DYNAMIC PROPERTIES AND NONLINEAR CONTROL OF ROBOTS WITH MIXED RIGID/ELASTIC JOINTS

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ABSTRACT

Compliance in motion transmission components is the main source of vibrational behavior in robot manipulators. Since the actuator/transmission design may differ from joint to joint, many robot arms have some joints that can be considered completely rigid and some other where elasticity is relevant. We consider dynamic modeling and control design for robot manipulators with mixed rigid/elastic joints. A nonlinear dynamic feedback controller is presented that allows to achieve exact linearization and input-output decoupling for the general class of robots having mixed rigid/elastic joints in any possible kinematic sequence. Simulation results are presented for a $2R$ planar arm having only the second joint elastic.

KEYWORDS: Elastic joints robots, Trajectory tracking, Dynamic feedback linearization

1. INTRODUCTION

It is well known that compliance in the motion transmission components is the main source of vibrational behavior in otherwise rigid industrial robot manipulators [1]. The use of harmonic drives, long shafts, or transmission belts introduce an elastic coupling between the actuators and the driven links, that can be modeled with linear/torsional springs of finite stiffness located at each joint, while doubling the number of system generalized coordinates [2, 3]. More recently, joint elasticity has been introduced on purpose in the design of lightweight robots in order to increase safety indices for potential collisions in a human-robot close interaction [4, 5].

Both regulation and trajectory tracking control problems have been considered for robots with all joints elastic. Global asymptotic stabilization to a constant equilibrium configuration can be obtained via PD control on the motor variables, plus the addition of a constant [3, 6] or on-line [7] gravity compensation term. For more demanding trajectory tracking tasks, a model-based nonlinear feedback control should be used. When inertial couplings between the motors and links in motion are absent (due to the kinematic robot structure) [8] or can be neglected [2], a *static* state feedback allows to exactly linearize and decouple the closed-loop dynamics. The obtained result mimics the one obtained with the computed torque method in the rigid robot case, except that the resulting linear system is made of separate chains of four (instead of two) input-output integrators from the new input to the link position coordinates. In the more general case of motor/link inertial couplings, it has been shown in [9] that the use of *dynamic* nonlinear state feedback yields a similar result. For a robot with N joints, all being elastic, the dimension of the exact linearizing dynamic compensator is upper bounded by $2N(N-1)$. An efficient implementation of this type of inverse-dynamics controller can be found in [10].

Since the actuator/transmission mechanical arrangement is usually different from joint to joint, many robot arms (e.g., those in the SCARA family) have some joints (say, $N_r < N$) that can be considered completely rigid while for the others ($N_e = N - N_r$) elasticity is relevant. Modeling and control issues of this mixed situation have been considered in [11, 12], but limited to the case of neglecting the motor/link inertial couplings (i.e., with the standing assumption of the so-called reduced dynamic model of Spong [2]). In particular, except when very restrictive conditions on the internal structure of the link inertia matrix hold, a dynamic feedback is needed for decoupling and linearization purposes [11]. The controller has dimension $2N_r$ and physically forces the rigid robot joints to behave as elastic ones.

In this paper, we consider the general dynamic model of robot manipulators for the mixed case of rigid/elastic joints appearing in any possible kinematic sequence. A coordinate transformation is introduced that puts the robot dynamic equations in a canonical form where control-oriented properties can easily be investigated. Control design based on exact linearization and input-output decoupling via nonlinear dynamic feedback is then presented, specializing the constructive algorithm given in [9] for deriving the actual expression and the dimension of the dynamic feedback linearizing law. Simulation results illustrating trajectory tracking performance are presented for a $2R$ planar robot with the first joint rigid and the second elastic.

2. DYNAMIC MODELING

Consider an open kinematic chain of rigid bodies, interconnected by N joints all undergoing elastic deformation. The robot is actuated by N electrical drives, the i-th being located at the i-th joint (or mounted on a previous link of index $j < i$). Let $\widehat{q} \in \mathbb{R}^N$ be the link position coordinates and $\theta \in \mathbb{R}^N$ be the motor (i.e., rotor) position coordinates as reflected through the gear ratios. Joint deformations are small, so that their elasticity can be modeled with (undamped) linear springs. Moreover, the rotors of the motors are uniform bodies with center of mass on their rotation axis [3]. From these standard assumptions, the inertia matrix and the gravity term in the dynamic model will be independent from θ.

Following a Lagrangian approach, the kinetic energy (including links and motors) is

$$T = \frac{1}{2} \begin{pmatrix} \dot{\widehat{q}}^{\mathrm{T}} & \dot{\theta}^{\mathrm{T}} \end{pmatrix} M_{\mathrm{elastic}}(\widehat{q}) \begin{pmatrix} \dot{\widehat{q}} \\ \dot{\theta} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \dot{\widehat{q}}^{\mathrm{T}} & \dot{\theta}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} B(\widehat{q}) & S(\widehat{q}) \\ S^{\mathrm{T}}(\widehat{q}) & J \end{pmatrix} \begin{pmatrix} \dot{\widehat{q}} \\ \dot{\theta} \end{pmatrix},$$

where all blocks of the inertia matrix $M_{\mathrm{elastic}}(\widehat{q})$ are $N \times N$ matrices: $B(\widehat{q}) > 0$ contains the inertial properties of the rigid links, $S(\widehat{q})$ is a strictly upper triangular matrix and accounts for the inertial couplings between motors and links, while the diagonal matrix $J > 0$, is the matrix of the effective motor inertias. For the ease of presentation, we will assume from now on that matrix S is constant. This holds true, e.g., for a spatial $3R$ elbow manipulator and for planar robots with any number of rotational joints. The potential (gravitational plus elastic) energy is

$$U = U_g(\widehat{q}) + \frac{1}{2}(\widehat{q} - \theta)^{\mathrm{T}} K (\widehat{q} - \theta),$$

where $K = \mathrm{diag}\{K_1, \ldots, K_N\} > 0$ is the stiffness matrix of the elastic joints.

The robot dynamic model is obtained from the Euler-Lagrange equations for the Lagrangian $L = T - U$. Under the above assumptions, the $2N$ second-order differential equations have the form (see [3] or [8] for a detailed derivation)

$$\begin{aligned} B(\widehat{q})\ddot{\widehat{q}} + S\ddot{\theta} + c(\widehat{q}, \dot{\widehat{q}}) + g(\widehat{q}) + K(\widehat{q} - \theta) &= 0 \qquad (1) \\ S^{\mathrm{T}}\ddot{\widehat{q}} + J\ddot{\theta} + K(\theta - \widehat{q}) &= \tau, \qquad (2) \end{aligned}$$

where $c(\widehat{q}, \dot{\widehat{q}})$ are Coriolis and centrifugal terms, $g(\widehat{q}) = (\partial U_g / \partial \widehat{q})^{\mathrm{T}}$ are gravity terms, and $\tau \in \mathbb{R}^N$ are the torques supplied by the motors. We define also $n(\widehat{q}, \dot{\widehat{q}}) = c(\widehat{q}, \dot{\widehat{q}}) + g(\widehat{q})$.

Suppose now that $N_r < N$ joints (in any order in the chain) are rigid and $N_e = N - N_r$ joints are elastic, with associated motor coordinates $\theta \in \mathbb{R}^{N_e}$. The $(N + N_e) \times (N + N_e)$ inertia matrix becomes

$$\widehat{M}_{\text{mixed}}(\widehat{q}) = \begin{pmatrix} \widehat{B}(\widehat{q}) & \widehat{S} \\ \widehat{S}^{\mathrm{T}} & J_{EE} \end{pmatrix}$$

with $\widehat{B}(\widehat{q}) \in \mathbb{R}^{N} \times \mathbb{R}^{N}$, $\widehat{S}(\widehat{q}) \in \mathbb{R}^{N} \times \mathbb{R}^{N_e}$ and $J_{EE} \in \mathbb{R}^{N_e} \times \mathbb{R}^{N_e}$. It is convenient to reorder the link position vector $\widehat{q}$ so that the first N_r components q_R are those driven through rigid joints while the remaining q_E are related to the N_e elastic joints, or

$$\widehat{q} = T_q\, q \qquad \Leftrightarrow \qquad q = \begin{pmatrix} q_R \\ q_E \end{pmatrix} = T_q^{\mathrm{T}} \widehat{q},$$

being T_q an orthonormal matrix. In the new coordinates, the dynamic model becomes

$$M_{\text{mixed}}(q) \begin{pmatrix} \ddot{q}_R \\ \ddot{q}_E \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} n_R(q, \dot{q}) \\ n_E(q, \dot{q}) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ K(q_E - \theta) \\ K(\theta - q_E) \end{pmatrix} = \begin{pmatrix} \tau_R \\ 0 \\ \tau_E \end{pmatrix}, \tag{3}$$

with

$$M_{\text{mixed}}(q) = \left(\begin{array}{cc||c} B_{RR}(q) + J_{RR} + S_{RR} + S_{RR}^{\mathrm{T}} & B_{RE}(q) + S_{ER}^{\mathrm{T}} & S_{RE} \\ B_{RE}^{\mathrm{T}}(q) + S_{ER} & B_{EE}(q) & S_{EE} \\ \hline\hline S_{RE}^{\mathrm{T}} & S_{EE}^{\mathrm{T}}(q) & J_{EE} \end{array} \right)$$

where the blocks are extracted from $B(\widehat{q})$, S, and J in eqs. (1-2) as

$$T_q\, B(\widehat{q})\, T_q^{\mathrm{T}} = \begin{pmatrix} B_{RR}(q) & B_{RE}(q) \\ B_{RE}^{\mathrm{T}}(q) & B_{EE}(q) \end{pmatrix} \; T_q\, S\, T_q^{\mathrm{T}} = \begin{pmatrix} S_{RR} & S_{RE} \\ S_{ER} & S_{EE} \end{pmatrix} \; T_q\, J\, T_q^{\mathrm{T}} = \begin{pmatrix} J_{RR} & 0 \\ 0 & J_{EE} \end{pmatrix},$$

while $\left(n_R^{\mathrm{T}}(q, \dot{q})\; n_E^{\mathrm{T}}(q, \dot{q})\right)^{\mathrm{T}} = T_q n(\widehat{q}, \dot{\widehat{q}})$. We note that the block S_{RE} (representing the inertial coupling between each rigid joint and the elastic joints that follow in the robot kinematic chain) differs from S_{ER} (representing the inertial coupling between each elastic joint and the rigid joints that follow in the robot kinematic chain). The matrices S_{RR} and S_{EE} inherit the strictly upper triangular structure of S. As an example, consider a $5R$ robot having the first and fourth joints rigid and the remaining three elastic. After reordering, we will have the following structure for the inertia matrix:

$$M_{\text{mixed}} = \left(\begin{array}{cc|ccc||ccc} B_{11} + J_{11} & B_{14} + S_{14} & B_{12} & B_{13} & B_{15} & S_{12} & S_{13} & S_{15} \\ B_{14} + S_{14} & B_{44} + J_{44} & B_{24} + S_{24} & B_{34} + S_{34} & B_{45} & 0 & 0 & S_{45} \\ \hline B_{12} & B_{24} + S_{24} & B_{22} & B_{23} & B_{25} & 0 & S_{23} & S_{25} \\ B_{13} & B_{34} + S_{34} & B_{23} & B_{33} & B_{35} & 0 & 0 & S_{35} \\ B_{15} & B_{45} & B_{25} & B_{35} & B_{55} & 0 & 0 & 0 \\ \hline\hline S_{12} & 0 & 0 & 0 & 0 & J_{22} & 0 & 0 \\ S_{13} & 0 & S_{23} & 0 & 0 & 0 & J_{33} & 0 \\ S_{15} & S_{45} & S_{25} & S_{35} & 0 & 0 & 0 & J_{55} \end{array} \right).$$

Defining now $\widetilde{B}_{RR}(q) = B_{RR}(q) + J_{RR} + S_{RR} + S_{RR}^{\mathrm{T}}$ and $\widetilde{B}_{RE}(q) = B_{RE}(q) + S_{ER}^{\mathrm{T}}$, the dynamic equations (3) of a robot with mixed rigid/elastic joints are rewritten as:

$$\widetilde{B}_{RR}(q)\, \ddot{q}_R + \widetilde{B}_{RE}(q)\, \ddot{q}_E + S_{RE}\, \ddot{\theta} + n_R(q, \dot{q}) = \tau_R \tag{4}$$

$$\widetilde{B}_{RE}^{\mathrm{T}}(q)\, \ddot{q}_R + B_{EE}(q)\, \ddot{q}_E + S_{EE}\, \ddot{\theta} + n_E(q, \dot{q}) + K(q_E - \theta) = 0 \tag{5}$$

$$S_{RE}^{\mathrm{T}}\, \ddot{q}_R + S_{EE}^{\mathrm{T}}\, \ddot{q}_E + J_{EE}\, \ddot{\theta} + K(\theta - q_E) = \tau_E. \tag{6}$$

We perform a partial feedback linearization of eqs. (4 6), by defining two nonlinear static state feedback laws for τ_R and τ_E, in terms of new control inputs u_R and u_E, as follows:

$$\begin{aligned} \tau_R &= \tilde{B}_{RR}(q)\,u_R + \tilde{B}_{RE}(q)\,\ddot{q}_E + S_{RE}\,u_E + n_R(q,\dot{q}) & (7)\\ \tau_E &= S_{RE}^{T}\,u_R + S_{EE}^{T}\,\ddot{q}_E + J_{EE}\,u_E + K(\theta - q_E). & (8)\end{aligned}$$

Note that the formal dependence on acceleration of the control expressions (7 8) can be eliminated by substituting for $\ddot{q}_E$ solved from eq. (5). The system dynamics becomes:

$$\begin{aligned} \ddot{q}_R &= u_R & (9)\\ \tilde{B}_{RE}^{T}(q)\,u_R + B_{EE}(q)\,\ddot{q}_E + S_{EE}\,u_E + n_E(q,\dot{q}) + K(q_E - \theta) &= 0 & (10)\\ \ddot{\theta} &= u_E. & (11)\end{aligned}$$

We note that if the matrix $S_{EE} = 0$, the result in [11] would apply and a dynamic feedback law of dimension $2N_r$ would then be enough to exactly linearize system (9 11). We shall consider hereafter the (most difficult) case in which *all* upper diagonal elements of S_{EE} are different from zero.

3. DYNAMIC FEEDBACK LINEARIZATION

The design of a feedback linearizing controller can be completed by specializing the algorithm given in [9] to the mixed rigid/elastic joint case. In doing so, the state of the linearizing dynamic compensator is built. We will use the differential notation $x^{[i]} = d^i x/dt^i$.

3.1 Linearization of the generalized force f_E

Define as temporary system output the generalized force

$$f_E = \tilde{B}_{RE}^{T}(q)\,\ddot{q}_R + B_{EE}\,(q)\ddot{q}_E + n_E(q,\dot{q}) + Kq_E,$$

that collects all terms depending on q within eq. (5), which can thus be rewritten as

$$f_E + S_{EE}\,\ddot{\theta} + K\theta = 0. \qquad (12)$$

Due to the strictly upper triangular structure of matrix S_{EE}, we can solve eq. (12) for each component f_{Ei} in terms of θ_i and $\ddot{\theta}_j$, $j = i+1,\ldots,N_e$, starting from the last scalar equation and moving backwards. Repeated differentiation of eq. (12) and use eq. (11) allows to express the resulting derivatives of f_{Ei} in terms of the components of the input u_E. In order to avoid differentiation of input u_E, we need to introduce a first *dynamic extension* in terms of compensator states $\phi_{E_{i,j}}$, for $i = 2,\ldots,N_e$, $j = 1,\ldots,2(i-1)$:

$$\begin{aligned} u_{E_i} &= \phi_{E_{i,1}} \\ (\dot{u}_{E_i} =)\,\dot{\phi}_{E_{i,1}} &= \phi_{E_{i,2}} \\ &\;\vdots \\ \left(u_{E_i}^{[2i-1]} =\right)\dot{\phi}_{E_{i,2i-1}} &= \phi_{E_{i,2(i-1)}} \\ \left(u_{E_i}^{[2(i-1)]} =\right)\dot{\phi}_{E_{i,2(i-1)}} &= \overline{w}_{E_i} \end{aligned} \qquad (13)$$

with the position $u_{E_1} = \overline{w}_{E_1}$ for notational consistency and where $\overline{w}_E \in I\!R^{N_e}$ is an intermediate control input. The introduced compensator state ϕ_E will be of dimension $N_e(N_e-1)$.

It is easy to see that the following control law defined recursively for $\overline{w}_E$, using *linear* feedback from ϕ_E and a new control input $w_E \in I\!R^{N_e}$,

$$
\begin{aligned}
K_{N_e}\overline{w}_{E_{N_e}} &= w_{E_{N_e}} \\
K_{N_e-1}\overline{w}_{E_i} &= S_{EE_{N_e-1,N_e}}\overline{w}_{E_{N_e}} + w_{E_{N_e-1}} \\
&\vdots \\
K_i\overline{w}_{E_i} &= S_{EE_{i,i+1}}\overline{w}_{E_{i+1}} + \sum_{j=i+2}^{N_e} S_{EE_{i,j}}\phi_{E_{j,2i+1}} + w_{E_i},
\end{aligned}
\tag{14}
$$

for $i = N_e-2, N_e-3, \ldots, 1$, will cancel the couplings due to elements of matrix S_{EE}, leading to the linear and decoupled relations

$$
\frac{d^{2i} f_{E_i}}{dt^{2i}} = w_{E_i}, \qquad i = 1, \ldots, N_e. \tag{15}
$$

3.2 Linearization of the link coordinates q

We resume now the original output $q = (q_R, q_E)$ and complete the control design by taking into account the original system (9–11) and the relations (15). For linearizing and decoupling the dynamics of q_E, a *dynamic balancing* of the different chains of integrators introduced at the previous step is needed. Differentiating $2(N_e - i)$ times the i-th equation in (15), we obtain

$$
\frac{d^{2(N_e-i)}}{dt^{2(N_e-i)}}\frac{d^{2i} f_{E_i}}{dt^{2i}} = \frac{d^{2(N_e-i)} w_{E_i}}{dt^{2(N_e-i)}} = \frac{d^{2N_e}}{dt^{2N_e}}\left(\tilde{b}^{\mathrm{T}}_{RE_i}(q)\, u_R + b^{\mathrm{T}}_{EE_i}(q)\, \ddot{q}_E + n_{E_i}(q,\dot{q}) + K_i\, q_{E_i}\right), \tag{16}
$$

for $i = 1, \ldots, N_e$, where $b_{**_i}(q)$ is the i-th column of matrix $B_{**}(q)$.

In order to avoid differentiation of input w_E, we need to introduce a second *dynamic extension* in terms of compensator states $\psi_{E_{i,j}}$, for $i = 1, \ldots, N_e - 1$, $j = 1, \ldots, 2(N_e - i)$:

$$
\begin{aligned}
w_{E_i} &= \psi_{E_{i,1}} \\
(\dot{w}_{E_i} =)\, \dot{\psi}_{E_{i,1}} &= \psi_{E_{i,2}} \\
&\vdots \\
\left(w_{E_i}^{[2(N_e-i)-1]} =\right) \dot{\psi}_{E_{i,2(N_e-i)-1}} &= \psi_{E_{i,2(N_e-i)}} \\
\left(w_{E_i}^{[2(N_e-i)]} =\right) \dot{\psi}_{E_{i,2(N_e-i)}} &= \overline{v}_{E_i},
\end{aligned}
\tag{17}
$$

with the position $w_{E_{N_e}} = \overline{v}_{E_{N_e}}$ and where $\overline{v}_E \in I\!R^{N_e}$ is an intermediate control input. The additional compensator state ψ_E introduced will be again of dimension $N_e(N_e - 1)$.

As a result, eqs. (16) become in vector form

$$
\frac{d^{2N_e}}{dt^{2N_e}}\left(\tilde{B}^{\mathrm{T}}_{RE}(q)\, u_R + B_{EE}(q)\, \ddot{q}_E + n_E(q,\dot{q}) + K q_E\right) = \overline{v}_E \tag{18}
$$

and would involve differentiation of the input u_R to the rigid joints whenever $\tilde{B}^{\mathrm{T}}_{RE}(q) \neq 0$. Therefore, assuming that every column of this matrix contains at least a non-zero element, we define a third *dynamic extension*, i.e., add a chain of $2N_e$ integrators on each input channel u_{R_i}, for $i = 1, \ldots, N_r$, with compensator states $\zeta_{R_{i,j}}$ ($i = 1, \ldots, N_r$, $j = 1, \ldots, 2N_e$):

$$
(\ddot{q}_{R_i} =)\, u_{R_i} = \zeta_{R_{i,1}}
$$

$$
\begin{aligned}
\left(q_{R_i}^{[3]} = \dot{u}_{R_i} =\right) \dot{\zeta}_{R_{i,1}} &= \zeta_{R_{i,2}} \\
&\vdots \\
\left(q_{R_i}^{[2N_e+1]} = u_{R_i}^{[2N_e-1]} =\right) \dot{\zeta}_{R_{i,2N_e-1}} &= \zeta_{R_{i,2N_e}} \\
\left(q_{R_i}^{[2(N_e+1)]} = u_{R_i}^{[2N_e]} =\right) \dot{\zeta}_{R_{i,2N_e}} &= v_{R_i},
\end{aligned}
\tag{19}
$$

where $v_R \in I\!R^{N_r}$ is the linearizing control input on the rigid joints, i.e.,

$$
\frac{d^{2(N_e+1)} q_{R_i}}{dt^{2(N_e+1)}} = v_{R_i}, \qquad i = 1, \dots, N_r. \tag{20}
$$

The introduced compensator state ζ_R is of dimension $2N_e N_r$. Note that this is a new operation needed for the mixed rigid/elastic joint case, as opposed to the algorithm in [9].

As for the elastic joints, developing differentiation in eq. (18) and using eqs. (19), yields

$$
\widetilde{B}_{RE}^{\mathrm{T}}(q) v_R + B_{EE}(q) q_E^{[2(N_e+1)]} + \nu = \overline{v}_E \tag{21}
$$

with

$$
\nu = \binom{2N_e}{p} \sum_{p=1}^{2N_e} B_{RE}^{\mathrm{T}[p]}(q)\, \zeta_{R_{:,2N_e-p+1}} + \binom{2N_e}{p} \sum_{p=1}^{2N_e} B_{EE}^{[p]}(q)\, q_E^{[2(N_e+1)-p]} + n_E^{[2N_e]}(q,\dot{q}) + K q_E^{[2N_e]},
$$

where $\zeta_{R_{:,2N_e-p+1}}$ denotes a column vector with elements $\zeta_{R_{i,2N_e-p+1}}$, $i = 1, \dots, N_r$. The final operation is the definition of an inverse dynamics control law

$$
\overline{v}_E = \widetilde{B}_{RE}^{\mathrm{T}}(q)\, v_R + B_{EE}(q)\, v_E + \nu \tag{22}
$$

which can be rewritten in the form of a nonlinear static feedback from the *extended state* of the robot and of the dynamic compensator. Combining eqs. (21) and (22) leads to

$$
\frac{d^{2(N_e+1)} q_{E_i}}{dt^{2(N_e+1)}} = v_{E_i}, \qquad i = 1, \dots, N_e. \tag{23}
$$

3.3 Overall controller and tracking error stabilization

Summarizing the algorithmic steps in eqs. (7–8), (13–14), (17), (19), and (22), we have designed a nonlinear dynamic compensator of the form

$$
\dot{\xi} = \alpha(q,\theta,\dot{q},\dot{\theta},\xi) + \beta(q,\theta,\dot{q},\dot{\theta},\xi) v \qquad \tau = \gamma(q,\theta,\dot{q},\dot{\theta},\xi) + \delta(q,\theta,\dot{q},\dot{\theta},\xi) v \tag{24}
$$

with state $\xi = (\zeta_R^{\mathrm{T}}\ \phi_E^{\mathrm{T}}\ \psi_E^{\mathrm{T}})^{\mathrm{T}} \in I\!R^P$ and linearizing input $v = (v_R^{\mathrm{T}}\ v_E^{\mathrm{T}})^{\mathrm{T}} \in I\!R^N$, such that the closed-loop equations consist of N balanced and decoupled chains of $2(N_e+1)$ integrators each, from the inputs v_i to the outputs q_i, $i = 1, \dots, N$ –see eqs. (20) and (23). The dimension of the dynamic compensator (24) is $P = 2N_e(N_e-1) + 2N_e N_r = 2N_e(N-1)$. Note that this is only an upper bound if some columns in the strictly upper triangular part of matrix S_{EE} and/or in matrix $\widetilde{B}_{RE}^{\mathrm{T}}(q)$ turn out to be zero.

In order to obtain stable tracking of a desired trajectory $q_d(t) = (q_{Rd}^{\mathrm{T}}(t)\ q_{Ed}^{\mathrm{T}}(t))^{\mathrm{T}}$, the control design is completed on the linear and decoupled side of the problem by pole assignment, i.e., choosing

$$
v_i = q_{d_i}^{[2(N_e+1)]} + \sum_{j=0}^{2N_e+1} k_{i,j}\left(q_{d_i}^{[j]} - q_i^{[j]}\right), \qquad i = 1, \dots, N, \tag{25}
$$

where $k_{i,j}$ are coefficients of Hurwitz polynomials with desired roots in the open left-hand side of the complex plane. Therefore, the tracking error can be exponentially stabilized to zero with any desired rate of convergence. It follows from eq. (25) that, in order to obtain perfect tracking for matched initial conditions, the desired trajectory $q_d(t)$ should be at least $2(N_e + 1)$ differentiable.

4. SIMULATION RESULTS

We have considered a $2R$ robot with the first joint rigid and the second being elastic, moving in the horizontal plane. The links have thin cylindrical form and uniformly distributed masses and the robot is characterized by the following data: link lengths $\ell_i = 0.5$ [m], $i = 1, 2$; link radius $r_i = 0.05$ [m], $i = 1, 2$; link masses $m_1 = 10$, $m_2 = 5$ [kg]; center of mass positions $d_i = 0.25$ [m], $i = 1, 2$; link barycentral inertias $I_{zzi} = \frac{1}{12}m_i\ell_i^2 + \frac{1}{4}m_i r_i^2$ [Nm·s^2/rad], $i = 1, 2$; mass of second motor $m_{m2} = 0.5$ [kg]; motor inertias $J_{mi} = 0.5$ [Nm·s^2/rad], $i = 1, 2$; stiffness of second joint $K_2 = 1000$ [Nm/rad].

The dynamic model is already in the form (4–6), with the following elements:

$$
\begin{aligned}
B_{RR}(q) &= I_{zz1} + I_{zz2} + J_{m2} + m_1 d_1^2 + m_2 d_2^2 + (m_2 + m_{m2})\ell_1^2 + 2m_2 d_2 \ell_1 \cos q_2 \\
B_{RE}(q) &= I_{zz2} + m_2 d_2^2 + m_2 d_2 \ell_1 \cos q_2 \qquad B_{EE} = I_{zz2} + m_2 d_2^2 \\
J_{RR} &= J_{m1} \qquad J_{EE} = J_{m2} \qquad S_{RR} = S_{ER} = S_{EE} = 0 \qquad S_{RE} = J_{m2} \\
n_R(q, \dot{q}) &= -m_2 d_2 \ell_1 \sin q_2\, \dot{q}_2(2\dot{q}_1 + \dot{q}_2) \qquad n_E(q, \dot{q}) = m_2 d_2 \ell_1 \sin q_2\, \dot{q}_1^2
\end{aligned}
$$

The desired trajectory $q_d(t)$ is a 9-th order polynomial for each link, interpolating the initial position $q_d(0) = (0, 0)$ to the final position $q_d(T) = (\pi/2, \pi/2)$ in $T = 10$ s and with zero boundary conditions up to the fourth time derivative at both ends. The robot arm is initially at rest, but with $q(0) = (\pi/12, \pi/12)$, i.e., with an initial position error on both links. Since $S_{EE} = 0$, the dynamic feedback linearizing controller has dimension $P = 2N_r = 2$ and the resulting linear system is made of two chains of four integrators. The stabilizer in eq. (25) assigns four collocated poles in -12 to each input-output channel.

Figures 1–2 show the obtained results. The initial error is recovered within 1 s with (absolute) peak torques of about 30 Nm and 11 Nm at the two joints, occurring in the first instants of motion. Once the desired trajectory is reached, the required torques are much lower (about 1 Nm and 0.4 Nm, respectively). The deformation of the elastic joint has a maximum of $\approx 7 \cdot 10^{-3}$ rad in the initial transient, while its behavior is a scaled and rather specular copy of the torque applied at the second joint.

5. CONCLUSION

We have presented the general dynamic model of robot arms having joints of the mixed rigid/elastic type and investigated its properties of exact linearizability via nonlinear dynamic state feedback. A feedback transformation converts the problem in a form where the dynamic linearization algorithm of [9] can be applied, limited to those link position outputs driven through elastic joints. The dimension of the linearizing compensator for a robot with N joints, N_e of which are elastic, is (at most) equal to $2N_e(N-1)$. In particular, it is possible to define a dynamic control law of order $2N_rN_e$ for the N_r rigid joints of the robot and then deal with the elastic ones in a cascaded form. Using the structure of the obtained model, conditions can be derived under which the dimension of the dynamic controller reduces or even vanishes into a static state feedback, generalizing the results in [11].

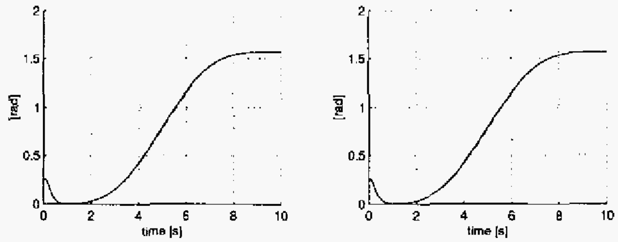

Figure 1: *Desired (- -) and actual (—) trajectories:* q_1 (left) *and* q_2 (right)

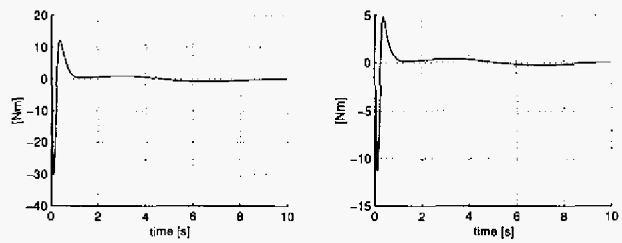

Figure 2: *Control torques:* τ_1 (left) *and* τ_2 (right)

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