# Control Systems

## Models

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## some concepts from the last lecture

- analysis & control of dynamical systems
- dynamics/motion
- models/mathematical models
- prediction/simulation
- linearity
- time-invariance
- feedback control scheme: principle and example

### this lecture

- models: from differential equations to state space representation
- input & output choice
- state
- similar transformations

# general mathematical model

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$
$$x(0) = x_0$$

state space representation

linear time invariant (LTI)
dynamical system (continuous time)

$x(t)$ **state** $x \in \mathbf{R}^n$

$u(t)$ **input** $u \in \mathbf{R}^m$ multi input (if $m = 1$, single input)

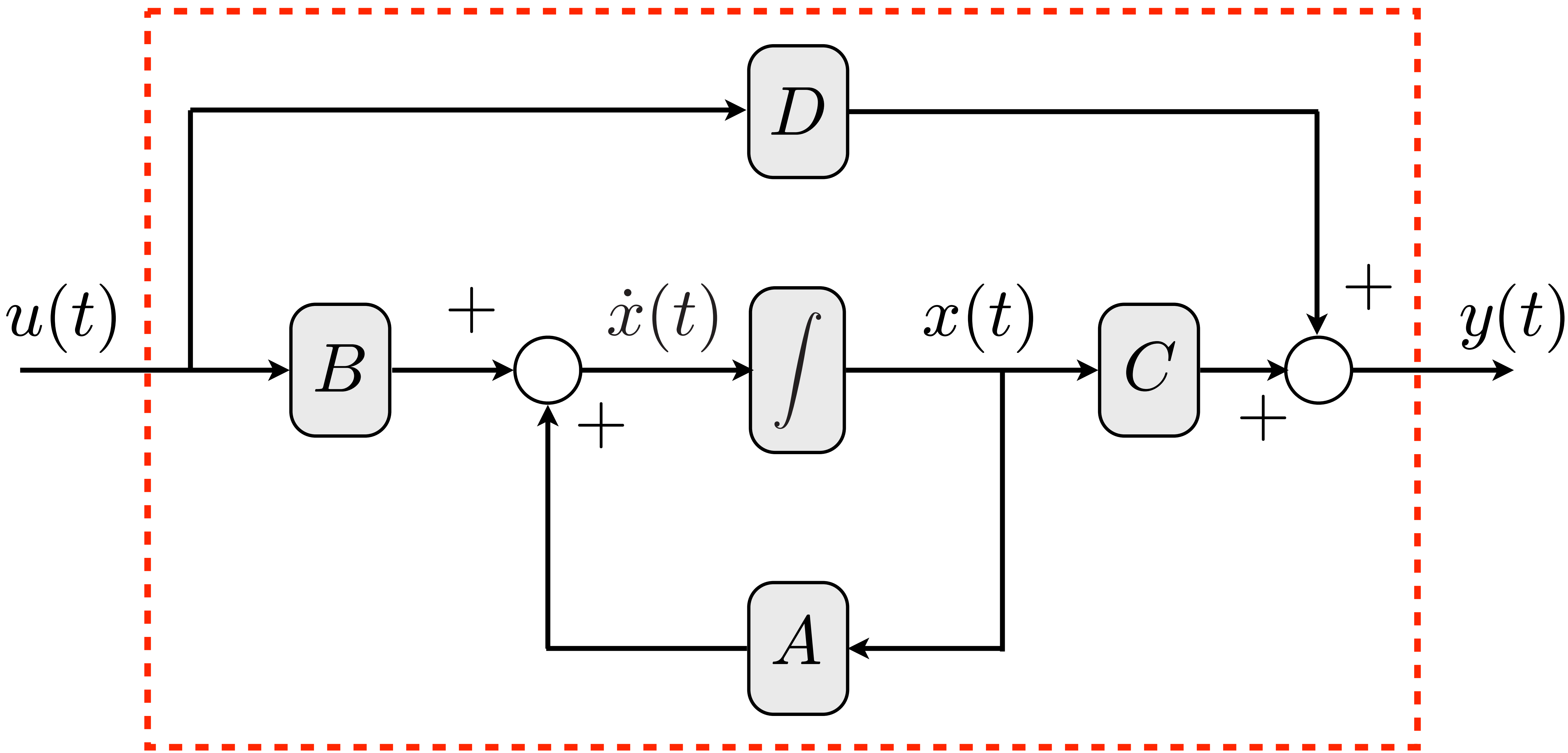
$y(t)$ **output** $y \in \mathbf{R}^p$ multi output (if $p = 1$, single output)

$A$ dynamics matrix ($n$x$n$)
$B$ input matrix ($n$x$m$)

$C$ output or sensor matrix ($p$x$n$)
$D$ feedthrough matrix ($p$x$m$)

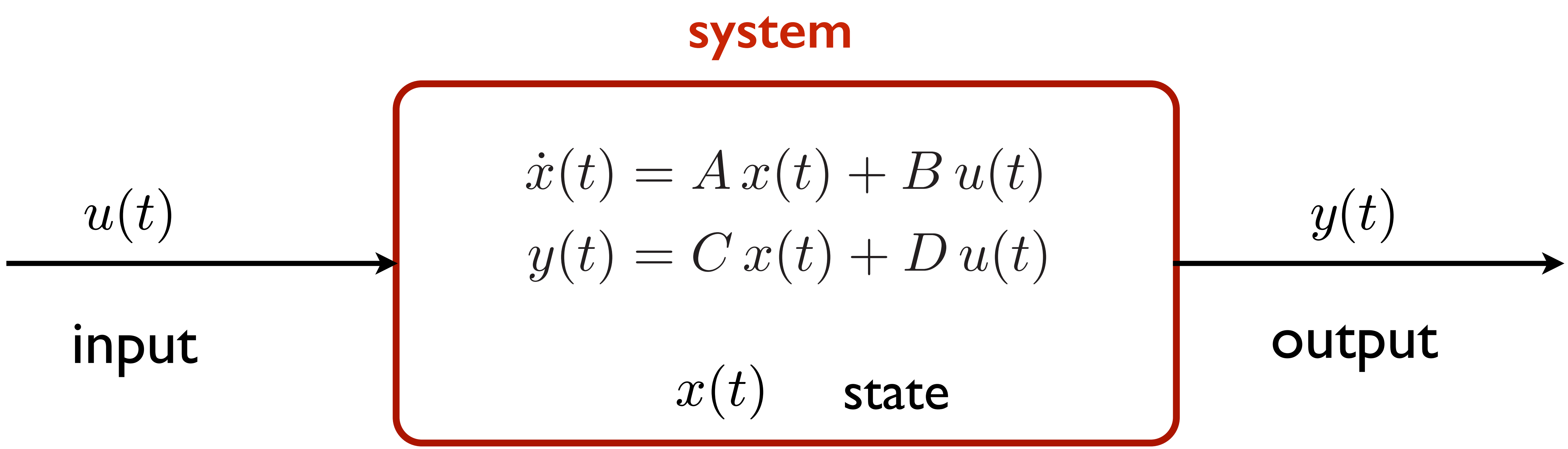
SISO system: $B$ ($n$x$1$) and $C$ ($1$x$n$)

# block diagram representation




simulation model

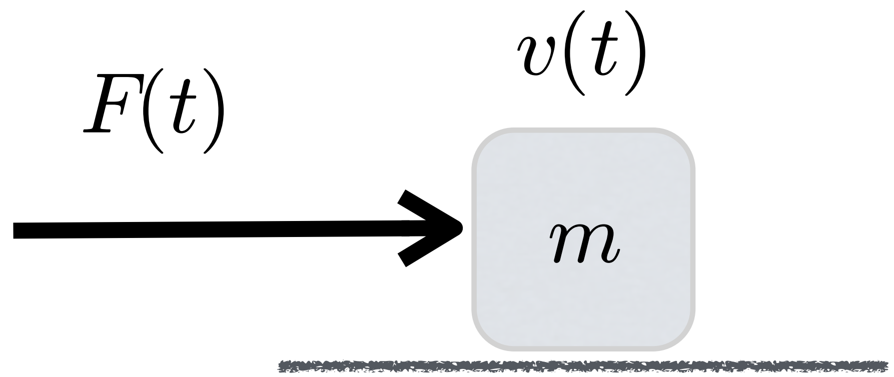
# mathematical model



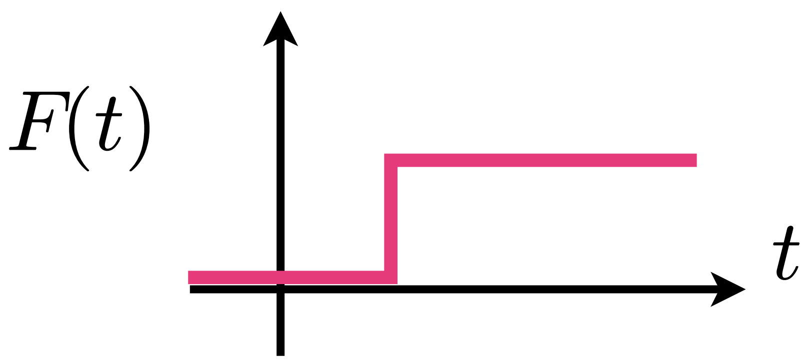

- the state evolution is influenced by the initial condition and the input
- the output displays the measurable effect of such state evolution (and potentially may also depend directly on the input when $D$ is non-zero)

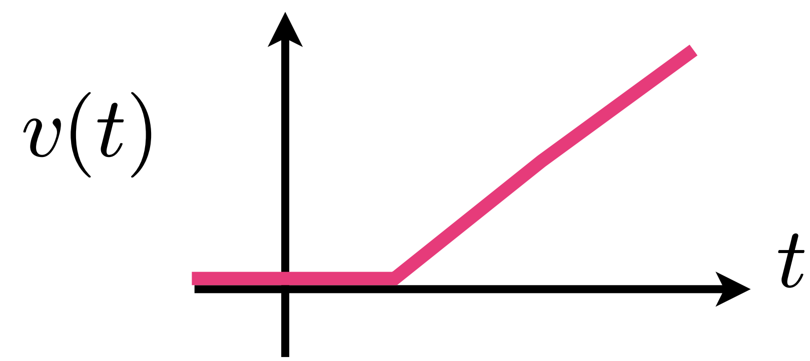
the system transforms an input signal into an output signal

# system as a signal transformer




no friction case
and $v_0 = 0$



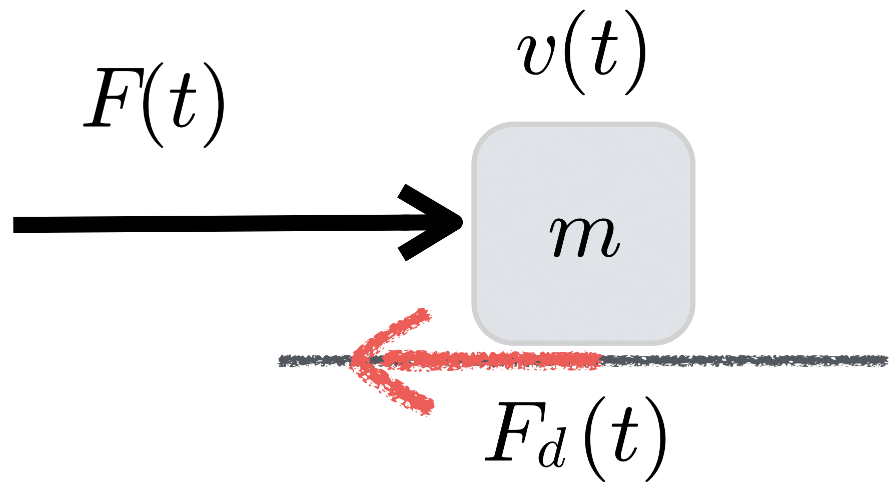


model
(system representation)

$$\dot{v}(t) = \frac{1}{m} F(t)$$

no friction

# mass model with viscous friction




with friction force $F_d(t)$

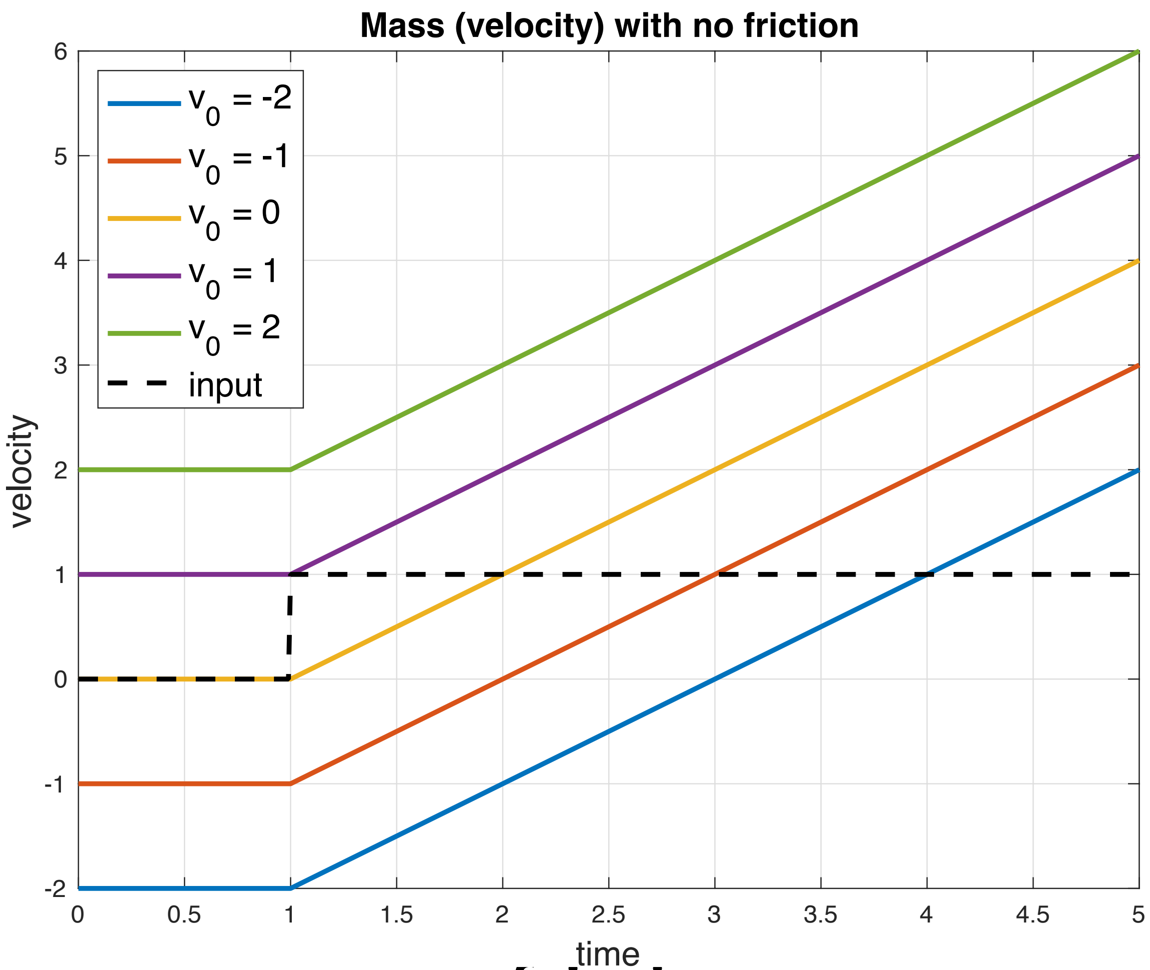
we add a viscous friction force $F_d(t)$, acting on the mass, which can be considered proportional to the velocity and acts in the opposite direction

$$F_d(t) = \mu \, v(t)$$

Newton's equation $$m\dot{v}(t) = -F_d(t) + F(t) = -\mu \, v(t) + F(t)$$

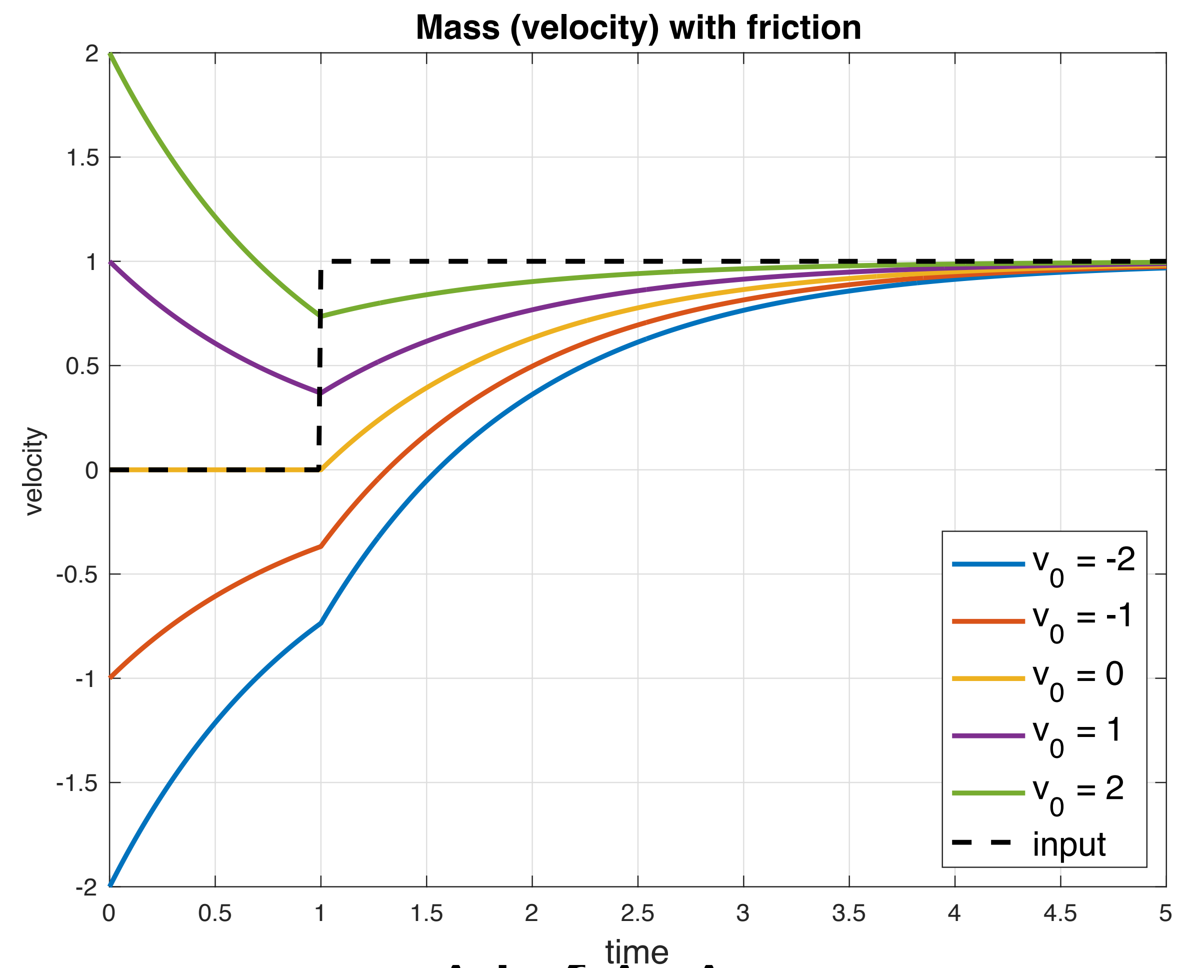
model is updated as (system representation) $$\dot{v}(t) = -\mu \, v(t) + \frac{1}{m} F(t)$$

# mass system - simulations




no friction

(different initial velocities)




with friction

(different initial velocities)

velocity when a constant $1$ N force from $t = 1$ sec is applied

(learn how to interpret plots)

suggested problem:
use only piece-wise constant $F(t) = \pm F$ in order to change velocity from $v_1$ to $v_2$ (you can switch value at any time)

# signals & systems perspective

MIT OpenCourseware Dennis Freeman 600.3 - Fall 2011 Signals & Systems

Lecture 1: Signals and Systems

analysis & design of systems via their signal transformation properties

system transforms an input signal into an output signal

**how**: system description (we saw mathematical model)

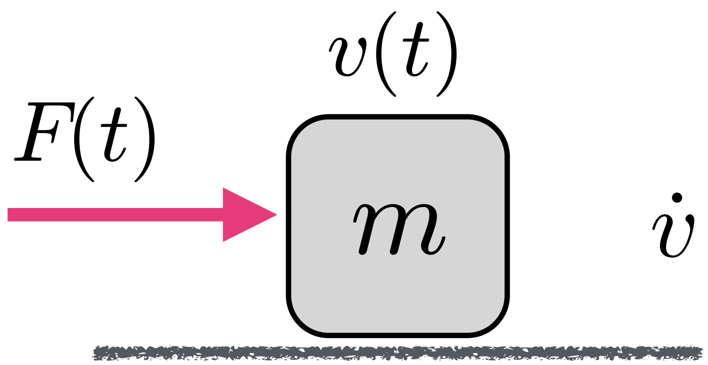
is independent from physical substrate (e.g., cell phone)

**focus**: flow of information

abstract, widely applicable, modular, hierarchical

# different systems may have similar models

- linear motion under the action of a force




$$\dot{v}(t) = \frac{1}{m} F(t)$$

choosing

$$x(t) = v(t) \quad b = 1/m$$

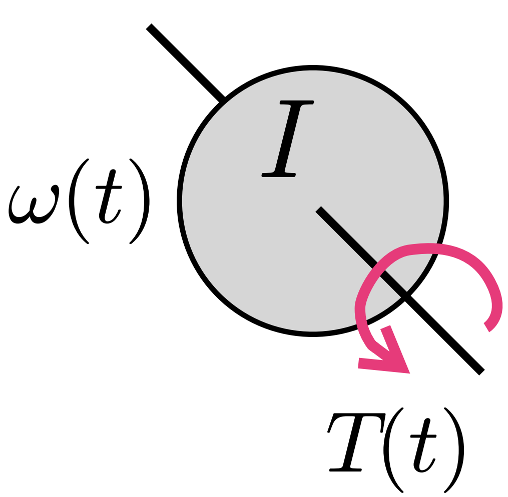
$$u(t) = F(t)$$

$$\dot{x}(t) = 0 \cdot x(t) + b \cdot u(t)$$

**same model structure**

$$\dot{x}(t) = b \cdot u(t)$$

- rotational motion under the action of a torque



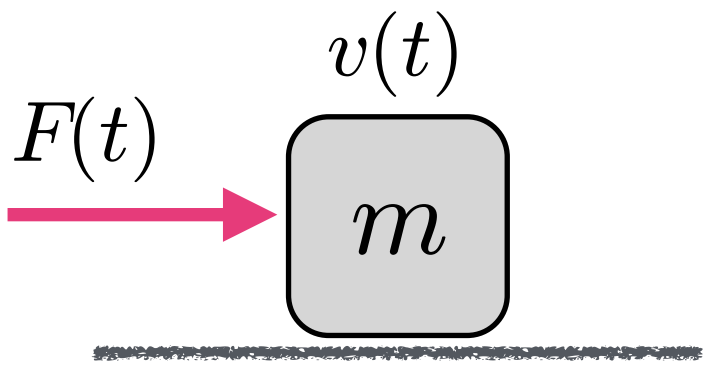

$$\dot{\omega}(t) = \frac{1}{I} T(t)$$

$$x(t) = \omega(t)$$

$$u(t) = T(t) \quad b = 1/I$$

# similar models and similar behavior

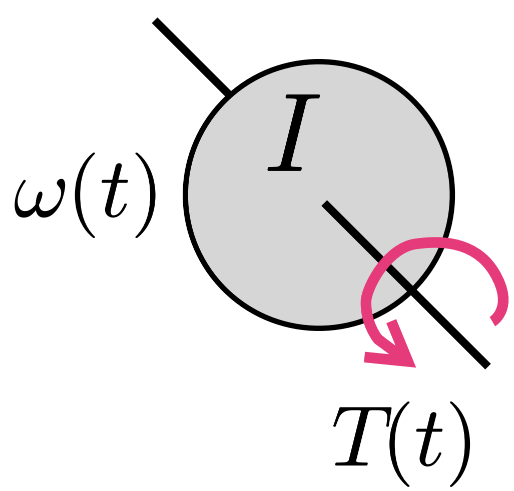
- linear motion under the action of a force




$$v(t) = v_0 + \frac{1}{m}\int_0^t F(\tau)d\tau$$

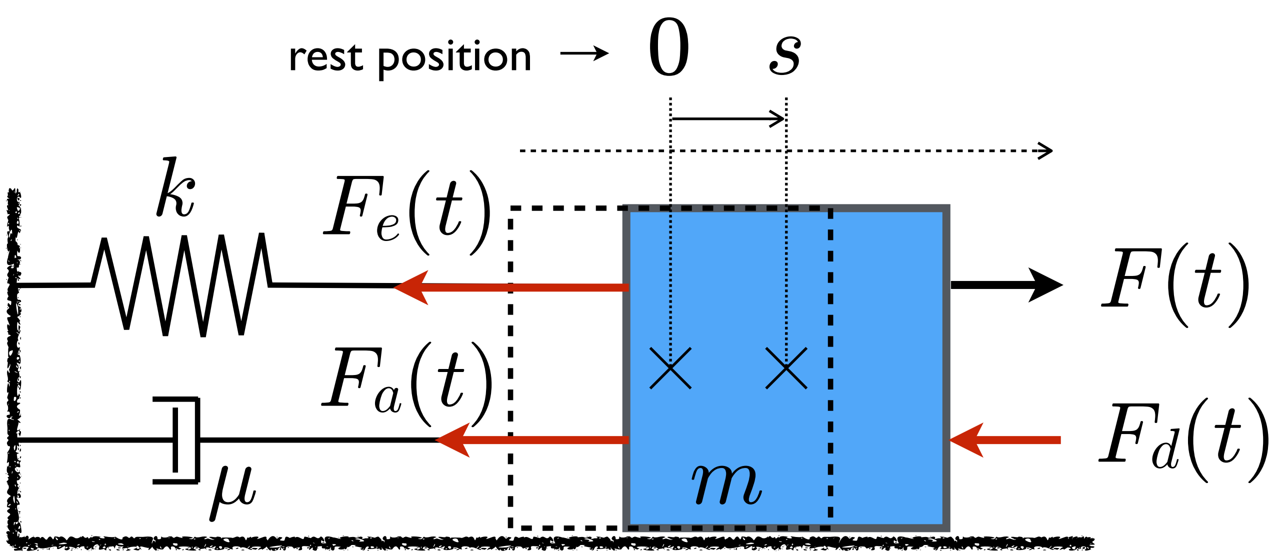
$$\dot{x}(t) = 0 \cdot x(t) + b \cdot u(t) \quad \dashrightarrow \quad x(t) = x_0 + b\int_0^t u(\tau)d\tau$$

- rotational motion under the action of a torque




$$\omega(t) = \omega_0 + \frac{1}{I}\int_0^t T(\tau)d\tau$$

# mass-spring-damper (MSD)




$F(t)$ control force

$F_d(t)$ disturbance force

$F_e(t)$ elastic force

$F_a(t)$ friction force

rest position: with zero velocity

Newton's second law of motion

$$m\, a(t) = F(t) - F_e(t) - F_a(t) - F_d(t)$$

$s(t) =$ deviation from rest position

$$v(t) = \dot{s}(t)$$
$$a(t) = \ddot{s}(t)$$

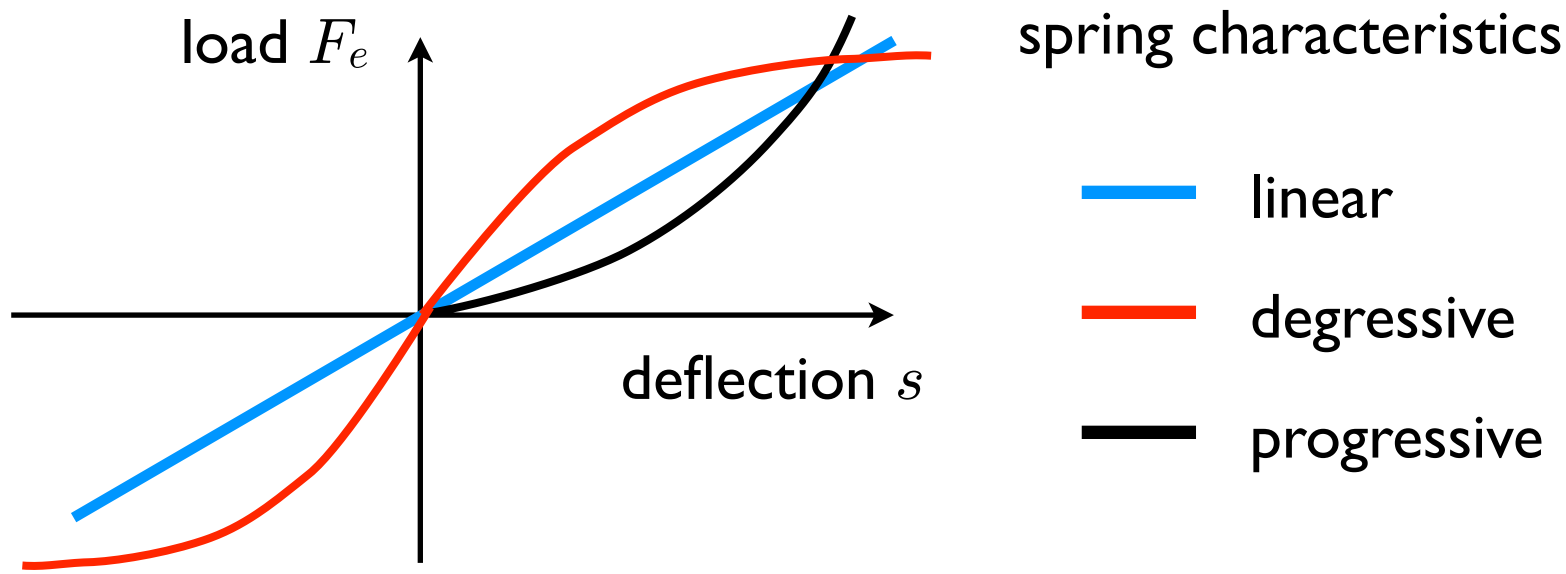
modeling hypothesis

$F_e(t) = k\ s(t)$ linear spring

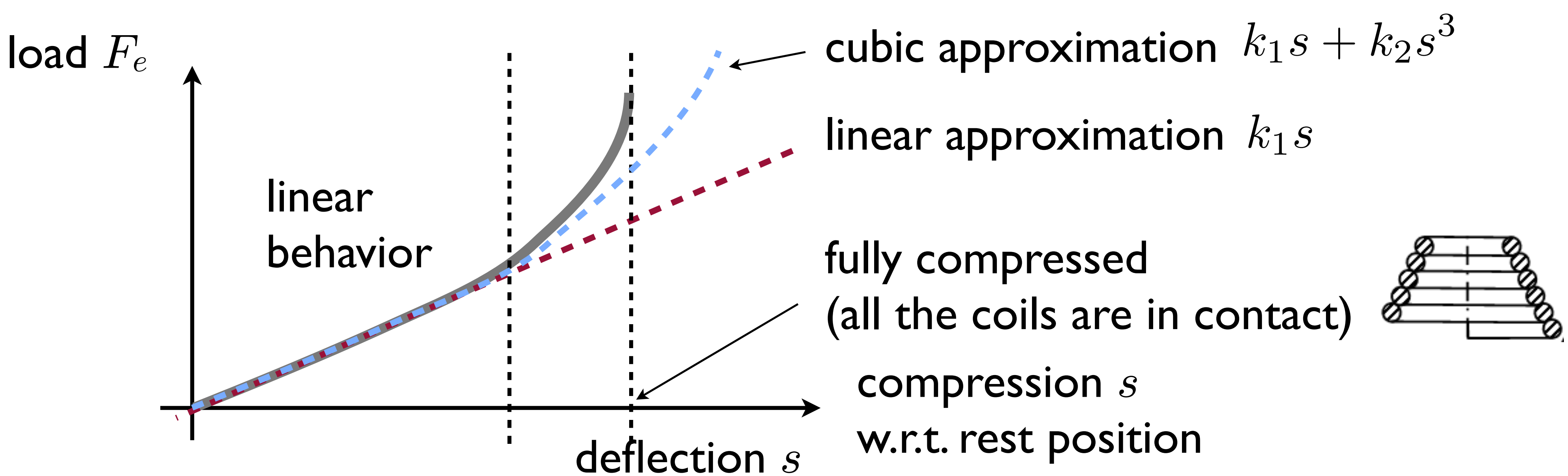
$F_a(t) = \mu\ v(t)$ linear viscous friction

# MSD - elastic force

modeling hypothesis




example: progressive spring

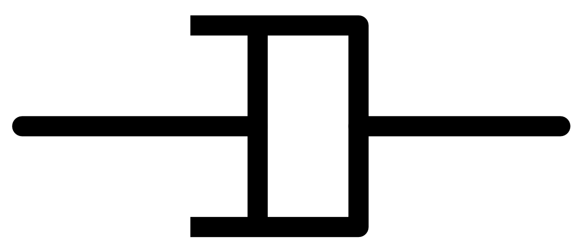



the (linear) approximation of the spring characteristic is part of the modeling phase
similarly for the friction force

# MSD - friction force

we assumed the viscous friction force $F_d(t)$ to be proportional to the velocity and acting in the opposite direction

we indicate schematically the presence of viscous friction with the symbol and we call it a **damper**



damper

a mechanical damper is also called **dashpot**

# MSD - state space representation

$$m\,a(t) = F(t) - F_e(t) - F_a(t) - F_d(t)$$

\+ modeling hypothesis

$$m\,\ddot{s}(t) = -k\,s(t) - \mu\,\dot{s}(t) + F(t) - F_d(t)$$

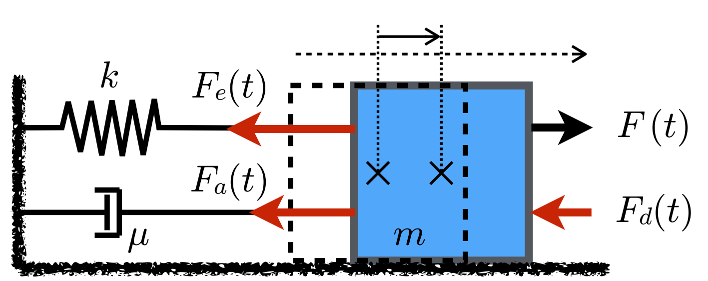
how do we rewrite this linear model in the standard state space form?

we need to

- define the state
- define the input(s) and output(s)
- rewrite this second order differential equation in terms of the state and its derivative

input(s) ?

how many ?




state ?

$n = ?$

output(s) ?

how many ?

can also be seen as a
signal transformer

# MSD - state space representation

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$||| \; ? \qquad \text{if yes, how ?}$$

$$m\,\ddot{s}(t) = -k\,s(t) - \mu\,\dot{s}(t) + F(t) - F_d(t)$$

**input**: 2 choices

- single (scalar) input $u(t) = F(t)$ - $F_d(t)$
  (if it is not necessary to distinguish between the control input $F(t)$ and the disturbance $F_d(t)$, for example in a pure analysis context)

- two distinctive inputs $F(t)$ and $F_d(t)$ become a unique two dimensional input vector $u(t)$

$$u(t) = \begin{pmatrix} F(t) \\ F_d(t) \end{pmatrix}$$

# MSD - state space representation

choosing as state the position displacement and the velocity of the mass

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} s(t) \\ \dot{s}(t) \end{pmatrix}$$

we can rewrite the second order differential equation as

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{\mu}{m}x_2(t) + \frac{1}{m}(F(t) - F_d(t))$$

from | $1$ second order differential equation | to | $2$ first order differential equations

and matrix form ?

# MSD - state space representation

in the form $$\dot{x}(t) = Ax(t) + Bu(t)$$

- single input case

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{B} (F(t) - F_d(t))$$

with $$u(t) = F(t) - F_d(t)$$

- input vector case

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{B_1} F(t) - \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{B_2} F_d(t)$$

with $u(t) = \begin{pmatrix} F(t) \\ F_d(t) \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & -B_2 \end{pmatrix}$

# MSD - state space representation

matrices $A$ and $B$ are characteristics of the given system,

while $C$ and $D$ depend upon the particular chosen **output**

examples:

$$y(t) = s(t) \longrightarrow \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D = 0$$

$$y(t) = \dot{s}(t) \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad D = 0$$

$$y(t) = s(t) - \pi \dot{s}(t) \quad C = \begin{pmatrix} 1 & -\pi \end{pmatrix} \quad D = 0$$

no special physical meaning

$$y(t) = \ddot{s}(t)$$

we use $\ddot{s} = -\dfrac{k}{m} s - \dfrac{\mu}{m} \dot{s} + \boxed{\dfrac{1}{m}} u$

$$C = \begin{pmatrix} -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}$$

feedthrough term $D$

# high order ODE

$$z^{(n)} + a_{n-1} z^{(n-1)} + \cdots + a_2 z^{(2)} + a_1 z^{(1)} + a_0 z^{(0)} = bu(t)$$

with $\quad z^{(i)}(t) = \dfrac{d^i z(t)}{dt^i}$

we can choose as **state** $\quad x(t) = \begin{pmatrix} z^{(0)} \\ z^{(1)} \\ \vdots \\ z^{(n-1)} \end{pmatrix}$ and find $(A, B)$

$n$ dimensional vector

dimension of state $n =$ number of initial conditions necessary to define a unique solution of the $n$-th order differential equation

MSD as a special case $\quad m\ddot{s} + \mu\dot{s} + ks = u$

# high order ODE

finding $(A,B)$

$$\dot{x}(t) = \begin{pmatrix} z^{(1)} \\ z^{(2)} \\ \vdots \\ z^{(n)} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & 1 \\ -a_0 & -a_1 & & \ddots & -a_{n-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} z^{(0)} \\ z^{(1)} \\ \vdots \\ z^{(n-1)} \end{pmatrix}}_{x(t)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}}_{B} \underbrace{u}_{u(t)}$$

$$= \quad A \quad x(t) \quad + \quad B \quad u(t)$$

output example: choose $y(t) = z(t)$

$$y(t) = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}}_{C} x(t)$$

$$= \quad C \quad x(t) \quad + \quad 0 \quad u(t)$$

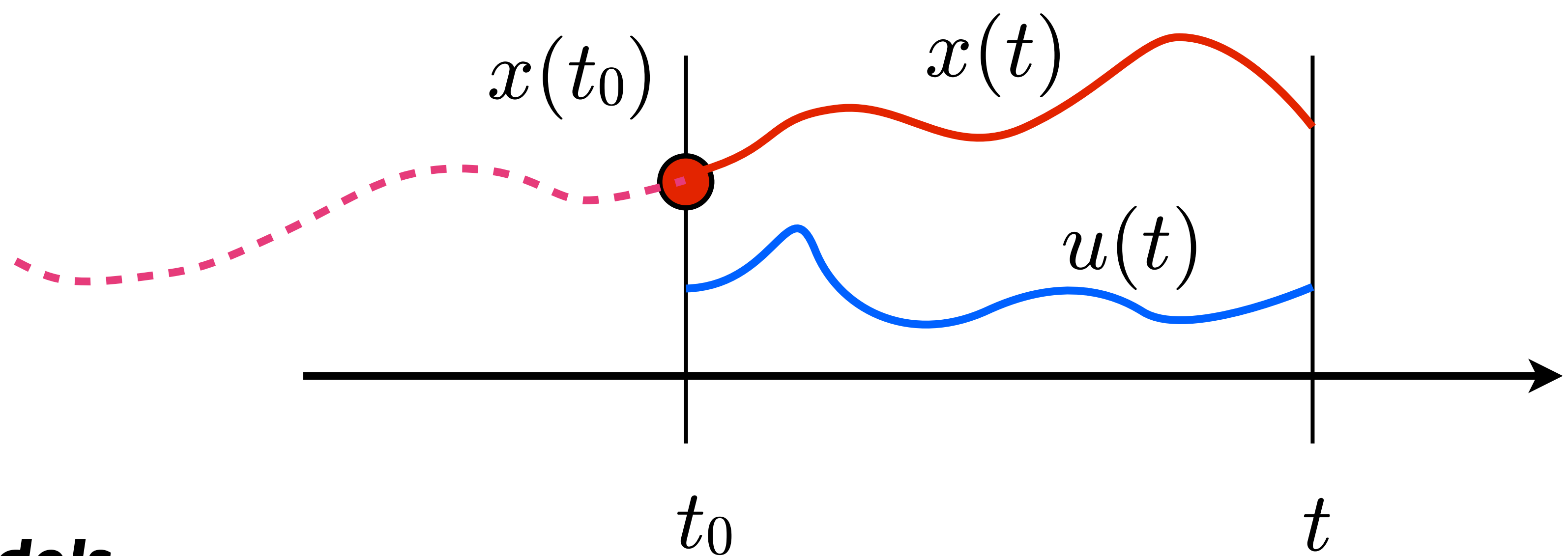
i.e. $D = 0$

# state

The **state** of a dynamical system is a set of physical quantities (**state variables**), the specification of which (in the absence of external excitation) completely determines the evolution of the system (B. Friedland)

Specific physical quantities that define the state are not unique, although their number (**system order**) is unique

Or the minimum set of variables such that their knowledge at time $t_0$, together with the knowledge of the external excitations (inputs) in $[t_0, t)$, allows the complete characterization of the system evolution in $[t_0, t)$



# state dimension - examples

- first order system $\dot{v}(t) = \frac{1}{m} F(t) \qquad x \in \mathbf{R}$
- second order system $m\ddot{s} + \mu\dot{s} + ks = u \qquad x \in \mathbf{R}^2$
- $n$-th order system $x \in \mathbf{R}^n$

$$z^{(n)} + a_{n-1} z^{(n-1)} + \cdots + a_2 z^{(2)} + a_1 z^{(1)} + a_0 z^{(0)} = bu(t)$$

- $2+1 = 3$rd order system

$$\begin{aligned} a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0 x_1 &= \alpha u \\ b_1\dot{x}_2 + b_0 x_2 &= \beta u \end{aligned} \qquad x = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^3$$

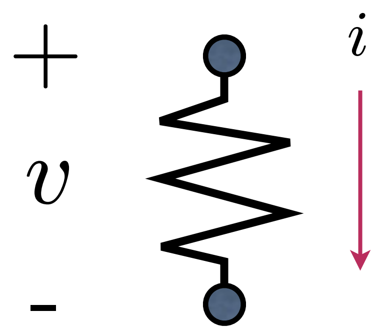
- 2nd order system although one 2nd order + one 1st order ODE

$$\begin{aligned} a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0 x_1 &= \alpha u \\ b_1\dot{x}_1 + b_0 x_1 &= \beta u \end{aligned} \qquad x = \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix} \in \mathbf{R}^2$$

substituting $\dot{x}_1$ from 2nd equation ito first: one 2nd order ODE

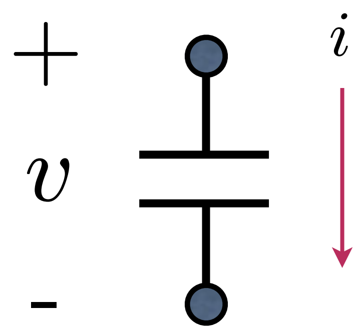
# models of electrical circuits

Resistor




$$v(t) = R\,i(t)$$

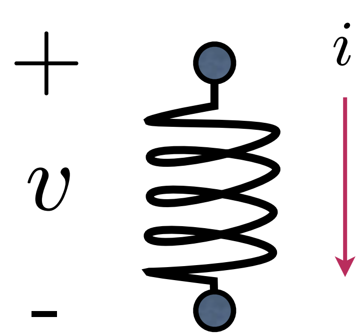
Capacitor




$$i(t) = C\,\frac{dv(t)}{dt}$$

or $\dot{v}(t) = \frac{1}{C}\,i(t)$

Inductor

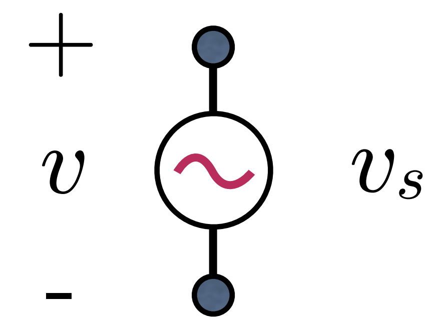



$$v(t) = L\,\frac{di(t)}{dt}$$

or $\dot{i}(t) = \frac{1}{L}\,v(t)$

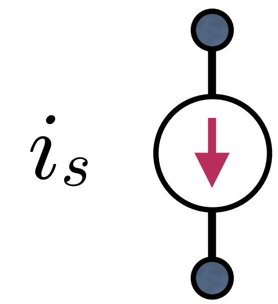
elements which store energy → **state components**

(one for each energy storing element)




voltage source

$$v(t) = v_s(t)$$




current source

$$i(t) = i_s(t)$$

# Kirchhoff's laws

conservation laws

KCL (**Kirchhoff current law**):

the algebraic sum of all the currents entering and leaving a node must be equal to zero

$$\sum_k i_k = 0$$

KVL (**Kirchhoff voltage law**):

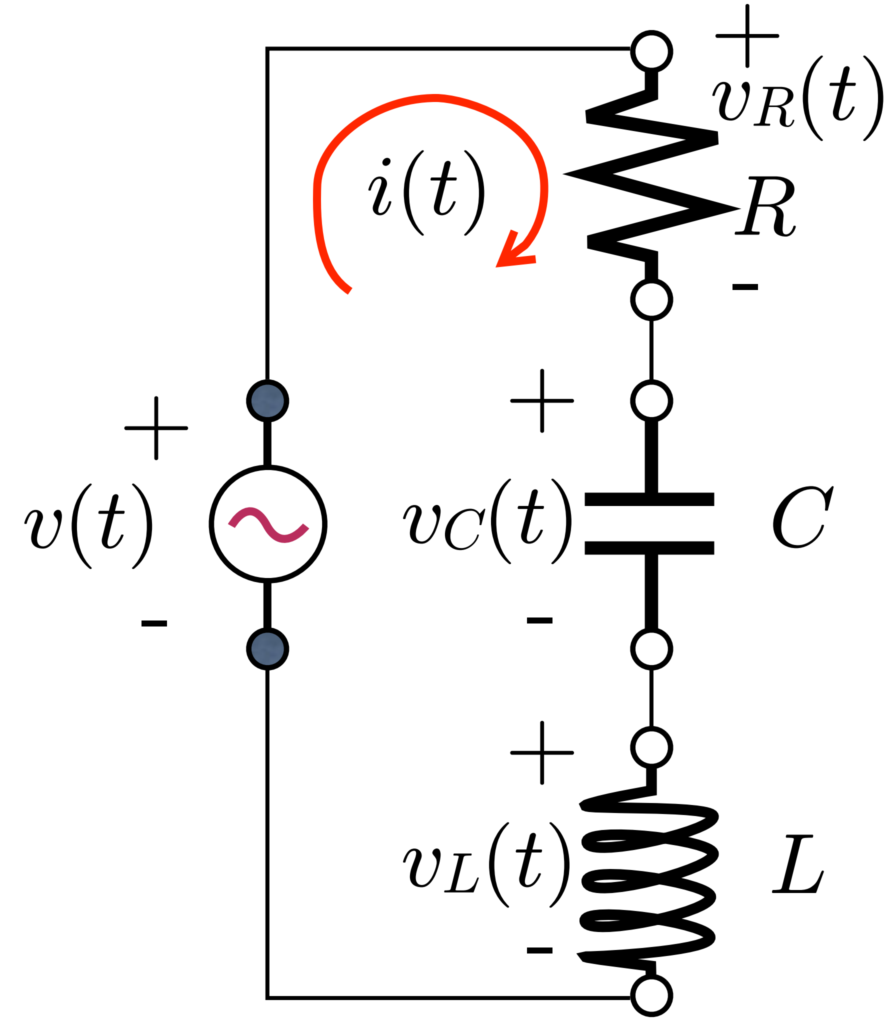
the algebraic sum of all the voltages within a closed circuit loop must be equal to zero

$$\sum_j v_j = 0$$

# models of electrical circuits (RLC example)

series RLC circuit (Resistor, Inductor, Capacitor):

2 energy storing elements




$$v_R(t) = Ri(t) \qquad v_L(t) = L\frac{di(t)}{dt} \qquad v_C(t)$$

KVL → $v_R + v_L + v_C = v$

Att.: this looks like a 1st order ODE

$$L\frac{d\,i(t)}{dt} + R\,i(t) + v_c(t) = v(t)$$

**state**

(one possible choice)

$$x(t) = \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix}$$

find $\dot{x} = Ax + Bv$ (sol.)

$$\begin{aligned} \frac{d\,i(t)}{dt} &= -\frac{R}{L}i(t) - \frac{1}{L}v_C(t) + \frac{1}{L}v(t) \\ \frac{d\,v_c(t)}{dt} &= \frac{1}{C}i(t) \end{aligned}$$

# models of electrical circuits

series RLC circuit (alternative model)

being $v_R(t) = Ri(t)$ $\quad v_L(t) = L\dfrac{di(t)}{dt}$ $\quad i(t) = C\dfrac{dv_C(t)}{dt}$

we rewrite the KVL equation as $LC\,\ddot{v}_C + RC\,\dot{v}_C + v_C = v$

**state**
(other possible choice)

$$z(t) = \begin{pmatrix} v_C(t) \\ \dot{v}_C(t) \end{pmatrix} \quad \text{instead of} \quad x(t) = \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix}$$

note the similitude between the two expressions

(**RLC**) $LC\,\ddot{v}_C + RC\,\dot{v}_C + v_C = v$

(**MSD**) $m\ddot{s} + \mu\dot{s} + ks = u$

→ "similar" structure/solution/behavior

# models of electrical circuits

series RLC circuit

- with state $x$ we had

$$A = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$$

- with state $z$ we have

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{LC} \end{pmatrix}$$

→ since these two different representations refer to the same RLC circuit, they must share the same important system characteristic

different dynamic matrices but with **same characteristics** (e.g., same eigenvalues - see algebra slides)

# models of electrical circuits

series RLC circuit: 2 different state vectors choice

note that $x(t)$ and $z(t)$ are related by a linear nonsingular transformation

$$z(t) \xleftrightarrow{\;T\;} x(t) \qquad\qquad z(t) = T\,x(t)$$

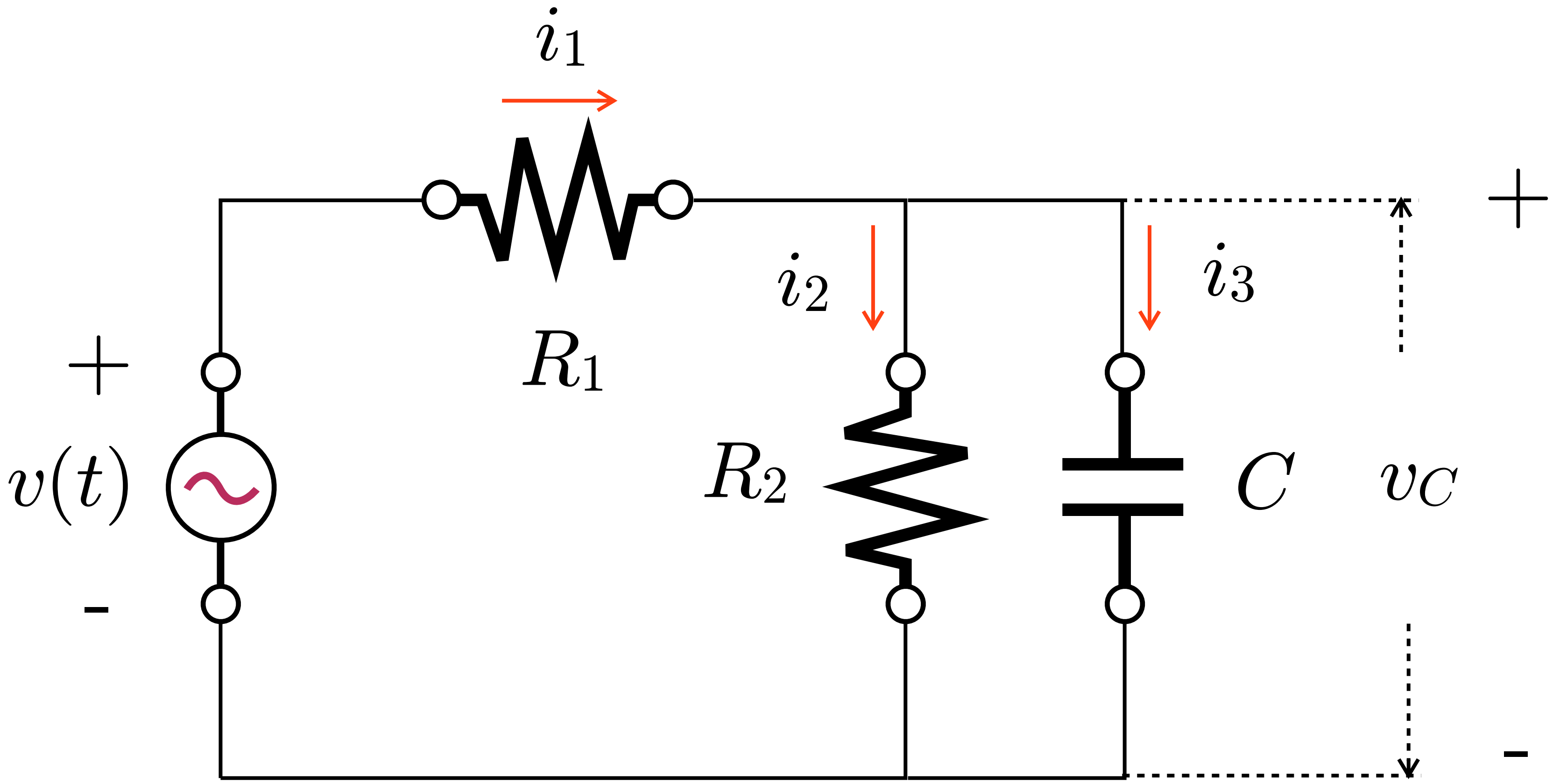
change of coordinates

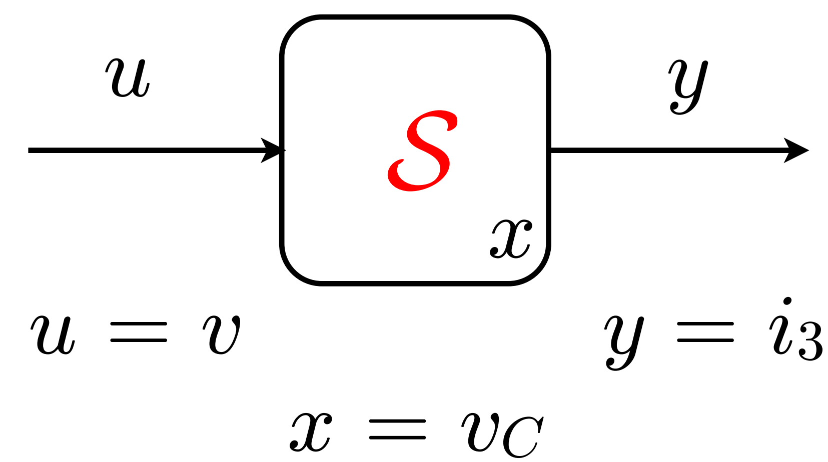
$$z(t) = \begin{pmatrix} v_C(t) \\ \dot{v}_C(t) \end{pmatrix} = \begin{pmatrix} v_C(t) \\ \frac{1}{C}i(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i(t) \\ v_C(t) \end{pmatrix} = Tx(t)$$

check $T$ nonsingular

$T$ nonsingular defines a similarity transformation
(see algebra slides)

# models of electrical circuits (other example)




$$
\begin{aligned}
\dot{x} &= -\frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) x + \frac{1}{R_1 C} u \\
y &= -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) x + \boxed{\frac{1}{R_1}} u
\end{aligned}
$$

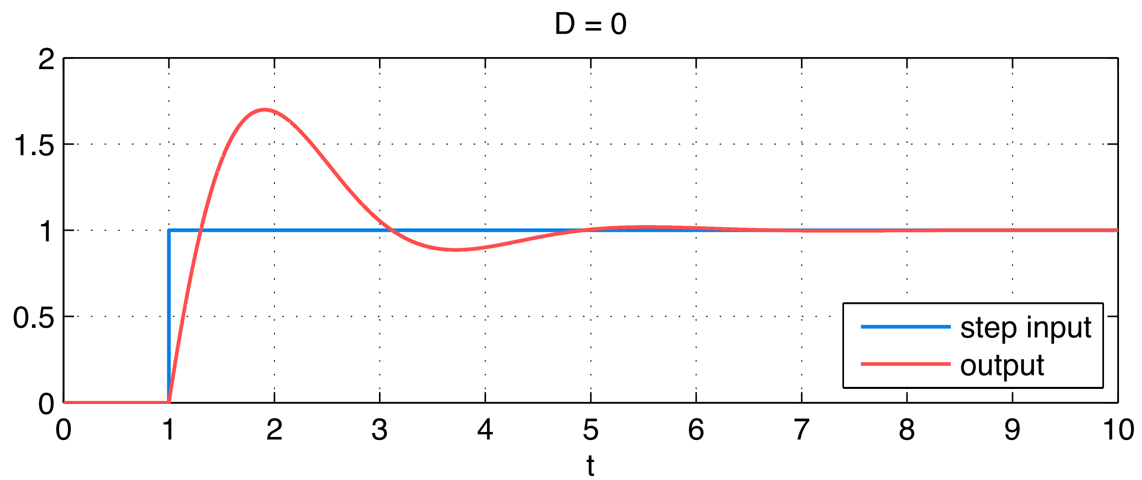
$D$ term

output may depend instantaneously from the input

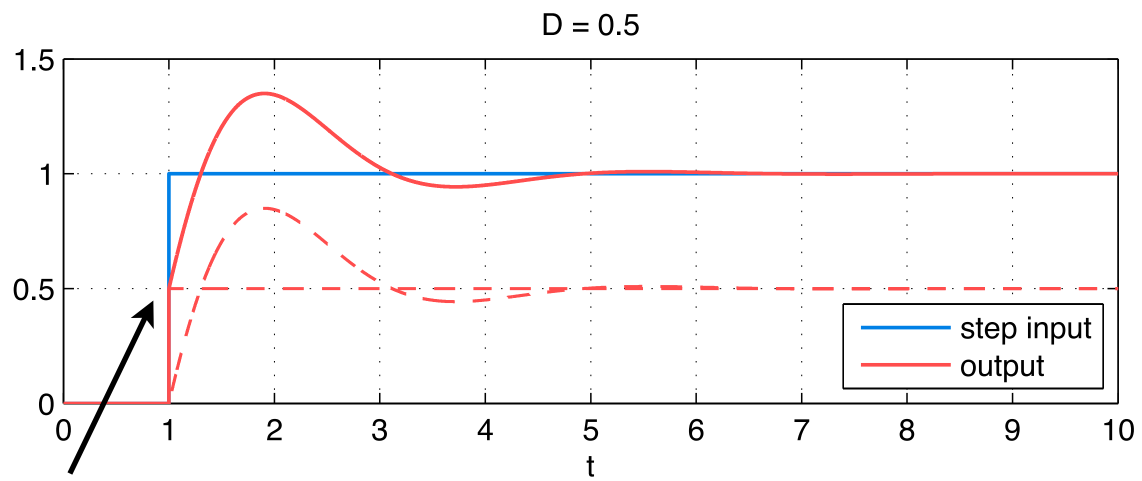
# feedthrough term $D$

numerical example

$$A_1 = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_1 = \begin{bmatrix} 4 & 4 \end{bmatrix} \quad D_1 = 0$$




$u(t)$ unit step input from $t = 1$




$D_2 = 0.5$

at time $t = 1$, the input switches from $0$ to $1$ and instantaneously the output switches from $0$ to $D_2\, u(1) = D_2$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad D_2 = 0.5$$

# heat flow models

lumped capacitance models

heat flow (variation of heat $Q$, Joule/s) through a resistance (wall)

$$\dot{Q} = \frac{1}{R}(T_e - T)$$

induces a change in temperature

$$\dot{T} = \frac{1}{C}\dot{Q}$$

rate of change of $T$ of box (thermal capacitance) is proportional to heat flow

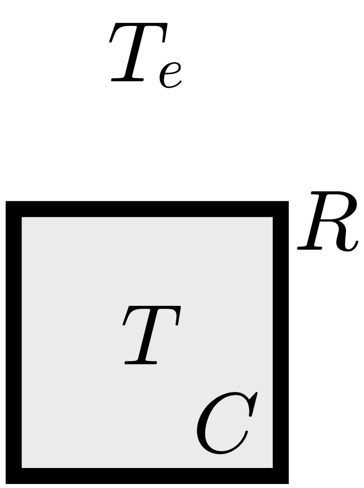
$Q$ heat (Joule)

$R$ thermal resistance

$T$ temperature

$T_e$ ambient temperature

$C$ heat capacity

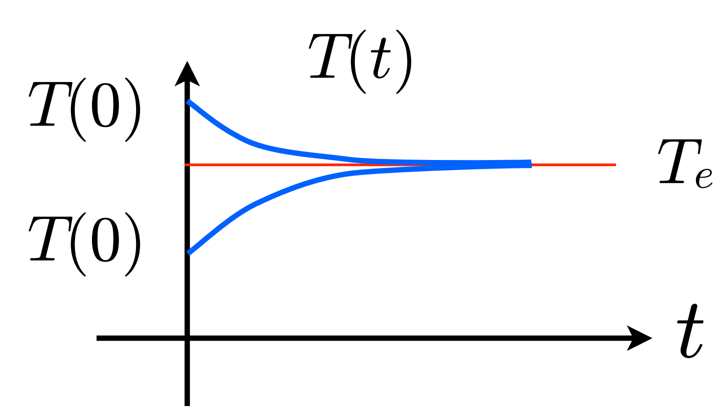



$$\dot{T} = -\frac{1}{RC}T + \frac{1}{RC}T_e$$

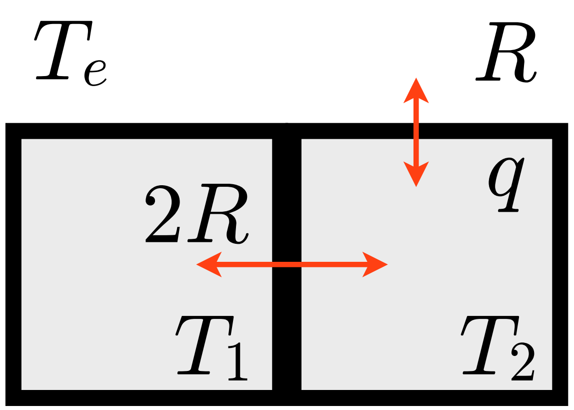
first order system

example: a box placed with internal temperature $T$ in an ambient at a temperature $T_e$

if $T_e$ constant



2 similar rooms




$C$ room thermal capacity

$R$ room thermal resistance

$2R$ room thermal resistance between rooms

$q$ heat flow source

writing the variation of heat in each room

$$C\dot{T}_1 = \frac{T_2 - T_1}{2R} - \frac{T_1 - T_e}{R}$$

$$C\dot{T}_2 = q - \frac{T_2 - T_e}{R} - \frac{T_2 - T_1}{2R}$$

second order system

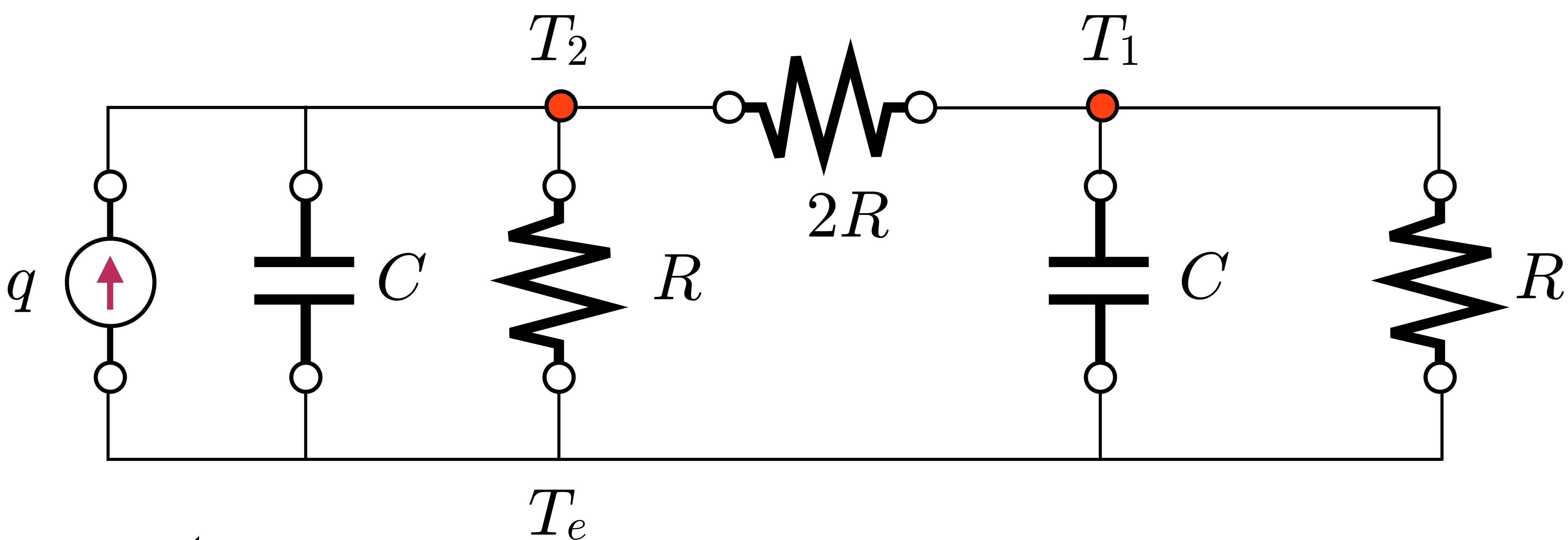
if $q$ is an input, the state components could be chosen as $T_1$ and $T_2$

→ find $(A,B)$ and choose an output of interest (and thus $C$)

# heat flow models

$$\begin{aligned} C\dot{T}_1 &= \frac{T_2 - T_1}{2R} - \frac{T_1 - T_e}{R} \\ C\dot{T}_2 &= q - \frac{T_2 - T_e}{R} - \frac{T_2 - T_1}{2R} \end{aligned}$$

same equations
as the following circuit
(prove it)




current
generator

voltage across first capacitor $T_1$ - $T_e$

voltage across second capacitor $T_2$ - $T_e$

## vocabulary

| English | Italiano |
|---|---|
| linear time-invariant system | sistema lineare stazionario |
| input/state/output | ingresso/stato/uscita |
| mass/spring/damper system | sistema massa/molla/smorzatore |
| state representation | rappresentazione nello spazio di stato |
| dynamics matrix | matrice dinamica |
| input (output) matrix | matrice di ingresso (uscita) |
| feedthrough matrix | matrice del legame diretto ingresso-uscita |