Control Systems

Introduction

L. Lanari

Dipartimento di Ingegneria Informatica
Automatica e Gestionale Antonio Ruberti

Sapienza
Università di Roma

course main topics

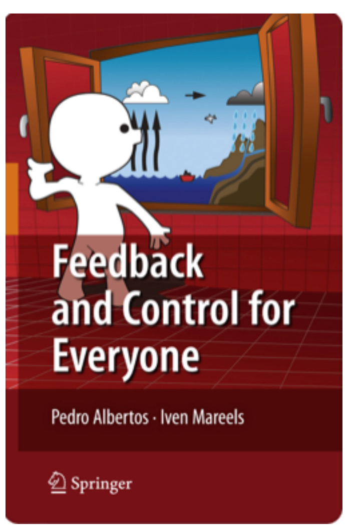
fundamental objective of the Control Systems course

analysis and control of dynamical systems

- analysis (time and frequency domain)
- general feedback control system
- controller design in the frequency domain (loop shaping)
- performance and limitations of a control system
- analysis and design using root locus
- state space design
- stability theory

control is a hidden technology

> Control and feedback are truly everywhere and touch all of us. They play a crucial role in both the natural and the engineered world. In fact neither nature nor our engineered world could possibly function without feedback. Yet, despite its prevalence we are often totally unaware of its presence. It is often said that control is a hidden technology, which is not such a bad thing for a technology, however it is not very good for the self-esteem of its engineers.



Feedback and Control for Everyone
P. Albertos - I. Mareels

this lecture: first basic concepts

- concept of dynamical system
- mathematical model of a dynamical system
- fundamental variables: state, input & output
- what do we seek when analyzing a dynamical system
- analysis and prediction
- linear dynamical systems
- time invariance of a linear dynamical system (LTI)
- state space representation of a LTI system
- control system examples

dynamical system and its characterization

dynamical system:

a system whose state variables evolve over time

state of a dynamical system:

minimum set of variables whose knowledge at the initial time t_0 univocally determine the system time evolution given also the knowledge of future inputs

e.g. in mechanical engineering, rigid body dynamics describes how future forces/torques (inputs) produce motion given an initial position and velocity, i.e., make positions and velocities (state variables) vary over time

- dynamical system: rigid body
- input: external applied torques and forces
- state variable: position and velocity

> how the state evolves in time is a function of the current state, and the exogenous variables (inputs) affecting the system

dynamical system and its characterization

- we need to describe (model) how the state $x(t)$ varies in time
- how do we represent such a variation?

variation in time of the state x

depending on the nature of the time variable

$t \in \mathbf{R}$

in continuous time: derivative

$$\frac{d\,x(t)}{dt} = \dot{x}(t) = \dot{x}$$

$t \in \mathbf{Z}$

in discrete time: difference

$$x(t+1) - x(t)$$

dynamical system representation:

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$x(t+1) = f(x(t), u(t), t)$$

continuous time system $t \in \mathbf{R}$

⇓

differential equations

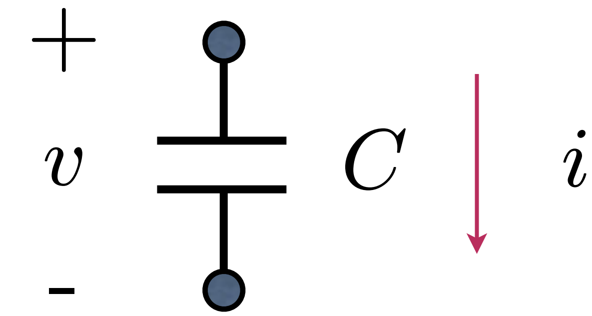
discrete time system $t \in \mathbf{Z}$

dynamical system and its characterization

examples of known relationships including time derivatives:

- capacitor: voltage (v) - current (i)

$$\frac{d\,v(t)}{dt} = \dot{v} = \frac{1}{C} i(t)$$

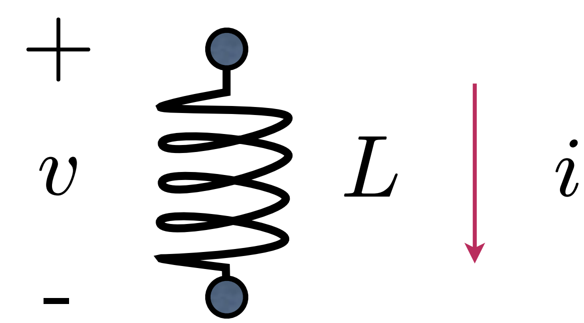


state of the system: $v(t)$

what is $i(t)$? It is an exogenous (external) variable influencing the evolution of the state $v(t)$: it is an input to the system.

- inductor: current (i) state - voltage (v) input

$$\frac{d\,i(t)}{dt} = \frac{1}{L} v(t)$$



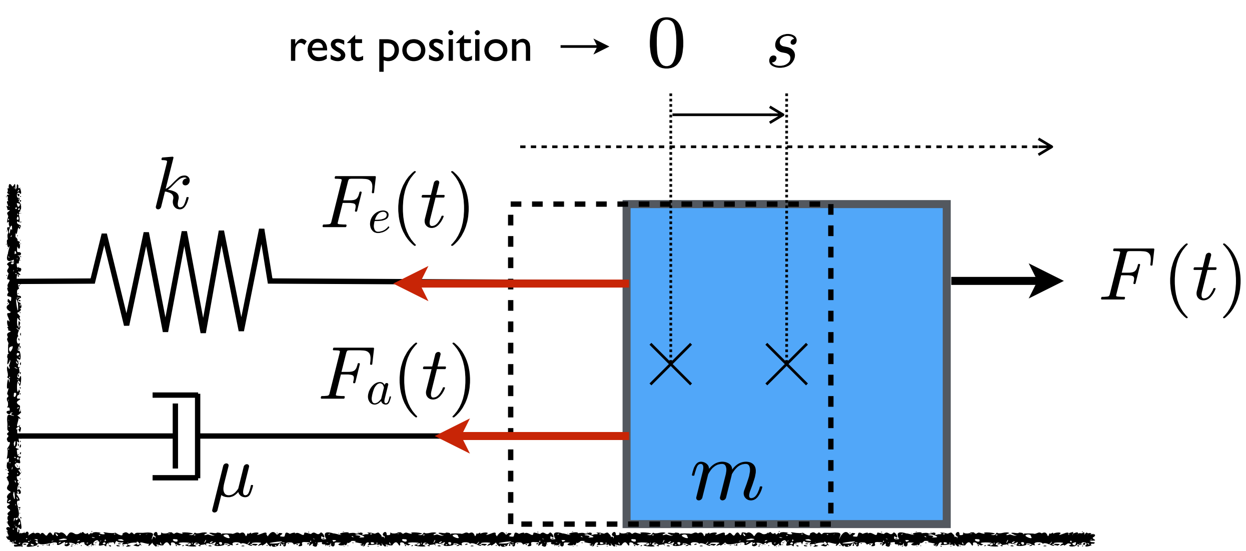
- velocity v (state) of a mass subject to a force F (input)

$$m\,\dot{v} = F$$

dynamical system and its characterization

not only first order differential equations:

Mass-Spring-Damper (MSD) system



$F(t)$ external force

$F_e(t)$ elastic force

$F_a(t)$ friction force

rest position, i.e., with zero velocity

Newton's second law of motion: mass x acceleration $=$ sum of forces

$$m\,a(t) = F(t) - F_e(t) - F_a(t)$$

defining $s(t) =$ deviation from rest position, we have $v(t) = \dot{s}(t)$ velocity

$a(t) = \ddot{s}(t)$ acceleration

+ modeling hypothesis

$F_e(t) = k\ s(t)$ linear spring

$F_a(t) = \mu\ v(t)$ linear viscous friction

$$m\,\ddot{s}(t) + \mu\,\dot{s}(t) + k\,s(t) = F(t)$$

second order differential equation

dynamical system and its characterization

$$m\,\ddot{s}(t) + \mu\,\dot{s}(t) + k\,s(t) = F(t)$$

MSD model

mathematical representation of the Mass-Spring-Damper system under the given hypothesis

- state of the mass-spring-damper system:
  position (s) and velocity ($\dot{s}$)

- input of the mass-spring-damper system:
  exogenous (external) force (F)

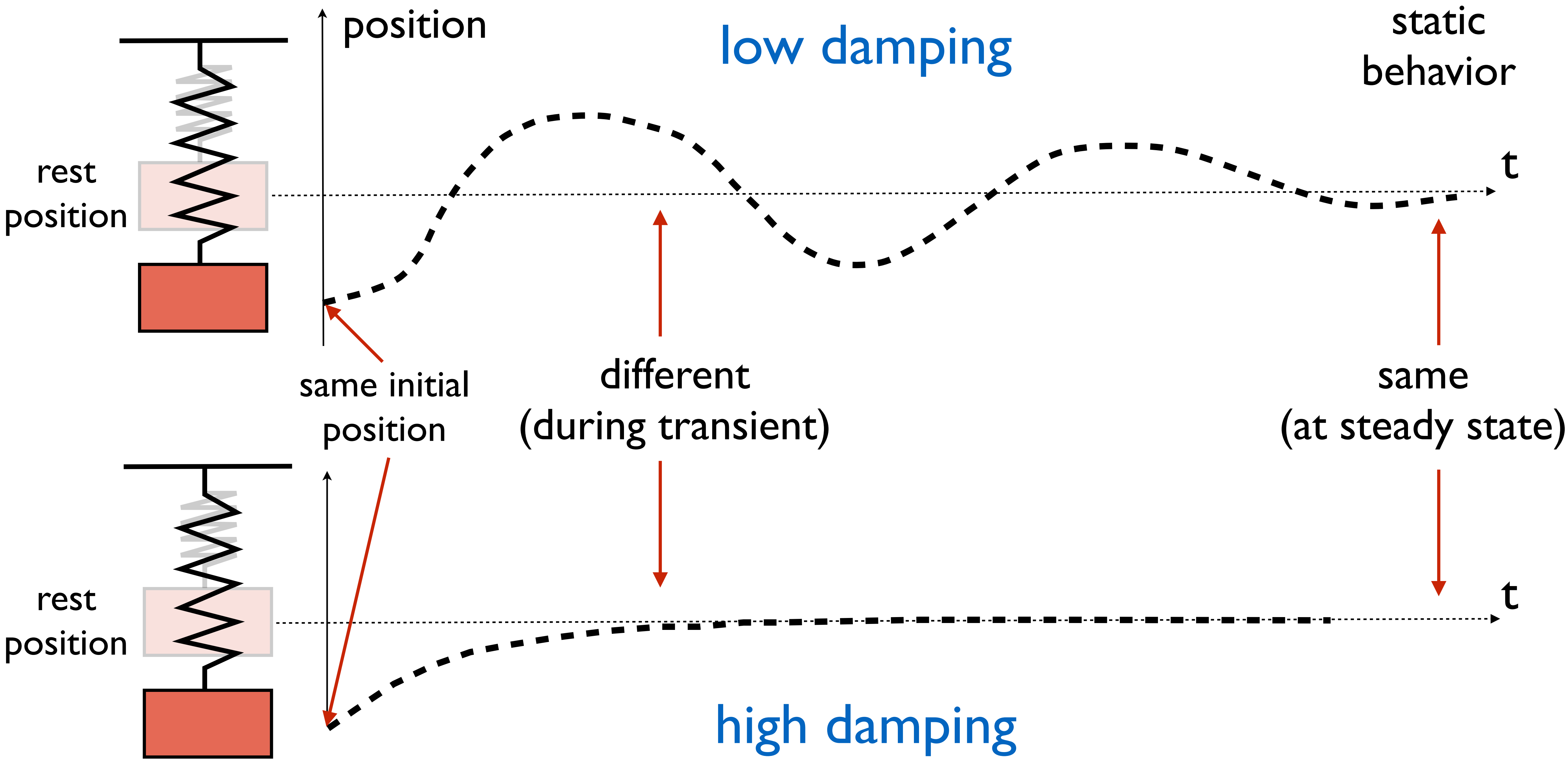
the state evolution is governed by a differential equation

we are going to study qualitatively differential equations in order to infer some basic properties (for example, divergence or convergence of the state evolution)

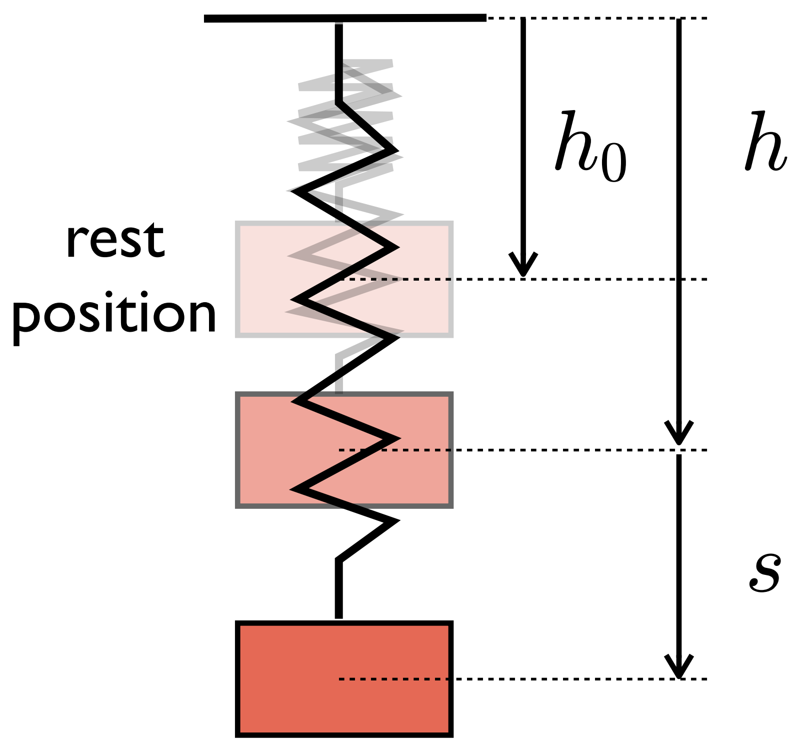
importance of dynamics

- same static behavior, different dynamic one

  two systems with same mass m and spring elastic coefficient k but with different damping coefficient μ (different friction force); starting from the same rest initial position will evolve differently but will end at the same rest position



importance of dynamics



Mass-Spring-Damper model under gravity

h_0 rest position of the spring with no mass (no load)

h (constant) rest position with mass corresponds to $s = 0$ → $k\,(h - h_0) = m\,g$

dynamic equation (or equation of motion)

$$m\,\frac{d^2(h+s)}{dt^2} = -\mu\,\frac{d(h+s)}{dt} - k(h+s-h_0) + mg$$

elastic force proportional to spring deformation

or, being h constant, $m\,\ddot{s} = -\mu\,\dot{s} - ks$

+ external vertical force $F(t)$ on the mass m

$$m\,\ddot{s}(t) + \mu\,\dot{s}(t) + k\,s(t) = F(t)$$

same structure, in these coordinates, as the MSD model

importance of dynamics

- description of the motion (e.g., satellite trajectory)
- simulation models (system behavior w.r.t. to inputs)
- prediction of future behavior
- predicting/avoiding undesirable behavior (e.g., during transient)
- capacity to understand some basic qualitative properties of the system itself (e.g., convergence or divergence of the state evolution as time increases): analysis of dynamic properties
- ability to understand the effect of inputs and how to modify them in order to obtain a desired behavior (basics of control)

w.r.t. means "with respect to"

analysis of dynamic properties

- infer important properties from few basic quantities
  - e.g., stability (from dynamic matrix eigenvalues)
  - characterization of the dynamic behavior as the transient or the steady-state (bandwidth, overshoot, poles, …)
  - study the possibility to influence the dynamics through the input (controllability analysis)
  - understand the internal dynamics through the observation of the output (observability analysis)

- these will allow a clear formulation of specifications for the control system design

- other uses: forecast, prediction

we will perform a qualitative analysis of the solution of systems of differential equations

analysis of a dynamic system model

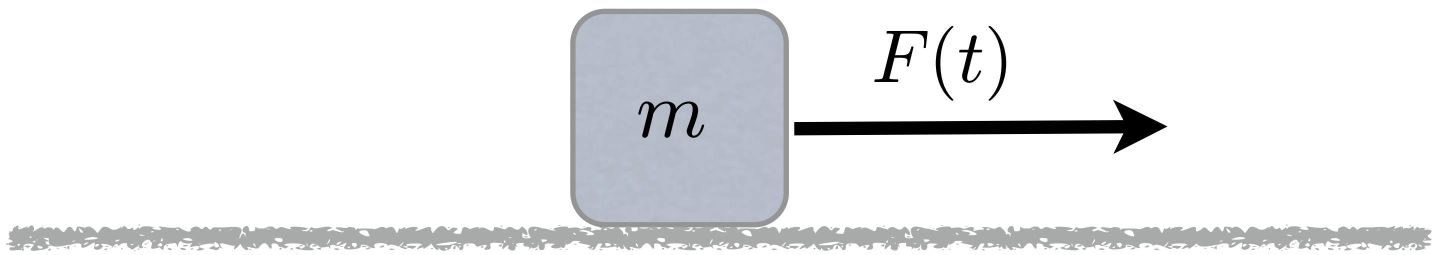
- model based approach:

  representation of the real system through a **model**

  (usually includes approximations)
- in particular we consider dynamical systems whose **mathematical model** is a set of differential equations
- primarily we will consider a set of **linear differential time invariant (LTI) differential equations**
- the **analysis** consists in the study of some characteristics of the system’s mathematical description with particular emphasis on quantities that identify the system qualitative motion

single mass example

mass m velocity v subject to a traction force F



- mass m moving on a line (one-dimensional motion) under the action of an applied force F (not necessarily constant)
- hyp: no friction

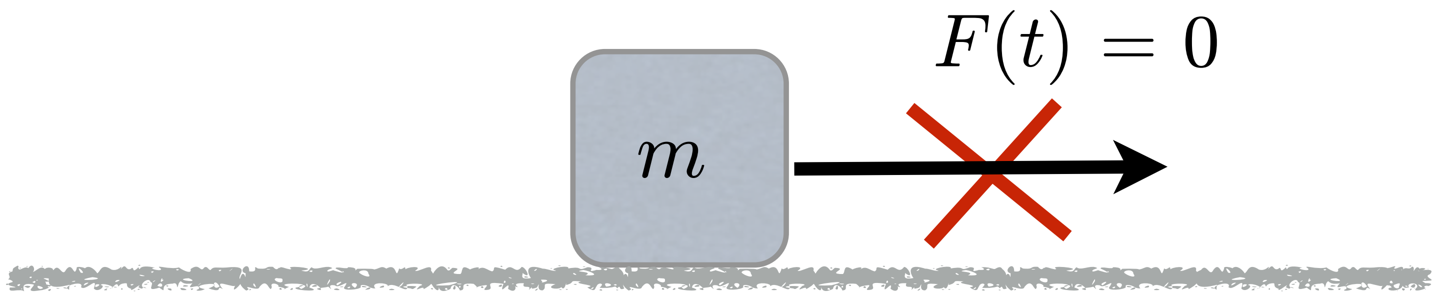
$$m\,\dot{v} = F$$ mathematical model

this mathematical relationship tells us how the variation of the mass velocity (i.e., the acceleration) is related to the applied force under the assumed hypothesis: it is our **model**

+ other implicit hypothesis
(ex. m constant otherwise linear momentum)

N.B. here the state is only v (we are not interested in the position of the mass and there is no relation involving the position in the dynamic equation)

single mass example (cont.)



- if $F = 0$ (no input) do we still have motion?

model becomes $\dot{v} = 0$ and the solution is $v(t) = v(0)$

therefore if we have a non-zero initial velocity $v(0)$, the mass moves (at constant speed) even without any input (force) applied

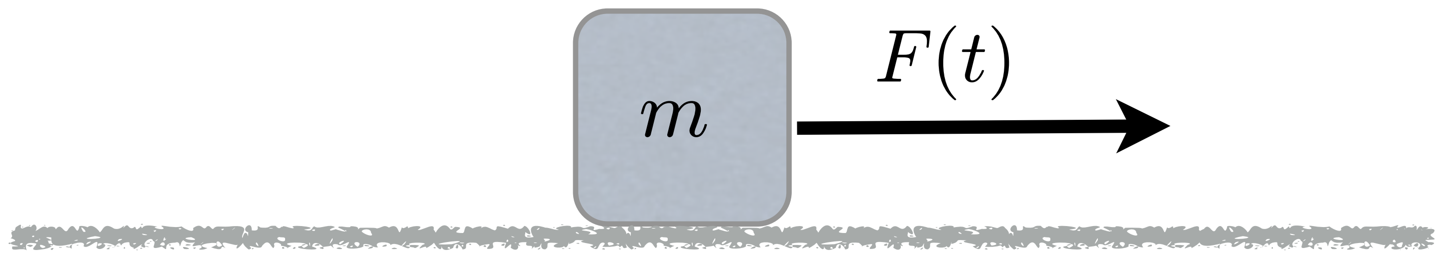
it is obvious in this case and it will be also in the general case

⇓

we need to learn how to read this information which is "hidden" in the mathematical model

N.B. here we solved the homogeneous differential equation $\dot{v} = 0$

single mass example (cont.)



model
(linear differential equation)

$$\dot{v} = \frac{1}{m} F$$
$$v(0) = v_0$$

the motion (here only the variation of v) is generated by

- forcing term $F(t)$ (the input to the system)
- initial condition v_0

here (v_0, F) represents the cause (of motion)
and $v(t)$ is the effect (motion) of such causes

single mass example (cont.)

model

$$\dot{v} = \frac{1}{m} F$$
$$v(0) = v_0$$

it is a first order linear differential equation with constant parameters of the form

$$\dot{x} = a\,x + b\,u \qquad \text{with} \qquad \begin{aligned} x &= v \\ u &= F \end{aligned} \qquad \text{and} \qquad \begin{aligned} a &= 0 \\ b &= 1/m \end{aligned}$$

general solution

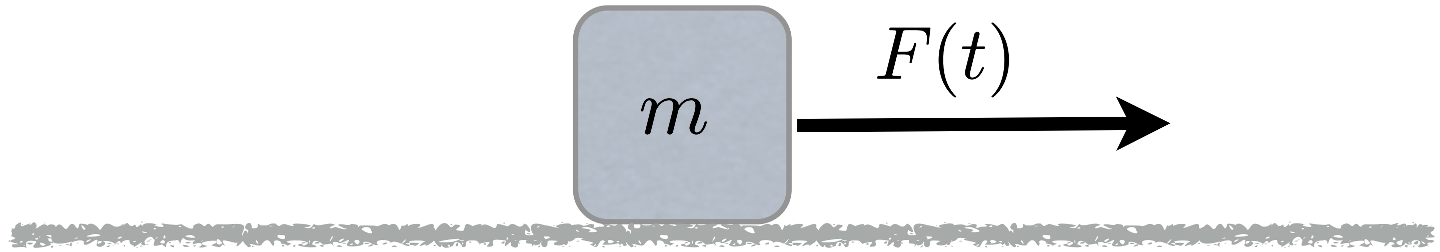
$$v(t) = \boxed{v_0} + \boxed{\frac{1}{m}\int_0^t F(\tau)d\tau}$$

solution of the homogeneous equation

$$\dot{v} = 0$$

particular solution

single mass example (cont.)



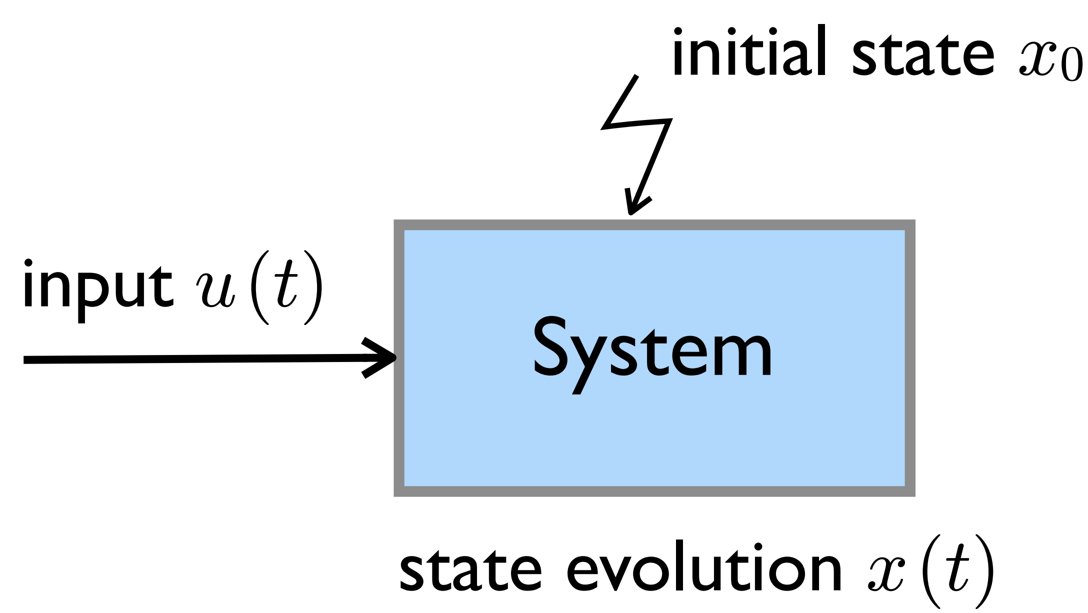
the solution of the differential equation (model)

$$v(t) = v_0 + \frac{1}{m} \int_0^t F(\tau) d\tau$$

tells us how the velocity depends upon the initial condition **and** the future applied force. Knowing the applied force (from $t = 0$) and the initial velocity we know how the velocity of the point mass will behave in the future

new capacity: **analysis** & **prediction**

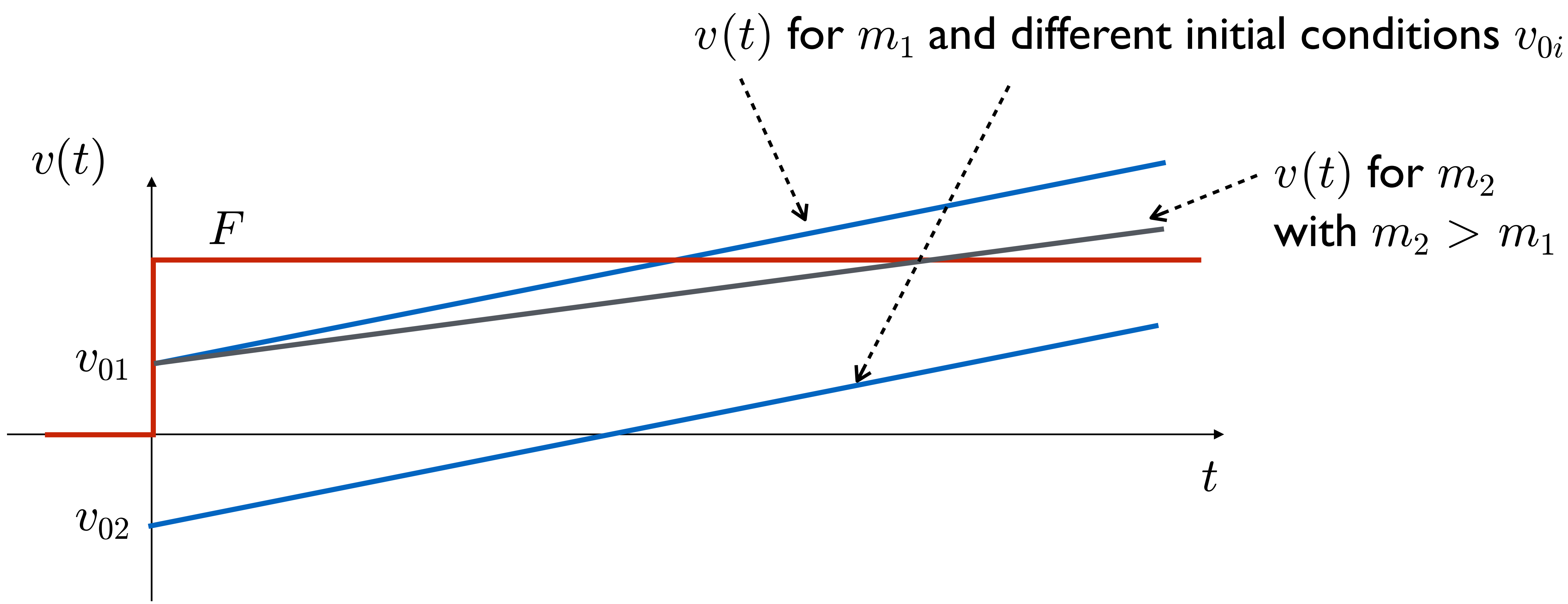
block diagram representation



single mass example (cont.)

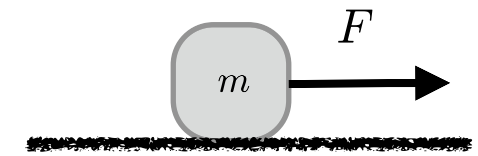
let's apply a constant input $F > 0$ from $t = 0$ and assume an initial velocity v_0

the solution of the differential equation is $$v(t) = v_0 + \frac{F}{m} t$$



exercise: give a physical interpretation of the plots:
is it true that with a larger mass and same applied constant force starting from the same initial velocity the evolution is as in the plot?

single mass example - linearity



linearity plays a fundamental (simplifying) role in our analysis

- with initial condition $v_0 = 0$ and $F \neq 0$ we have velocity v

  if we apply $2F$ instead of F what happens to velocity?

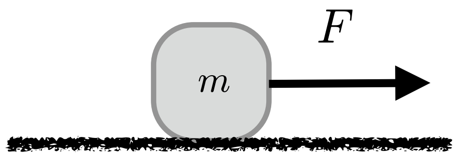
$$\tilde{v}(t) = \cancel{v_0} + \frac{1}{m}\int_0^t 2\, F(\tau)d\tau$$

  the resulting velocity will also double to $2\,v$

  linear behavior wrt to F

In general for a linear system, starting from the initial condition $x(0) = 0$, if an input $u(t)$ generates a state evolution $x_u(t)$, and an input $v(t)$ generates a state evolution $x_v(t)$, then the input $\alpha\, u(t) + \beta v(t)$ will generate the state evolution $\alpha\, x_u(t) + \beta x_v(t)$ (same linear combination)

single mass example - linearity



similarly

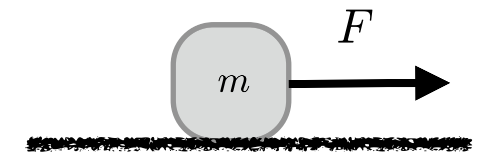
- with no force $F = 0$ and start with non-zero $v_0 \neq 0$, the velocity is $v = v_0$
  clearly, if the initial velocity changes to $3v_0$ the velocity will also triple

$$\tilde{v}(t) = 3\,v_0 + \frac{1}{m}\int_0^t F(\tau)d\tau$$

linear behavior wrt to the initial condition v_0

In general for a linear system, when no input is applied ($u(t) = 0$), if when we start from the initial condition $x_a(0)$ we have a state evolution $x_a(t)$, and from $x_b(0)$ we have $x_b(t)$ then starting from $\gamma\, x_a(0) + \mu\, x_b(0)$ the state evolution will be $\gamma\, x_a(t) + \mu\, x_b(t)$ (same linear combination)

single mass example - linearity



we saw separately linearity w.r.t. the input and w.r.t. the initial condition, what happens when we have simultaneously an initial condition and an input?

- Att. if $F \neq 0$ and $v_0 \neq 0$ simultaneously

if $F \longrightarrow 2F$

and $v_0 \longrightarrow 3v_0$

what happens to velocity?

previous examples

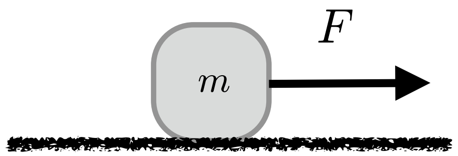
if $(v_0, 0) \longrightarrow v_1(t)$, $(0, F) \longrightarrow v_2(t)$ then $(3v_0, 0) \longrightarrow 3v_1(t)$, $(0, 2F) \longrightarrow 2v_2(t)$

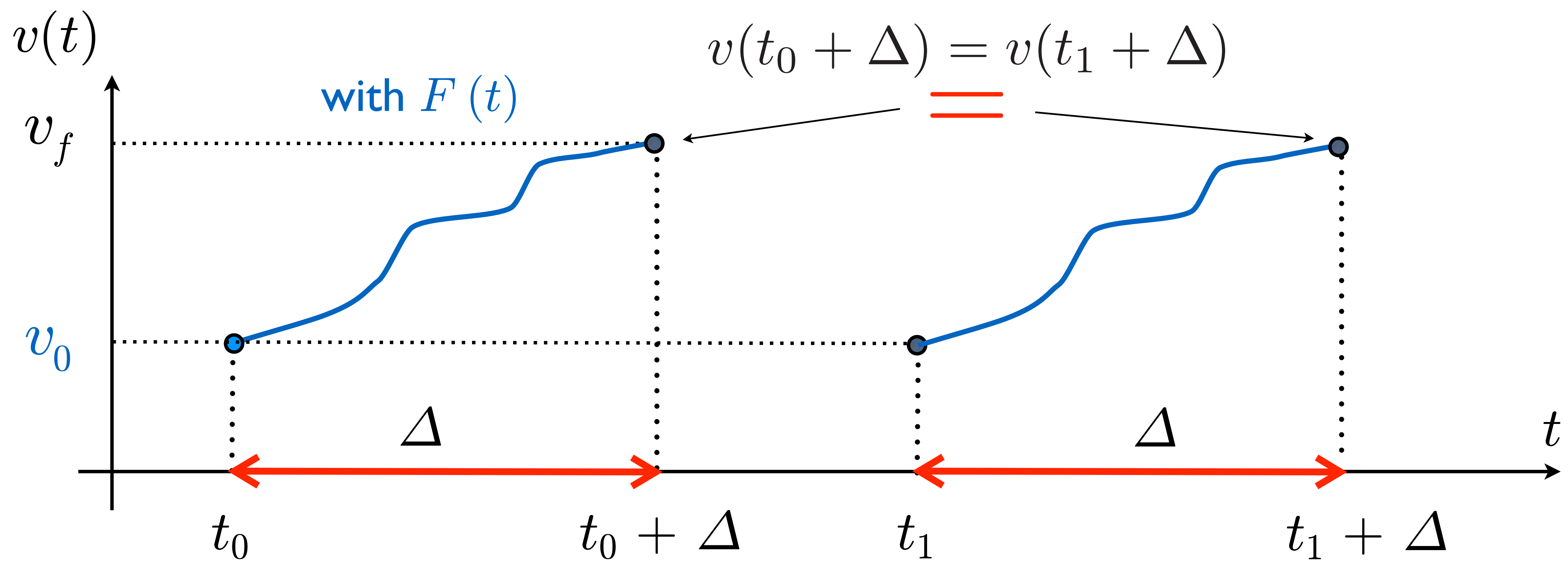
linear behavior wrt the motion cause (v_0, F)

simultaneous causes	→	sum of effects
$\alpha\ v_0$ and $\beta\ F$		$\alpha\ v_1(t) + \beta\ v_2(t)$

this linearity comes from the **differential equation** being **linear**

single mass example - time invariance





starting from the same initial condition v_0 and applying the same input (force) $F(t)$ after same time interval Δ leads to the same state

- state evolution does not depend on the initial time t_0 but only on the elapsed time Δ (then, without loss of generality, we can assume that $t_0 = 0$)
- this time invariance translates into the linear differential equation having constant coefficients

general mathematical model

the state evolution is described by a set (x is in general a vector) of first order linear differential equations with constant coefficients

state equation (general form) $\rightarrow$

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$x(0) = x_0$$

Linear Time Invariant (LTI) system

$x(t)$ **state** $\quad x \in \mathbf{R}^n$

$u(t)$ **input** $\quad u \in \mathbf{R}^q$ multi input case (if $q = 1$ then single input)

all the continuous time LTI systems admit this state space representation

N.B. we will see how to rewrite a differential equation of higher order as a set of first order differential equations

we can add an extra piece of information: what variable(s) can we measure and observe? These are defined as output(s) of the system

to remain in a linear setting we assume the following linear relation

output $\quad y(t) = C\,x(t) + D\,u(t)$

general mathematical model

state space representation

state equation → $\dot{x}(t) = A\,x(t) + B\,u(t)$

output equation → $y(t) = C\,x(t) + D\,u(t)$

$x(0) = x_0$

Linear Time Invariant (LTI) dynamical system (Continuous Time)

$x(t)$ **state** $x \in \mathbf{R}^n$

$u(t)$ **input** $u \in \mathbf{R}^q$ multi input (we consider $q = 1$, single input)

$y(t)$ **output** $y \in \mathbf{R}^p$ multi output (we consider $p = 1$, single output)

$q = p = 1$ SISO (single input/single output) linear time-invariant system

state, input and output dimensions determine the 4 matrices (A, B, C, D) dimensions

SISO	A: n x n	B: n x 1	C: 1 x n	D: 1 x 1
general	A: n x n	B: n x q	C: p x n	D: p x q

general mathematical model

$n = 2$ SISO example $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ $D = d$

linear combination

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t) \\ a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t) \end{bmatrix}$$

$$y(t) = Cx(t) + Bu(t) \longrightarrow y(t) = c_1x_1(t) + c_2x_2(t) + d\,u(t)$$

matrix form

linear combination

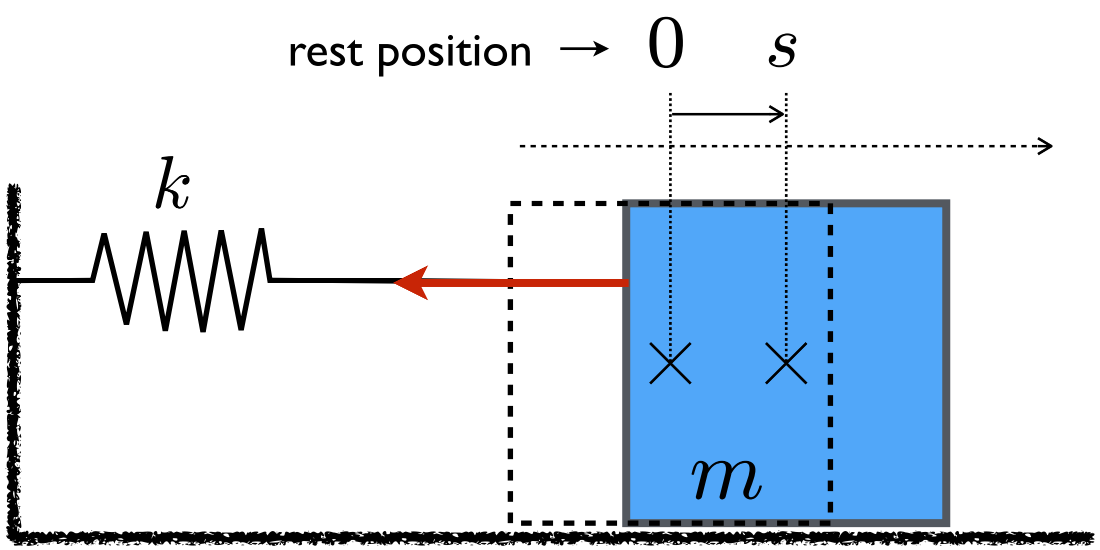
A how the states influence each other's evolution (**dynamic matrix**)

B how the input affects the state evolution (**input matrix/vector**)

C how the states affect (or combine) to give the output (**output matrix/vector**)

D how the input can directly affect the output (**feedthrough matrix/scalar**)

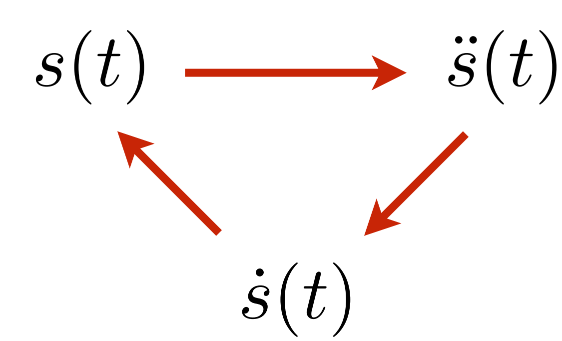
mass-spring example



equation of motion $m\,\ddot{s}(t) = -k\,s(t)$

or $\ddot{s}(t) = -\frac{k}{m}s(t)$

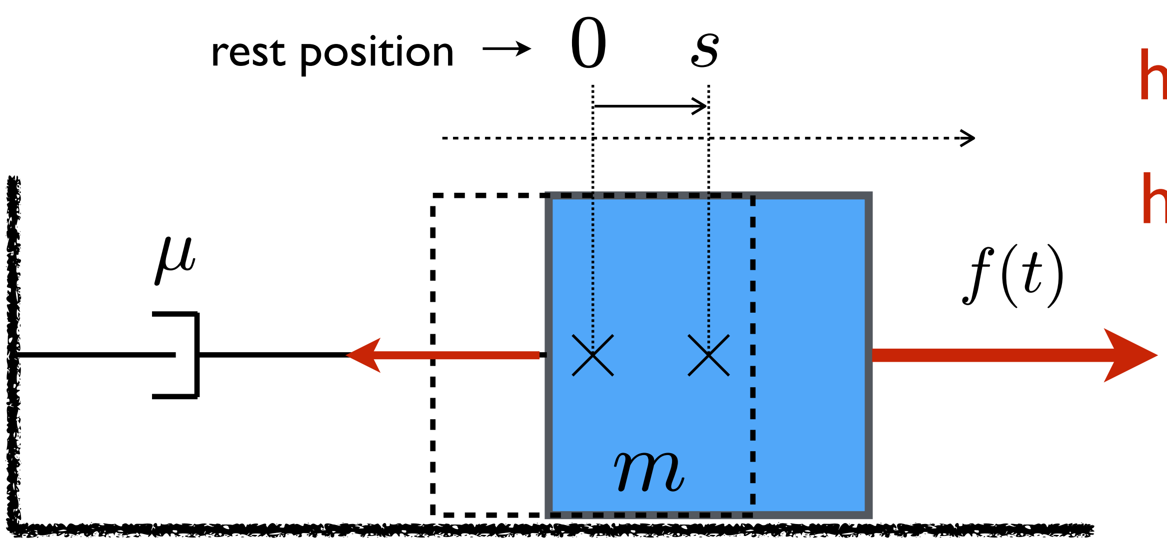
an initial displacement (for example stretching the spring and then letting the system evolve) produces an acceleration, which produces a change in velocity which in turn changes the position and back to the acceleration.



- state: $x = \begin{bmatrix} s \\ \dot{s} \end{bmatrix}$ (see slides “Models”) acceleration is a function of $(s, \dot{s})$
- no input if we add an external horizontal force $\ddot{s}(t) = -\frac{k}{m}s(t) + \frac{1}{m}f(t)$
- output if we are interested (and measure) the position $y(t) = s(t)$

  if we are interested (and measure) the velocity $y(t) = \dot{s}(t)$

control example: mass-damper



hyp 1 : we can measure the mass position $s(t)$

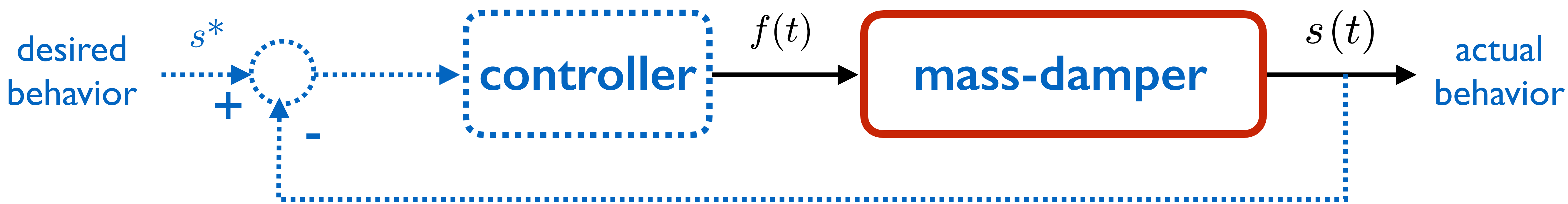
hyp 2 : the applied force $f(t)$ can be manipulated

dynamical system: input, state and output

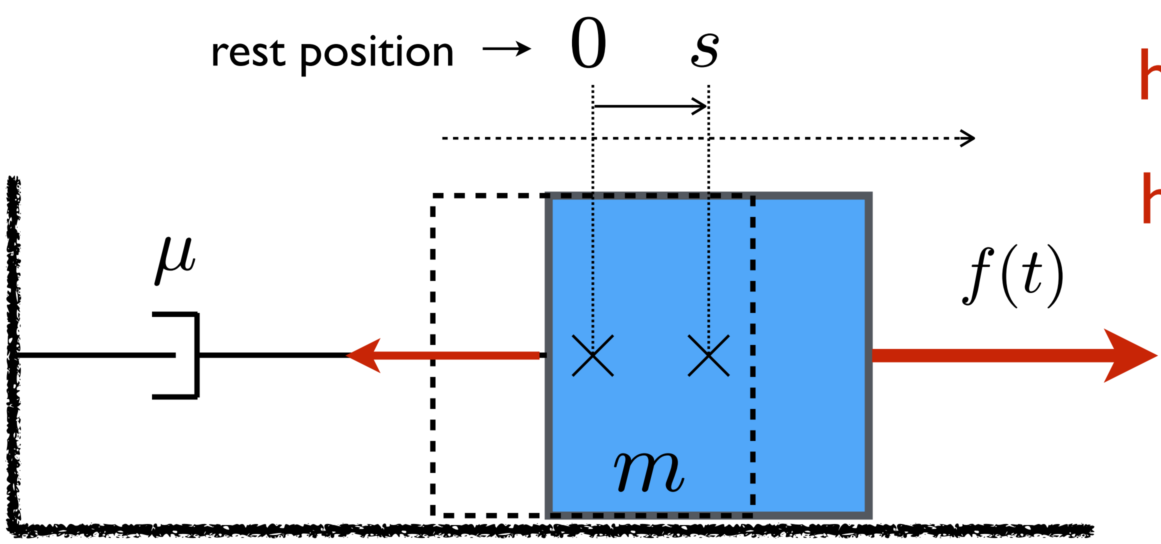
model:
$$m\,\ddot{s}(t) + \mu\,\dot{s}(t) = f(t)$$
$$y(t) = s(t)$$

goal: given the measure of the displacement $s(t)$ we want to design an automatic controller which guarantees that the mass will asymptotically tend (and stop) at the desired constant position s^*

feedback control scheme



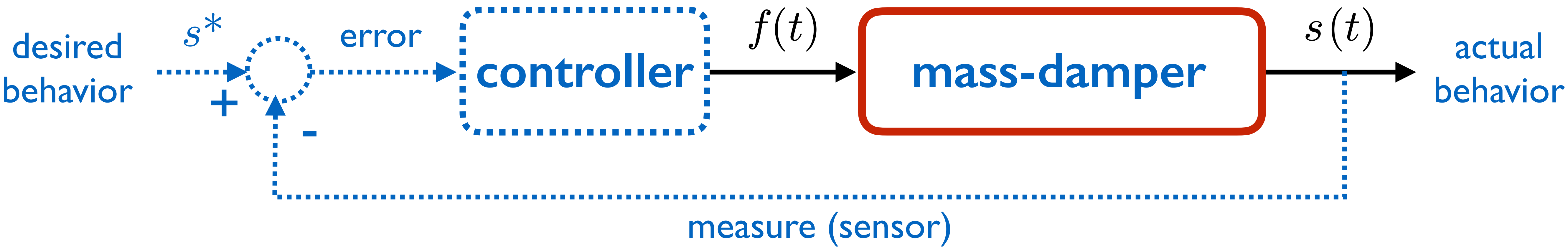
control example: mass-damper



hyp 1 : we can measure the mass position $s(t)$

hyp 2 : the applied force $f(t)$ can be manipulated

feedback control scheme



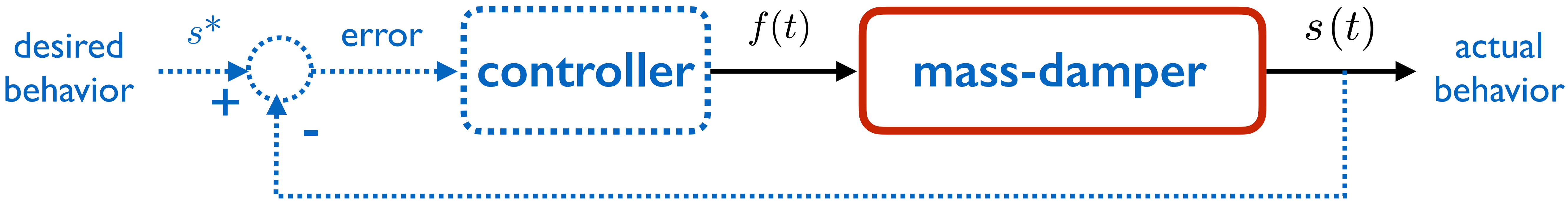
$f(t)$ is a force (produced by an actuator)

$s(t)$ needs to be measured (e.g. by a transducer)

this is an ideal (conceptual) control scheme

control example: mass-damper

feedback control scheme



we assume that the controller produces a force $f(t)$ proportional to the error (desired behavior – actual behavior) that is $f(t) = k_p\,(s^* - s(t))$ with s^* constant

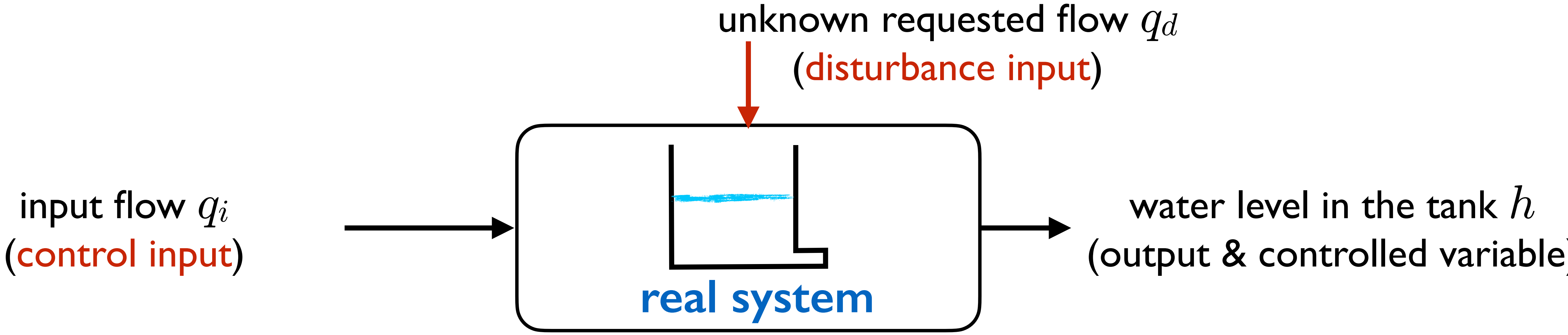
from the model we have $m\,\ddot{s} + \mu\,\dot{s} + k_p\,(s - s^*) = 0$ which with $p = s - s^*$ becomes

$$m\,\ddot{p} + \mu\,\dot{p} + k_p\,p = 0$$

this equation behaves as an unforced mass-spring-damper system that will stabilize in $p = 0$ (provided $k_p > 0$) that is in $s = s^*$; in other words the position will tend (with the dynamics governed by the second order differential equation) asymptotically to the desired position s^*

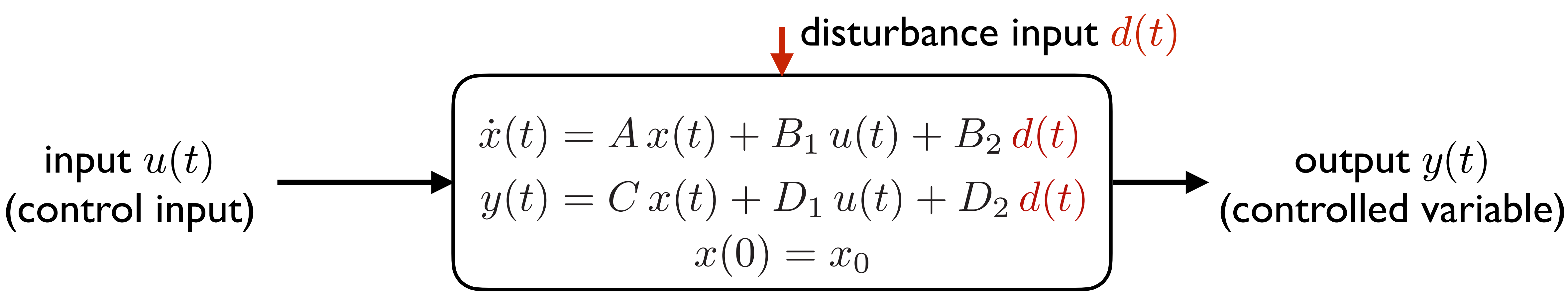
note that this conclusion is based on the system model; when implemented, the controller will be applied to the real system, not to the model

control example: water level in a tank



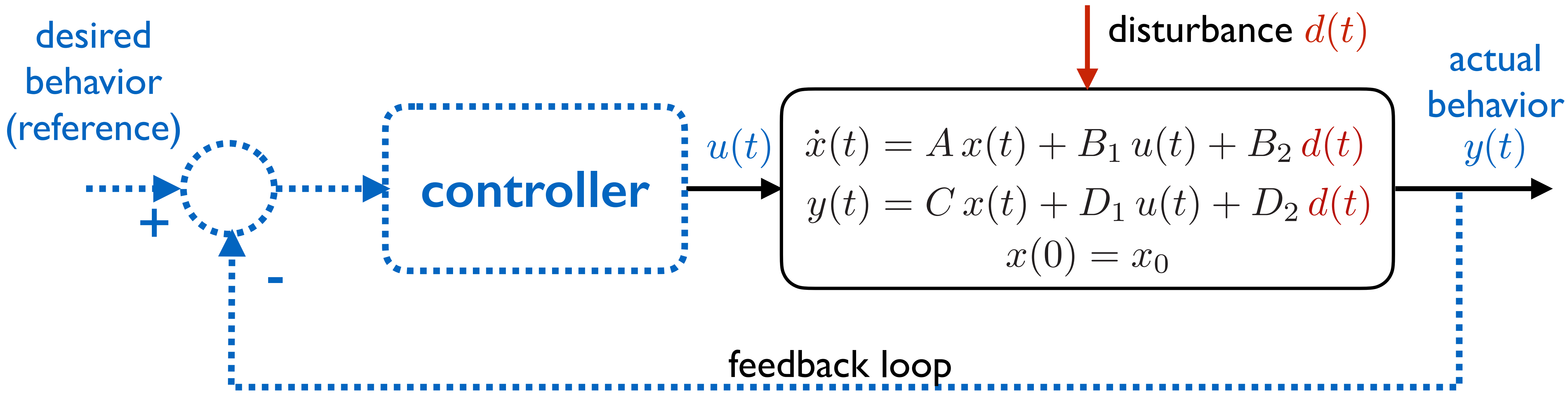
problem: we want to maintain the water level at a desired height regardless of the unknown output flow and any other disturbance (evaporation, loss in the tank,…)

- note the distinction of the 2 inputs: control (which can be manipoulated) and disturbance (which cannot be manipulated)
- understand how to automatically determine the (control) input in order to guarantee a **desired behavior** of the output



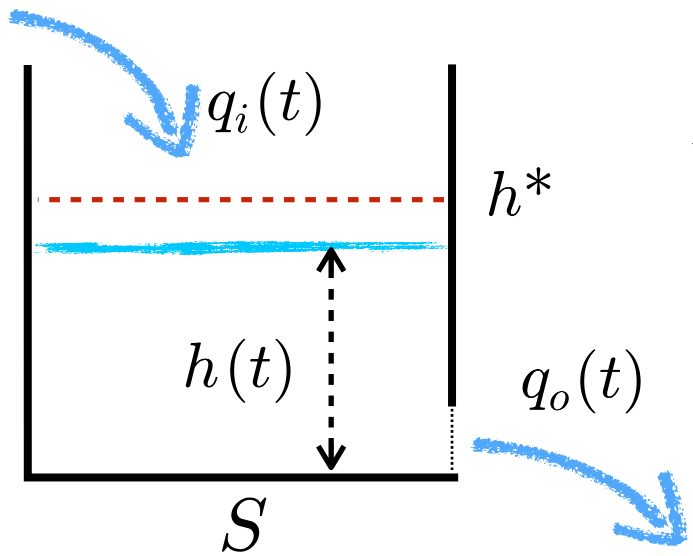
model

control example: water level in a tank



- schematic diagram of an **automatic control system** based on **feedback**
- the design of such a control system requires the determination (design) of the **controller**
- **we** need a systematic procedure in order to design the controller
- the design of the controller will be in general based on the **plant model**

control example: water level in a tank



since the variation of volume $V(t) = h(t)\ S$ is given by the flows (volume/s) which enter and/or exit, we have

$$\dot{V}(t) = S\,\dot{h}(t)$$
$$\dot{V}(t) = q_i(t) - q_o(t) \rightarrow$$

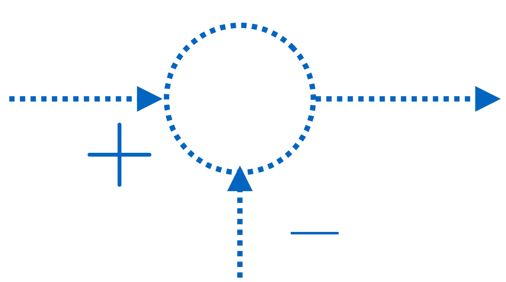
dynamic model

$$\dot{h} = \frac{1}{S}(q_i - q_o)$$

$q_i(t)$ input flow (volume/s): can be manipulated (control input)

$q_o(t)$ output flow (volume/s): cannot be manipulated, unknown (disturbance)

what happens with a controller which acts proportionally to the error, that is as $q_i(t) = K(h^* - h(t))$ with $K > 0$



at a given instant t_a

if $h(t_a) > h^*$ then $h^* - h(t_a) < 0$ and $q_i < 0$

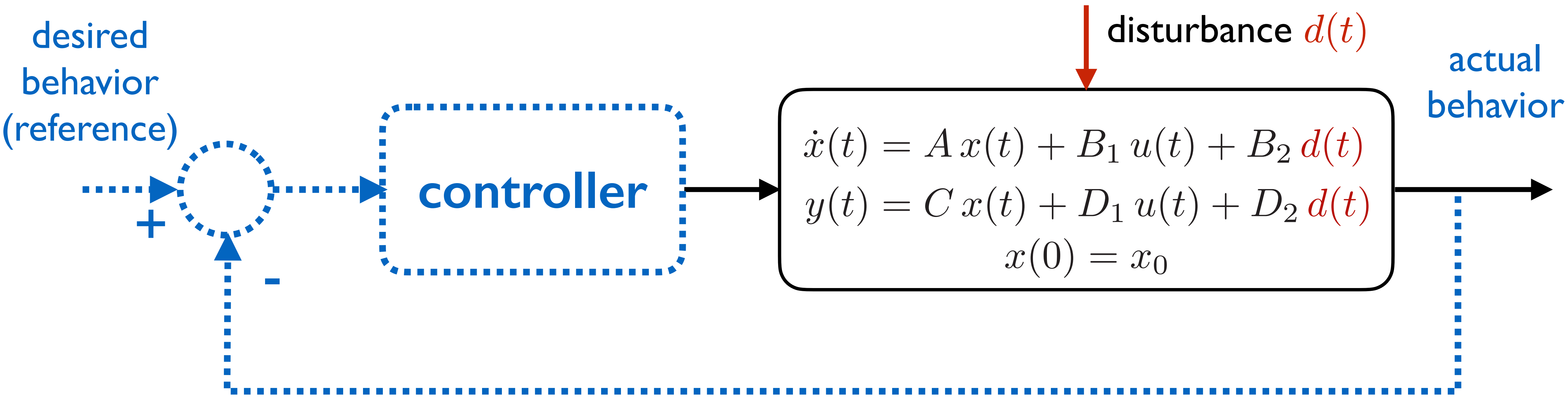
if $h(t_a) < h^*$ then $h^* - h(t_a) > 0$ and $q_i > 0$

with this choice we have

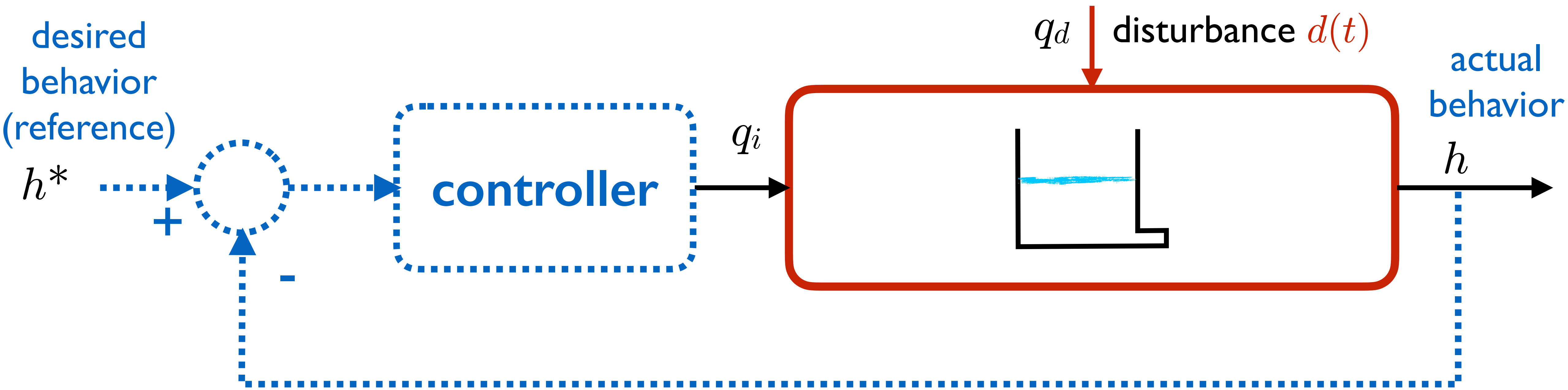
$$\dot{h}(t) = -\frac{K}{S}h(t) + \frac{K}{S}h^*$$

will the solution converge to h^* ?

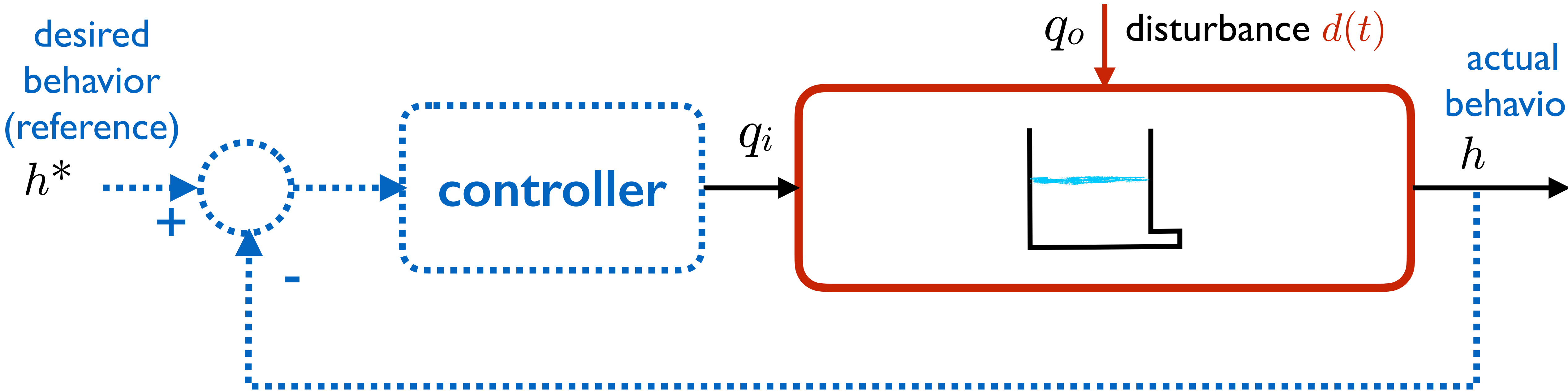
control example: water level in a tank



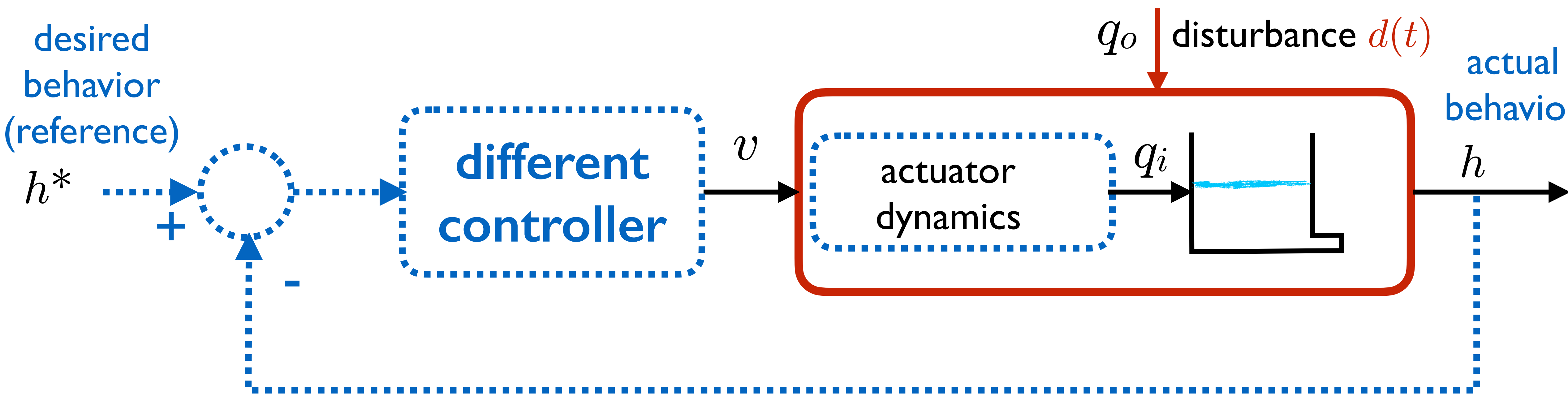
control scheme is implemented on the real system



control example: water level in a tank



may be q_i is not the real input, but the output of a valve commanded by a voltage v: we may need to include the **actuator dynamics**



typical flow in a control problem

- problem definition (real system)
- mathematical model (+ hypothesis & simplifications)
- real specifications translated into control system language
- design of the control system
- simulation on the more complete available model
- implementation on the real system

example: humanoid gait generation

- generate a gait (steps and timing) for the humanoid guaranteeing equilibrium
- simplest mathematical model: Linear Inverted Pendulum
- specifications (follow a plan and “do not fall”)
- control design (for example Model Predictive Control - MPC)
- simulation on a general purpose physics engine
  (e.g. MuJoCo - Multi-Joint dynamics with Contact)
- experiment