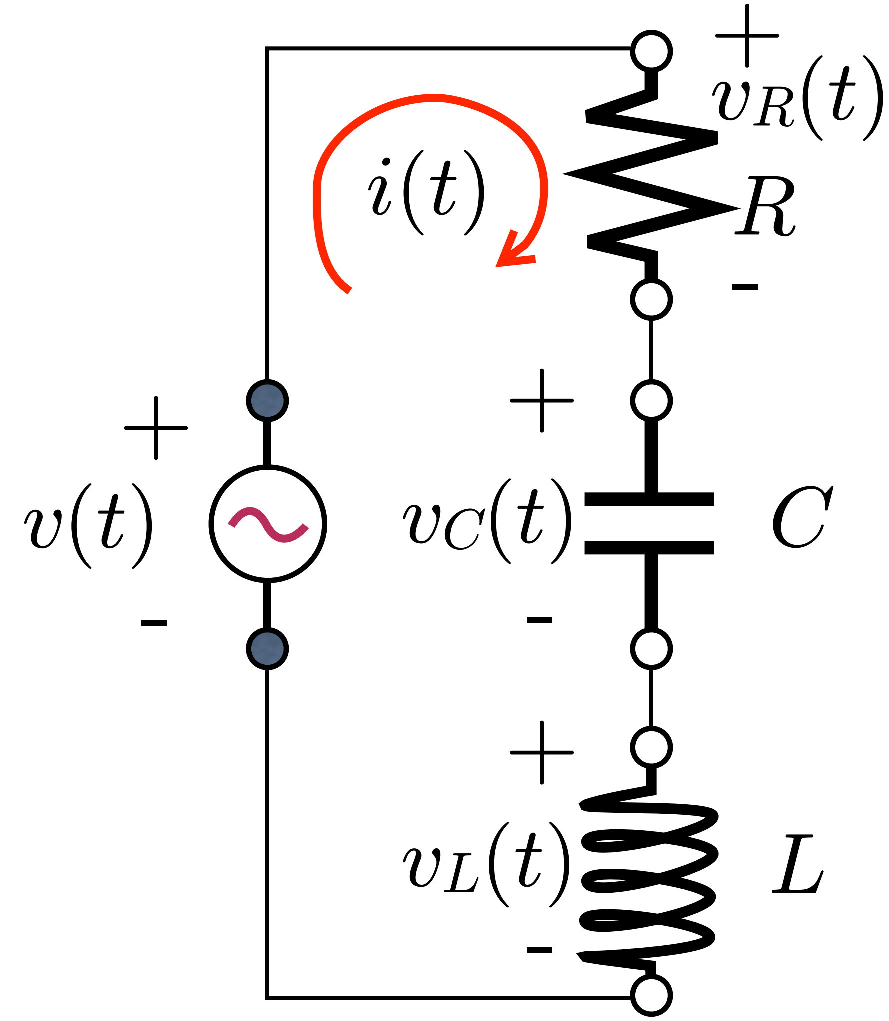
$$\sum_j v_j = 0$$

these are conservation laws

models of electrical circuits (RLC example)

series RLC circuit (Resistor, Inductor, Capacitor):

2 energy storing elements therefor state is 2-dimensional



$$v_R(t) = Ri(t) \qquad v_L(t) = L\frac{di(t)}{dt} \qquad v_C(t)$$

KVL $\longrightarrow$ $v_R + v_L + v_C = v$

that is $L\,\dfrac{d\,i(t)}{dt} + R\,i(t) + v_C(t) = v(t)$

Att.: this looks like a first order ODE

state one possible choice is $x(t) = \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix}$

$$\boxed{\begin{aligned} \frac{d\,i(t)}{dt} &= -\frac{R}{L}\,i(t) - \frac{1}{L}\,v_C(t) + \frac{1}{L}\,v(t) \\ \frac{d\,v_C(t)}{dt} &= \frac{1}{C}\,i(t) \end{aligned}}$$

equivalently $\dot{x} = Ax + Bv$ with

$$A = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$$

models of electrical circuits

series RLC circuit (alternative model)

being $v_R(t) = Ri(t)$ $\quad v_L(t) = L\dfrac{di(t)}{dt}$ $\quad i(t) = C\dfrac{dv_C(t)}{dt}$

we rewrite the KVL equation as $LC\,\ddot{v}_C + RC\,\dot{v}_C + v_C = v$

we can therefore have a different choice for the state (for the same physical system)

new state $z(t) = \begin{pmatrix} v_C(t) \\ \dot{v}_C(t) \end{pmatrix}$ instead of $x(t) = \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix}$

note the similitude between the two expressions

(**RLC**) $\quad LC\,\ddot{v}_C + RC\,\dot{v}_C + v_C = v$

(**MSD**) $\quad m\ddot{s} + \mu\dot{s} + ks = u$

$\rightarrow$ "similar" structure/solution/behavior

models of electrical circuits

series RLC circuit

- with state x we had

$$A = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$$

- with state z we have

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{LC} \end{pmatrix}$$

since these two different representations refer to the same RLC circuit, they must share the same important system properties

different dynamic matrices but with same characteristics (e.g., same eigenvalues - see algebra slides)

models of electrical circuits

series RLC circuit: 2 different state vectors choice

note that $x(t)$ and $z(t)$ are related by a linear transformation $z(t) = T\,x(t)$ with T nonsingular

$$z(t) \xleftrightarrow{\quad T \quad} x(t) \qquad\qquad z(t) = T\,x(t)$$

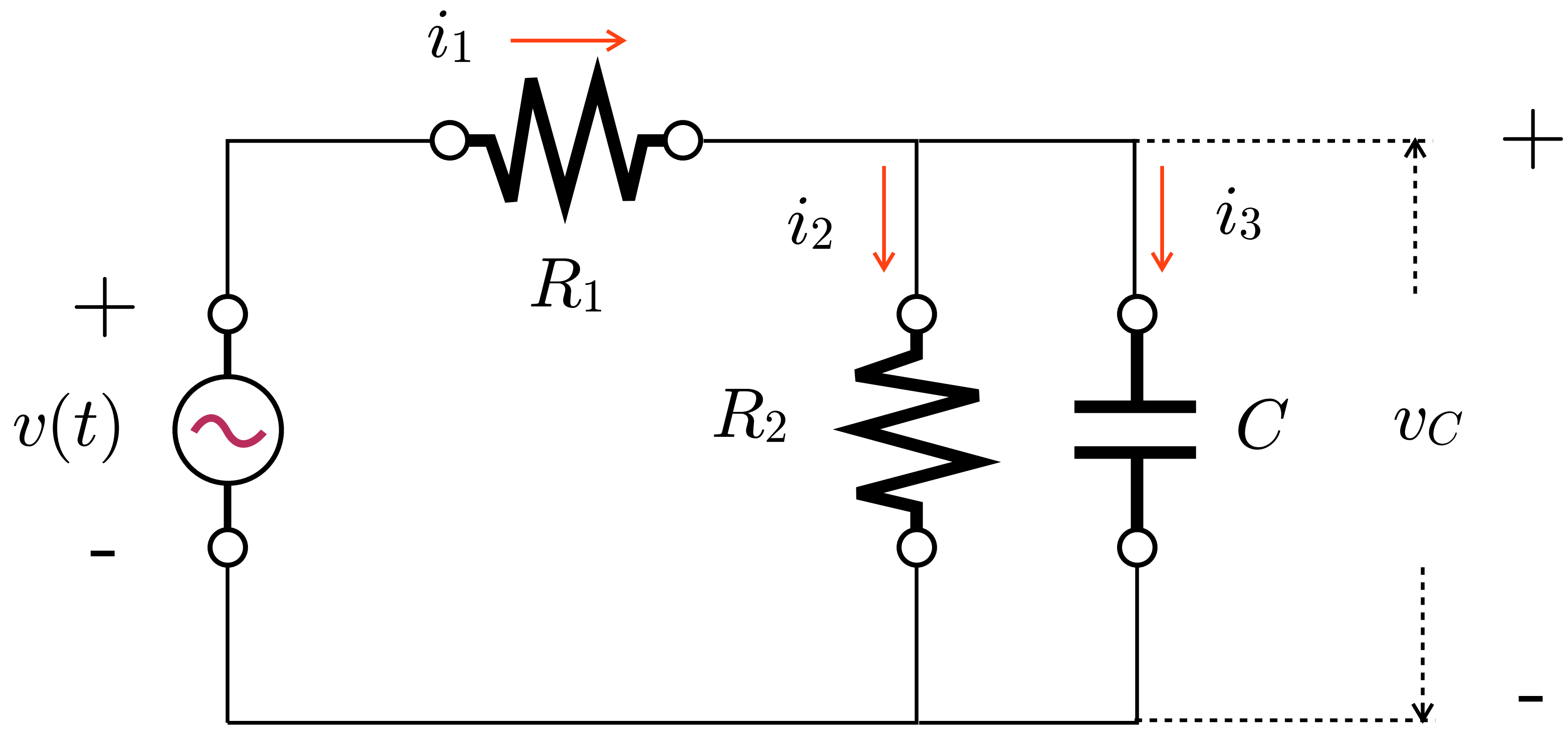
change of coordinates

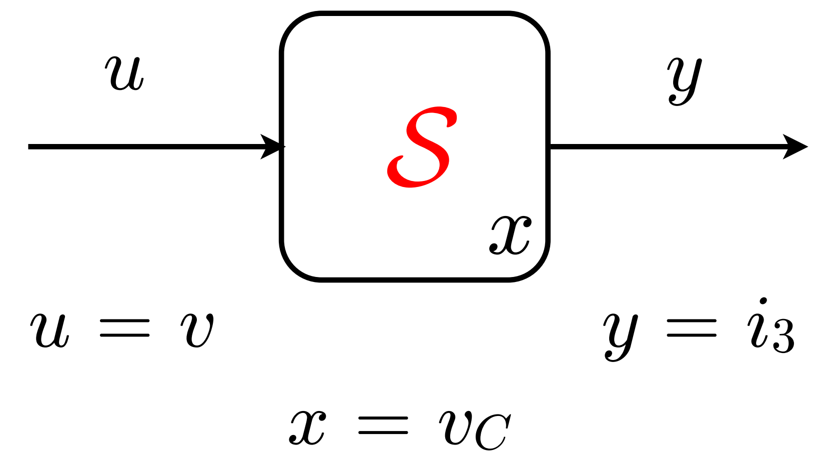
$$z(t) = \begin{pmatrix} v_C(t) \\ \dot{v}_C(t) \end{pmatrix} = \begin{pmatrix} v_C(t) \\ \frac{1}{C}\,i(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{1}{C} & 0 \end{pmatrix}}_{\text{check } T \text{ nonsingular}} \begin{pmatrix} i_C(t) \\ v_C(t) \end{pmatrix} = T\,x(t)$$

T nonsingular (det(T) different from 0) defines a **similarity transformation** (see algebra slides)

we have two equivalent state space representations of the same system

models of electrical circuits (other example)





one energy storage component and therefore the state is a scalar (here $x = v_C$)

$$
\begin{aligned}
\dot{x} &= -\frac{1}{C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)x + \frac{1}{R_1 C}u \\
y &= -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)x + \boxed{\frac{1}{R_1}}u
\end{aligned}
$$

D term

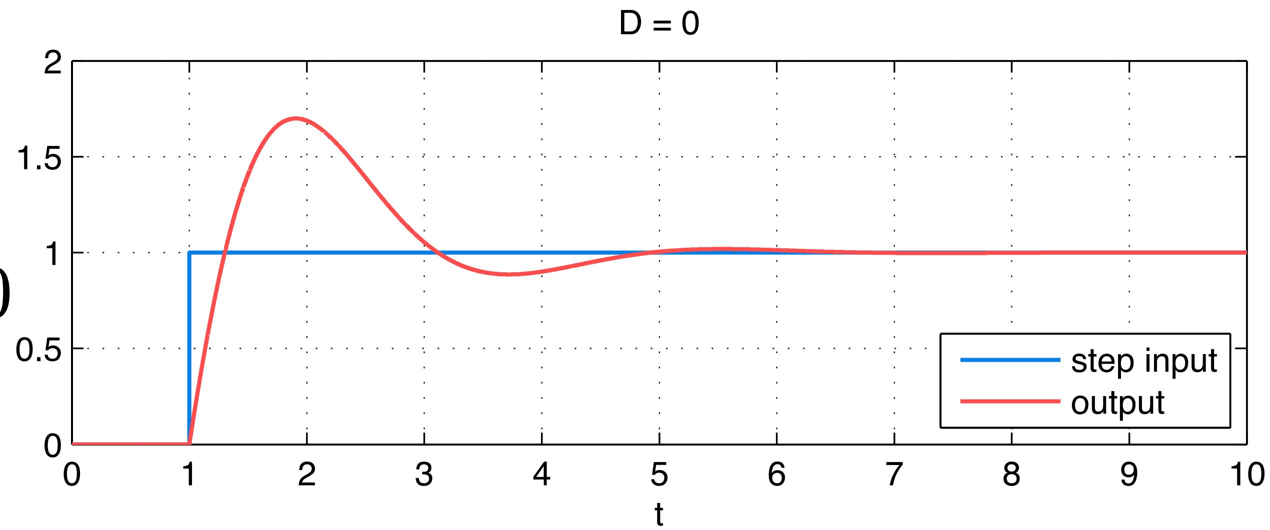
output may depend instantaneously from the input (non-zero D term)

feedthrough term D

numerical example

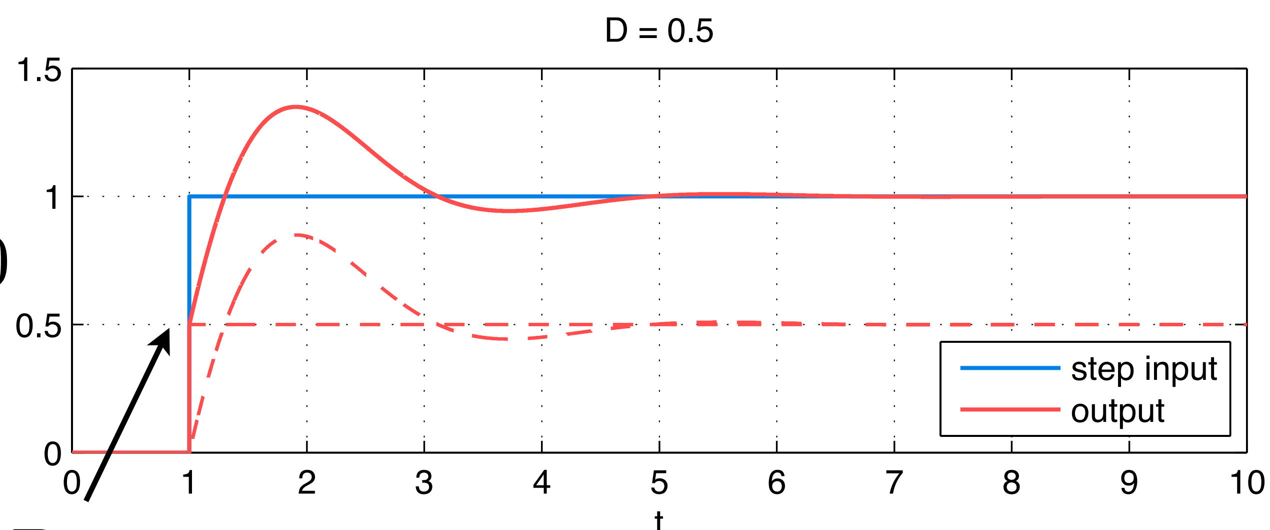
$$A_1 = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_1 = \begin{bmatrix} 4 & 4 \end{bmatrix} \quad \boxed{D_1 = 0}$$

$D = 0$



$u(t)$ unit step input from $t = 1$

$D \neq 0$



$D_2 = 0.5$

at time $t = 1$, the input switches from 0 to 1 and instantaneously the output switches from 0 to $D_2\, u(1) = D_2$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad \boxed{D_2 = 0.5}$$

heat flow models

lumped capacitance models

heat flow (variation of heat Q, Joule/s) through a resistance (wall)

$$\dot{Q} = \frac{1}{R}(T_e - T)$$

induces a change in temperature

rate of change of T of box (thermal capacitance) is proportional to heat flow

$$\dot{T} = \frac{1}{C}\dot{Q}$$

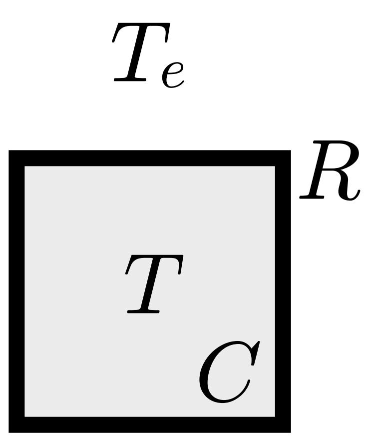
Q heat (Joule)

R thermal resistance

T temperature

T_e ambient temperature

C heat capacity

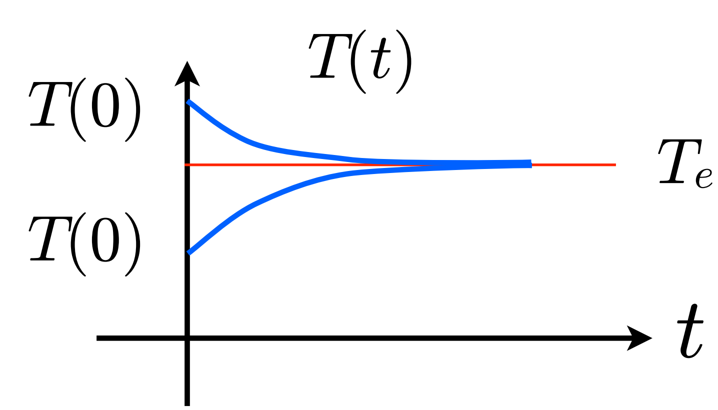


$$\boxed{\dot{T} = -\frac{1}{RC}T + \frac{1}{RC}T_e}$$

first order system

example: a box placed with internal temperature T in an ambient at a temperature T_e

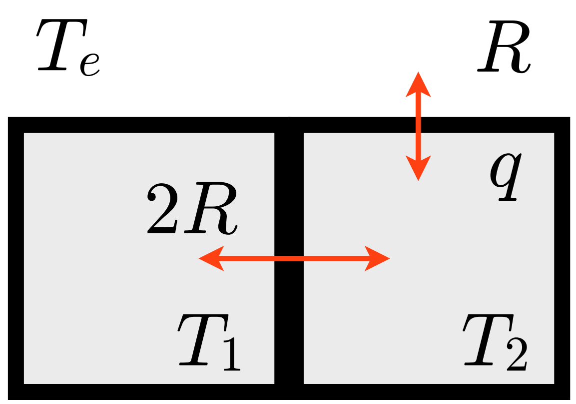
if T_e constant



heat flow models

lumped capacitance models

2 similar rooms



C room thermal capacity

R room thermal resistance

$2R$ room thermal resistance between rooms

q heat flow source

writing the variation of heat in each room

$$
\begin{aligned}
C\dot{T}_1 &= \frac{T_2 - T_1}{2R} - \frac{T_1 - T_e}{R} \\
C\dot{T}_2 &= q - \frac{T_2 - T_e}{R} - \frac{T_2 - T_1}{2R}
\end{aligned}
$$

second order system

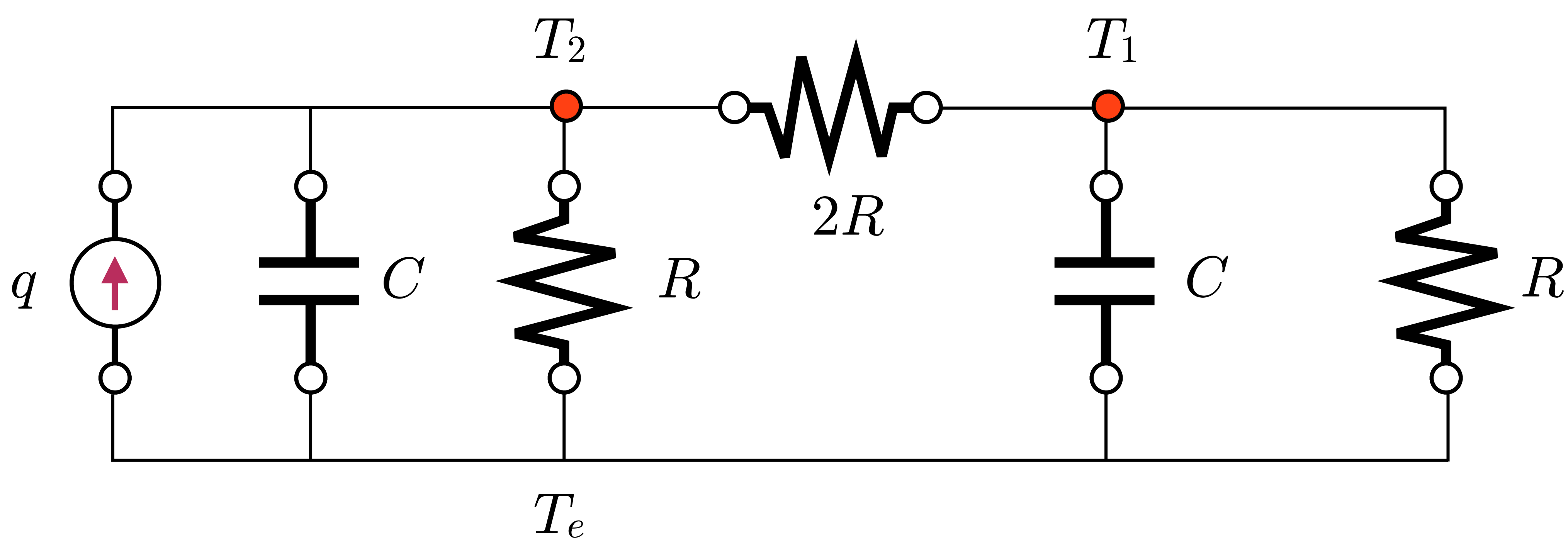
if q is an input, the state components could be chosen as T_1 and T_2

→ find (A,B) and choose an output of interest (and thus C)

heat flow models

$$
\begin{aligned}
C\dot{T}_1 &= \frac{T_2 - T_1}{2R} - \frac{T_1 - T_e}{R} \\
C\dot{T}_2 &= q - \frac{T_2 - T_e}{R} - \frac{T_2 - T_1}{2R}
\end{aligned}
$$

same equations
as the following circuit
(prove it)



current
generator

voltage across first capacitor T_2 - T_e

voltage across second capacitor T_1 - T_e

vocabulary

English	Italiano
linear time-invariant system	sistema lineare stazionario
input/state/output	ingresso/stato/uscita
mass/spring/damper system	sistema massa/molla/smorzatore
state representation	rappresentazione nello spazio di stato
dynamics matrix	matrice dinamica
input (output) matrix	matrice di ingresso (uscita)
feedthrough matrix	matrice del legame diretto ingresso-uscita