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Stabilization of an Underactuated Planar 2R Manipulator

Alessandro De Luca   Raffaella Mattone   Giuseppe Oriolo*

Dipartimento di Informatica e Sistemistica
Università degli Studi di Roma "La Sapienza"
Via Eudossiana 18, 00184 Roma, Italy
{deluca,mattone,oriolo}@labrob.ing.uniroma1.it

Abstract

We describe a technique for the stabilization of a 2R robot moving in the horizontal plane with a single actuator at the base, an interesting example of underactuated mechanical system that is not smoothly stabilizable. The proposed method is based on a recently introduced iterative steering paradigm, which prescribes the repeated application of an error contracting open-loop control law. In order to compute efficiently such a law, the dynamic equations of the robot are transformed via partial feedback linearization and nilpotent approximation. Simulation and experimental results are presented for a laboratory prototype.

* Corresponding Author

1 Introduction

Underactuated robotic systems (i.e., with less control inputs than generalized coordinates) are attracting a lot of attention, consistently with the *minimalistic* trend in the field [1]. Mechanisms that can perform complex tasks with a small number of actuators are desirable in view of their reduced cost, weight and failure rate. On the other hand, innovative approaches are required in order to synthesize effective control strategies for such systems.

In general, underactuated mechanical systems may be controllable via either *kinematic* or *dynamic* coupling. Typical examples of the first class are first-order nonholonomic systems, such as wheeled mobile robots and dextrous robotic hands under pure rolling constraint (e.g., see [2] and the references therein). The equations of these systems are nonlinear and driftless when generalized velocities are considered as control inputs. As a consequence, controllability of the linear approximation is lost, and smooth time-invariant stabilization is not possible in view of a celebrated result by Brockett [3]. Use of standard feedback techniques is then ruled out; the stabilization problem for such systems has been solved using time-varying [4] and/or discontinuous feedback [5, 6, 7].

The second class includes, among others, overhead cranes [8], manipulators with flexible elements [9] and gymnast robots, e.g., the Acrobot [10]. The corresponding system equations are still nonlinear, but a drift term accounting for gravitational or elastic forces is now present. All the above systems are smoothly—in particular, linearly—stabilizable.

However, some underactuated mechanisms that are controllable via dynamic coupling inherit the limitations of kinematic nonholonomic systems, namely the lack of smooth stabilizability. This situation arises whenever the drift term tends to zero when the generalized velocities do. Examples are provided by manipulators with some passive joints in the absence of gravity [11] or redundant manipulators driven by end-effector generalized forces [12]. The aforementioned control techniques for first-order nonholonomic systems cannot be applied in these cases, essentially due to the presence of a nontrivial drift. Another hint at the intrinsic difficulty of the control problem for these mechanisms comes from the observation that they are subject to *second-order* differential constraints which are not integrable [11]. Other examples of systems of this kind can be found in [13, 14].

In this paper, we address the stabilization problem for an underactuated 2R robot moving in the horizontal plane. Control methods for this specific mechanism have been presented in [15] based on a Poincaré map analysis, and in [16] using averaging techniques. Our solution relies on the following *general* scheme: devise an open-loop control which can steer the system closer to the desired equilibrium point in finite time, and apply it in an iterative fashion (i.e., from the state attained at the end of the previous iteration). Under appropriate hypotheses, this strategy provides robust exponential convergence to the equilibrium [17]. To perform the computation of an open-loop control, we approximate the system equations by a nilpotent form [18, 19], which can be easily integrated and, at the same time, preserves the controllability properties of the original system. Nilpotent approximations have been used for nonholonomic motion planning in [20].

The paper is organized as follows. In the next section, we outline the main steps of our approach to the control of underactuated manipulators, which include a partial feedback linearization, a nilpotent approximation and an iterative stabilization procedure. In Sect. 3, we apply the proposed approach to a 2R planar robot equipped with a single actuator at the base and present simulation as well as experimental results. Possible extensions are briefly mentioned in the concluding section. For the reader's convenience, the main features of the nilpotent approximation procedure and of the iterative steering technique are summarized in two appendices.

2 The general approach

Consider a manipulator with n joints, only m of which are actuated. Denote by $q \in I\!R^n$ the joint coordinates vector, and by $\tau \in I\!R^m$ the vector of generalized forces.

2.1 Partial feedback linearization

Partition vector q as (q_a, q_b), being $q_a \in I\!R^m$ the *active* joints and $q_b \in I\!R^{n-m}$ the *passive* joints. The dynamic model of the system can be written as

$$\begin{bmatrix} B_{aa} & B_{ab} \\ B_{ab}^T & B_{bb} \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_b \end{bmatrix} + \begin{bmatrix} h_a \\ h_b \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}, \tag{1}$$

with the corresponding partitions of the $n \times n$ inertia matrix $B(q)$ and of the n-vector $h(q, \dot{q})$, which collects centrifugal, Coriolis and possibly gravitational terms. The last $n - m$ equations represent a second-order differential constraint which is satisfied by the robot during its motion. Conditions under which such constraint is non-integrable (i.e., nonholonomic) are given in [11].

Choosing the generalized forces τ as

$$\tau = \left(B_{aa} - B_{ab}B_{bb}^{-1}B_{ab}^T\right) u + h_a - B_{ab}B_{bb}^{-1}h_b, \tag{2}$$

with $u \in I\!R^m$ an auxiliary input vector, one obtains

$$\ddot{q}_a = u, \tag{3}$$

$$\ddot{q}_b = -B_{bb}^{-1}h_b - B_{bb}^{-1}B_{ab}^T u = f_b(q, \dot{q}) + G_b(q)u. \tag{4}$$

In the absence of gravity, vector h in eq. (1) is a pure quadratic form in $\dot{q}$, and the same is true for the drift term f_b in eq. (4); as a consequence, the linear approximation of system (3–4) around equilibrium points turns out to be not controllable [11]. Besides, accessibility of the system—which may be tested via the Lie algebra rank condition [21]—does not imply controllability, due to the presence of the nontrivial drift f_b. Hence, the only way to prove controllability is to apply the sufficient conditions for small-time local controllability (STLC) given in [22] and then refined in [23]. Based on these results, STLC tests for systems in the form (3–4) have been given in [12] and [24]; however, relying on sufficient conditions, such tests may not be conclusive (as in the example of Sect. 3).

2.2 Nilpotent approximation

Nilpotent approximations [18] of control systems are higher-order approximations that prove useful when linearization does not preserve the original controllability properties.

In particular, in [20] a systematic approximation procedure is proposed, which can be applied to any driftless system provided that the accessibility property is satisfied. The procedure is briefly summarized in Appendix 1. The extension to systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \qquad x \in I\!R^n, \tag{5}$$

i.e., containing a nonzero drift term $f(x)$, can be worked out in a straightforward fashion.

The procedure is based on the existence of a set of *privileged* coordinates $z = T(x)$, locally defined around any point x^0 where the system is accessible. In these coordinates, the approximation is obtained by expanding each component of the system vector fields in Taylor series and truncating it at a proper order. Thus, the approximating vector fields $\hat{f}, \hat{g}_1, \ldots, \hat{g}_m$ are polynomial. Moreover, they generate a nilpotent Lie algebra which is full rank around x^0, so that also the approximating system is locally accessible.

As the i-th component ($i = 1, \ldots, n$) of the vector fields $\hat{f}, \hat{g}_1, \ldots, \hat{g}_m$ depends at most on $z_1, \ldots, z_{i-1}$, the approximating polynomial system has the *triangular* form

$$\begin{aligned} \dot{z}_i &= \hat{f}_i + \sum_{j=1}^{m} \hat{g}_{j_i} u_j, \qquad i = 1, \ldots, \nu, & (6) \\ \dot{z}_k &= \hat{f}_k(z_1, \ldots, z_{k-1}) + \sum_{j=1}^{m} \hat{g}_{j_k}(z_1, \ldots, z_{k-1}) u_j, \quad k = \nu + 1, \ldots, n, & (7) \end{aligned}$$

being ν the dimension of span$\{f, g_1, \ldots, g_m\}$ at x^0, and $\hat{f}_i, \hat{g}_{1_i}, \ldots, \hat{g}_{m_i}$ constant values, for $i = 1, \ldots, \nu$. Eqs. (6–7) generalize eqs. (27–28) of Appendix 1 by including a drift term.

One can prove that, if the original system (5) contains a linear subsystem (e.g., eq. (3)), the latter is preserved by the approximation (6–7). This suggests to perform the partial feedback linearization of Sect. 2.1 before proceeding with the nilpotent approximation.

2.3 Stabilization

We now address the problem of finding a feedback controller (necessarily time-varying and/or discontinuous) that transfers the system from an initial point $x^0 = (q^0, \dot{q}^0) = (q_a^0, q_b^0, \dot{q}_a^0, \dot{q}_b^0)$ to a desired equilibrium $x^d = (q^d, 0) = (q_a^d, q_b^d, 0, 0)$.

Our method prescribes the execution of two phases:

I. Drive in finite time T_1 the active joint variables q_a to their desired values q_a^d. At the end of this phase it will be $q_a(T_1) = q_a^d$ and $\dot{q}_a(T_1) = 0$. Correspondingly, $q_b(T_1) = q_b^I$ and $\dot{q}_b(T_1) = \dot{q}_b^I$, being in general $q_b^I \neq q_b^d$ and $\dot{q}_b^I \neq 0$.

II. Obtain asymptotic convergence of the passive joint variables q_b to their desired values q_b^d while guaranteeing that q_a returns to q_a^d.

The first phase, referred to as *alignment*, can be performed in feedback using a standard *terminal* controller [25] for the m chains of double integrators represented by eq. (3).

For the second phase, called *contraction*, we adopt the *iterative state steering* approach [17], whose main features are summarized in Appendix 2. The basic tool is a *contracting* open-loop control, that steers the system closer to the desired equilibrium x^d

in a finite time T. If such a control can be computed, its iterated application guarantees exponential convergence to x^d, provided that T is bounded and that the open-loop control is Hölder-continuous with respect to the initial conditions (see eq. (32)). Moreover, non-persistent perturbations are rejected, while ultimate boundedness of the error is guaranteed in the presence of persistent perturbations. The resulting control is given by a time-varying law whose expression depends on a sampled feedback action.

To apply the above technique, one should compute a contracting open-loop control $u(t)$ for system (3–4). One possibility is to perform a *cyclic* motion of duration T_2 on the q_a variables (i.e., a motion such that $q_a^{II} = q_a(T_1+T_2) = q_a(T_1)$ and $\dot{q}_a^{II} = \dot{q}_a(T_1+T_2) = 0$) resulting in a final passive joints position $q_b^{II} = q_b(T_1 + T_2)$ closer to q_b^d than the initial condition q_b^I, with final velocity $\dot{q}_b^{II}$ smaller in norm than $\dot{q}_b^I$. If such a cycle can be produced by a control law u that is Hölder-continuous with respect to the initial conditions, the passive joints will converge exponentially over the iterations to their desired value q_b^d.

The search for u may be conveniently performed within a parameterized class of inputs. In some cases (e.g., when the system can be put in second-order triangular or Čaplygin form [12]), the computation of the parameters identifying u in the chosen class can be directly performed by forward integration of the passive joints equation (4). In general, however, one can resort to the nilpotent approximation (6–7) of the dynamic equations, which is polynomial and hence always integrable.

In the next section, we illustrate the above approach by designing a stabilizing controller for an underactuated 2R robot moving in the horizontal plane.

3 Case study: A planar 2R robot

Consider the planar robot of Fig. 1, having two revolute joints and a single actuator at the base. The dynamic model is

$$\begin{bmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -a_2 \sin q_2 (\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) \\ a_2 \sin q_2 \, \dot{q}_1^2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}, \tag{8}$$

with the three dynamic parameters

$$\begin{aligned} a_1 &= m_1 d_1^2 + m_2(l_1^2 + d_2^2) + I_1 + I_2, \\ a_2 &= m_2 \ell_1 d_2, \\ a_3 &= m_2 d_2^2 + I_2, \end{aligned}$$

where I_i is the baricentral inertia of link i, m_i is the mass of link i, d_i is the distance between the center of mass of link i and the joint axis i, and ℓ_1 is the length of the first link. Note that neither gravity nor friction are present at the joints.

3.1 Design of a stabilizing controller

Assume now that we wish to steer the underactuated 2R robot from $q^0 = (q_1^0, q_2^0)$ to $q^d = (q_1^d, q_2^d)$, with final zero velocity. We apply the stabilization strategy proposed in Sect. 2.3, with $q_a = q_1$ and $q_b = q_2$.

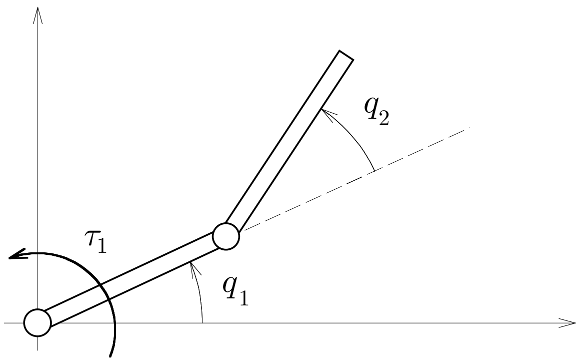

Figure 1: A 2R planar robot with a single actuator at the base

Partial feedback linearization

Defining the state vector $x = (q_1, q_2, \dot{q}_1, \dot{q}_2) \in I\!R^4$, and choosing the first joint torque in accordance with eq. (2) as

$$\tau_1 = \left(a_1 + 2a_2 \cos q_2 - \frac{(a_3 + a_2 \cos q_2)^2}{a_3} \right) u - a_2 \sin q_2 \left((\dot{q}_1 + \dot{q}_2)^2 + \frac{a_2}{a_3} \cos q_2 \, \dot{q}_1^2 \right), \quad (9)$$

with $u \in I\!R$, we obtain a partially linearized model in the form

$$\dot{x} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ 0 \\ -K \sin q_2 \dot{q}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 - K \cos q_2 \end{bmatrix} u = f(x) + g(x)u, \quad (10)$$

having set $K = a_2/a_3$.

Since the vector fields $\{g, [f,g], [g,[f,g]], [g,[g,[f,g]]]\}$ span $I\!R^4$ at any x such that $q_2 \neq k\pi/2$, $k = 0, 1 \ldots$, the system is locally accessible. However, one may verify that the sufficient conditions of [12, Prop. 2] for STLC are not satisfied.

Nilpotent approximation

In order to devise a contracting open-loop controller to be applied iteratively after the alignment phase, we need to compute the nilpotent approximation of the system at states such that $\dot{q}_1^I = 0$ and $\dot{q}_2^I \neq 0$. The nilpotent approximation technique of Appendix 1 (modified so as to account for the drift vector) has been applied to eq. (10), with the vector fields $\{f, g, [f,g], [g,[f,g]]\}$ spanning $I\!R^4$ at the points of interest. The change of coordinates $x = T^{-1}(z)$ required to transform the system in privileged coordinates is

$$\begin{aligned} q_1 &= q_1^I - z_3 \\ q_2 &= q_2^I + \dot{q}_2^I z_1 + \beta z_3 \\ \dot{q}_1 &= z_2 \\ \dot{q}_2 &= \dot{q}_2^I - \beta z_2 + \gamma z_3 - \delta z_4 + \gamma z_1 z_2, \end{aligned} \quad (11)$$

with $\beta = 1 + K \cos q_2^I$, $\gamma = K \dot{q}_2^I \sin q_2^I$, $\delta = K^2 \sin 2q_2^I$, while the nilpotent approximation (6–7) is obtained as

$$
\begin{aligned}
\dot{z}_1 &= 1 \\
\dot{z}_2 &= u \\
\dot{z}_3 &= -z_2 \\
\dot{z}_4 &= \frac{1}{2K \cos q_2^I}\, z_2^2 - \left(\frac{(\dot{q}_2^I)^2}{4K \sin q_2^I}\, z_1^2 + \frac{\beta}{2K \cos q_2^I}\, z_3 \right) u.
\end{aligned}
\tag{12}
$$

As expected, the dynamics of q_1 and $\dot{q}_1$ (which correspond to the dynamics of $-z_3$ and z_2, respectively) is exactly recovered, thanks to the partial feedback linearization. Instead, the use of the nilpotent dynamics (12), in place of the exact model (10), for computing the final value of q_2 and $\dot{q}_2$ after the application of a command $u(t)$ for a period T_2 will induce an approximation error. However, the magnitude of this error can be made arbitrarily small by reducing T_2. By enforcing sufficient contraction on the approximate system, one can guarantee that the contraction property is preserved for the original one.

Synthesis of an open-loop contracting control law

The above nilpotent approximation is now used to compute a contracting control law u. To simplify the notation, we reset time so that $t = 0$ at the start of the contraction phase.

The first requirement on u is that after one period T_2 the first joint position and velocity must go back at the values $(\dot{q}_1^I, 0)$ attained at the end of the alignment phase. In the following, we call *cyclic* this kind of open-loop control. In view of the partially linearized model (10), u must satisfy the conditions

$$
\int_0^{T_2} u(t)dt = 0 \qquad \text{and} \qquad \int_0^{T_2} \int_0^t u(\tau) d\tau dt = 0.
$$

If u is cyclic, eqs. (11) give

$$
\dot{q}_1^{II} = \dot{q}_1^I = 0 \implies z_2(T_2) = 0 \qquad \text{and} \qquad q_1^{II} = q_1^I \implies z_3(T_2) = 0.
$$

Hence,

$$
\Delta q_2 = q_2^{II} - q_2^I = \dot{q}_2^I z_1(T_2) = \dot{q}_2^I T_2, \tag{13}
$$

since $z_1(t) = t$ from the first of eqs. (12). This shows that the variation Δq_2 of the passive joint position along the cycle does not depend on the particular cyclic control used, but only on its period and on the initial velocity $\dot{q}_2^I$. As for the passive joint velocity, we have

$$
\Delta \dot{q}_2 = \dot{q}_2^{II} - \dot{q}_2^I = -\delta z_4(T_2).
$$

From the last of eqs. (12), we get

$$
z_4(T_2) = \int_0^{T_2} \frac{1}{2K \cos q_2^I} z_2^2(t) dt - \int_0^{T_2} \left(\frac{(\dot{q}_2^I)^2}{2K \sin q_2^I} z_1^2(t) + \frac{\beta}{2K \cos q_2^I} z_3(t) \right) u(t) dt.
$$

Integrating by parts we obtain

$$
\int_0^{T_2} z_1^2(t) u(t)\, dt = -2 \int_0^{T_2} z_3(t)\, dt, \qquad \int_0^{T_2} z_3(t) u(t)\, dt = \int_0^{T_2} z_2^2(t)\, dt,
$$

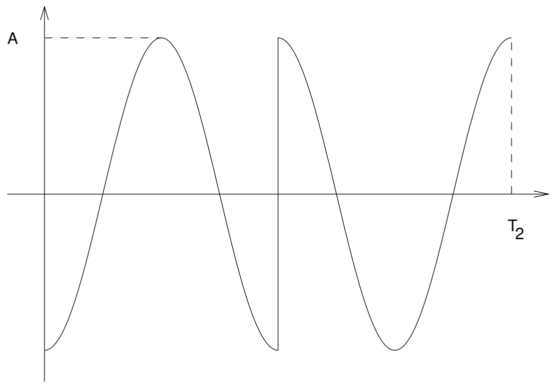

Figure 2: Profile of the cyclic open-loop control u used in the contraction phase

and finally

$$\Delta\dot{q}_2 = K^2 \sin q_2^I \cos q_2^I \int_0^{T_2} z_2^2(t)\, dt - 2K \cos q_2^I (\dot{q}_2^I)^2 \int_0^{T_2} z_3(t)\, dt. \tag{14}$$

The sign of the first term in the above expression does not depend on the choice of the specific cyclic input, but only on q_2^I, while the second term is $o((\dot{q}_2^I)^2)$.

We now adopt a specific class of cyclic control inputs as

$$u(t) = \begin{cases} -A\cos\dfrac{4\pi t}{T_2}, & t \in \left[0, \dfrac{T_2}{2}\right), \\[2ex] A\cos\dfrac{4\pi\left(t - \dfrac{T_2}{2}\right)}{T_2}, & t \in \left[\dfrac{T_2}{2}, T_2\right], \end{cases} \tag{15}$$

with duration T_2 and amplitude A (see Fig. 2). From eqs. (12) we get $\ddot{z}_3 = -u$, and thus

$$\int_0^{T_2} z_3(t)\, dt = -\int_0^{T_2}\int_0^t\int_0^\sigma u(\rho)\, d\rho\, d\sigma\, dt = 0,$$

having used eq. (15). Moreover

$$\int_0^{T_2} z_2^2(t)\, dt = \int_0^{T_2}\left(\int_0^t u(\sigma)\, d\sigma\right)^2 dt = \frac{T_2^3}{32\pi^2}A^2,$$

so that eq. (14) implies

$$\Delta\dot{q}_2 = \frac{A^2 T_2^3 K^2}{64\pi^2}\sin 2q_2^I. \tag{16}$$

This shows that, at each iteration, we can only obtain $\Delta\dot{q}_2$ of the same sign of $\sin 2q_2^I$, i.e., positive for q_2^I in the interior of the first and third quadrant and negative in the interior of the second and the fourth (see Fig. 3).

In order to meet the iterative steering paradigm, we must guarantee that the error contracts, i.e.,

$$|q_2^d - q_2^{II}| \leq \eta_1 |q_2^d - q_2^I|, \tag{17}$$

$$|\dot{q}_2^{II}| \leq \eta_2 |\dot{q}_2^I|, \tag{18}$$

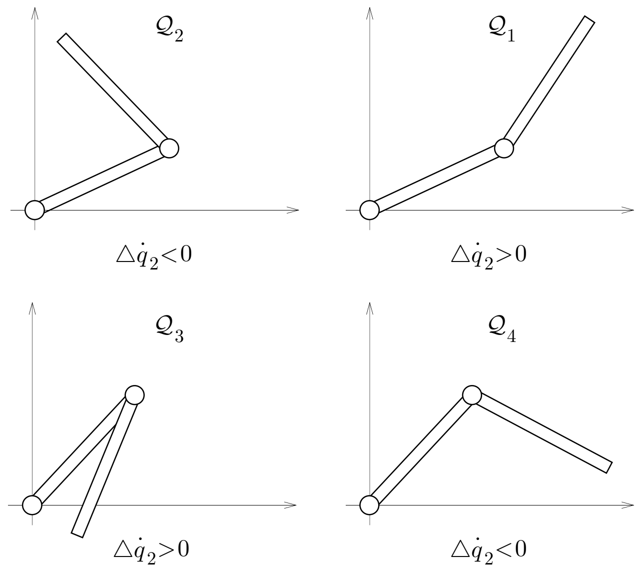

Figure 3: The sign of the achievable velocity variation $\Delta\dot{q}_2$ depends on the second link posture at the beginning of the control iteration

with η_1, $\eta_2 \in [0,1)$. In view of eqs. (13) and (16), we expect that the above conditions can be directly satisfied only in particular situations.

Assume the period and the amplitude of u in eq. (15) are chosen as

$$\begin{aligned} T_2 &= (1-\eta_1)\frac{q_2^d - q_2^I}{\dot{q}_2^I}, \qquad 0 \le \eta_1 < 1 &(19)\\ A &= \frac{8\pi}{KT_2}\sqrt{\frac{\dot{q}_2^I(\eta_2-1)}{T_2 \sin 2q_2^I}}, \qquad 0 \le \eta_2 < 1. &(20)\end{aligned}$$

Straightforward calculations give

$$\begin{aligned} q_2^d - q_2^{II} &= \eta_1(q_2^d - q_2^I), &(21)\\ \dot{q}_2^{II} &= \eta_2\dot{q}_2^I, &(22)\end{aligned}$$

i.e., the required contraction. However, for eqs. (19–20) to be well posed, we must require $T_2 > 0$, that is

$$\begin{cases} q_2^I < q_2^d \\ \dot{q}_2^I > 0 \end{cases} \quad \text{OR} \quad \begin{cases} q_2^I > q_2^d \\ \dot{q}_2^I < 0 \end{cases} \tag{23}$$

as well as the argument of the square root in eq. (20) to be positive, which implies

$$\begin{cases} q_2^I \in \mathcal{Q}_1 \text{ OR } q_2^I \in \mathcal{Q}_3 \\ \dot{q}_2^I < 0 \end{cases} \quad \text{OR} \quad \begin{cases} q_2^I \in \mathcal{Q}_2 \text{ OR } q_2^I \in \mathcal{Q}_4 \\ \dot{q}_2^I > 0 \end{cases} \tag{24}$$

Putting together conditions (23–24) one obtains the conditions under which contraction can be obtained using eqs. (19–20):

$$\begin{cases} q_2^d \in \mathcal{Q}_1 \\ q_2^I \in \mathcal{Q}_1 \\ q_2^I > q_2^d \\ \dot{q}_2^I < 0 \end{cases} \quad \text{OR} \quad \begin{cases} q_2^d \in \mathcal{Q}_2 \\ q_2^I \in \mathcal{Q}_2 \\ q_2^I < q_2^d \\ \dot{q}_2^I > 0 \end{cases} \quad \text{OR} \quad \begin{cases} q_2^d \in \mathcal{Q}_3 \\ q_2^I \in \mathcal{Q}_3 \\ q_2^I > q_2^d \\ \dot{q}_2^I < 0 \end{cases} \quad \text{OR} \quad \begin{cases} q_2^d \in \mathcal{Q}_4 \\ q_2^I \in \mathcal{Q}_4 \\ q_2^I < q_2^d \\ \dot{q}_2^I > 0 \end{cases} \tag{25}$$

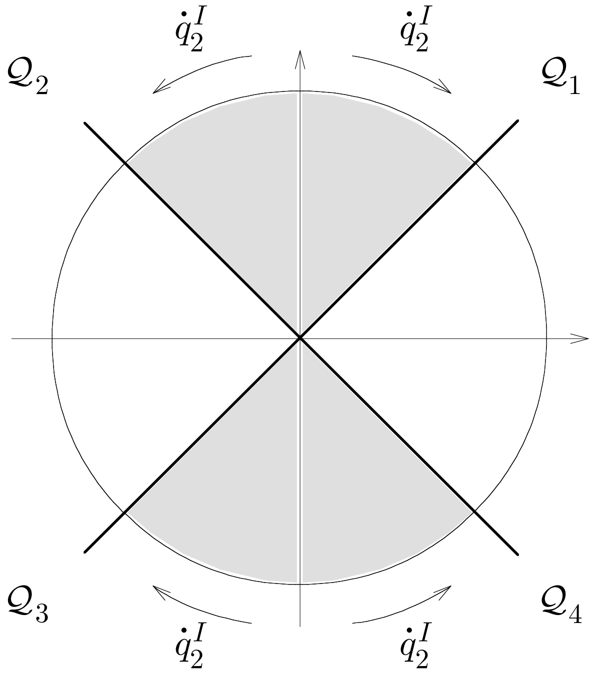

Figure 4: The conditions under which contraction is possible. For each quadrant, the bold line represents the desired q_2^d while the gray area shows the admissible initial positions q_2^I. The direction of the admissible velocities is also shown.

A compact picture of these is given in Fig. 4. In view of eqs. (21–22), which show that the position and the velocity error do not change sign, we also conclude that each of the conditions (25), once satisfied, holds continuously over the iterations. Note that the design of a contracting law is not possible if $q_2^d = \pm\pi/2$.

The requirement that the control law u is Hölder-continuous with respect to the initial state is always guaranteed under the contraction conditions (25). Moreover, boundedness of T_2 is ensured by letting $\eta_1 \leq \eta_2$, so that the fraction in eq. (19) admits a finite limit as $\dot{q}_2^I$ tends to zero. These two properties imply that the contraction phase produces exponential convergence to the desired equilibrium point $(q_2^d, 0)$.

If the conditions in eq. (25) do not hold, it is not possible to satisfy both eqs. (17–18) while approaching the desired configuration. Therefore, it is necessary to attain a modified initial condition $(q_2^I, \dot{q}_2^I)$ that satisfies eq. (25) *before* switching to the contraction phase. This *transition* phase can be always executed in finite time. For example, assume that $q_2^d \in \mathcal{Q}_1$ but at least one of the relative conditions for contraction does not hold. An admissible situation can be recovered as follows: if the initial velocity of the second joint is negative, keep it constant until q_2 enters $\mathcal{Q}_1$, else keep it constant until q_2 enters $\mathcal{Q}_2$ or $\mathcal{Q}_4$, where $\dot{q}_2$ can be made negative. Note that in order to keep $\dot{q}_2$ constant one simply sets $u = 0$ in eq. (10), resulting also in zero motion for the first joint. Similarly, one may devise simple transition phases for the other cases $q_2^d \in \mathcal{Q}_2$, $\mathcal{Q}_3$ or $\mathcal{Q}_4$. As a result, the convergence domain of the proposed control strategy can be made global.

The resulting control strategy

We summarize the overall structure of our stabilizing controller in pseudocode as follows.

```
program Stabilization
  begin
    Alignment                    /* ends with q1 = q1^d, q̇1 = 0, q2 = q2^I, q̇2 = q̇2^I */
    if Need_Transition then      /* verifies the contraction conditions */
      Transition
    Contraction                  /* computes iteratively u using eqs. (19) and (20) */
  end
```

We also give a partial coding of the procedure which implements the transition phase.

```
procedure Transition
  begin
    if q2^d ∈ Q1 then
      begin
        if q̇2^I > 0 then
          u = 0 until q2 ∈ Q2 or Q4
          apply u using eq. (13) so as to obtain q̇2^II < 0     /* regardless of q2^II */
        u = 0 until q2 ∈ Q1                                    /* now with negative q̇2 */
      end
    ⋮                                    /* similar maneuvers for q2^d ∈ Q2, Q3 or Q4 */
  end
```

3.2 Simulation results

To illustrate the performance of the proposed method, we present first a simulation for the partially linearized model (10) of the 2R robot with $K = 0.5$. We assume that, at the end of the alignment phase, it is $q_1^I = q_1^d = 0$, $\dot{q}_1^I = 0°/\text{s}$, $q_2^I = 22.5°$ and $\dot{q}_2^I = 13.2°/\text{s}$. The desired configuration of the passive joint is $q_2^d = 45°$.

Being $q_2^d \in \mathcal{Q}_1$ but $\dot{q}_2^I > 0$, the control strategy of Sect. 3.1 prescribes the execution of a transition phase, in which $\dot{q}_2$ is first kept constant until q_2 enters $\mathcal{Q}_2$, where $\dot{q}_2$ can be made negative. When q_2 returns in $\mathcal{Q}_1$, the contraction phase takes over.

While the control amplitude A is computed by eq. (20), for ease of implementation we used a constant T_2 during the contraction phase. By doing so, it is not possible to choose arbitrarily the position contraction rate, for η_1 will depend on $\dot{q}_2^I$ only (see eq. (13)). However, applying iteratively eqs. (13) and (22) (which is still valid), one can easily verify that, if a sufficiently small T_2 is used and the velocity contraction rate η_2 is chosen as

$$\eta_2 = 1 + \frac{T_2\, \dot{q}_2(0)}{q_2(0) - q_2^d} < 1,$$

one gets $\eta_1 = \eta_2 < 1$ constant over the iterations. Here, $q_2(0)$ and $\dot{q}_2(0)$ are respectively the second joint position and velocity at the beginning of the first iteration of the contraction phase. In particular, we could use $T_2 = 1$ s as an admissible value in our simulation.

The time history of the joint position errors $q_i - q_i^d$, $(i = 1, 2)$, during transition and contraction is reported in Fig. 5. Note the constant velocity of the second joint during the first part of the transition phase and the exponential convergence rate during the contraction phase. The long time needed to complete the reconfiguration is due to the fact that motion of the passive joint is not damped by friction in the simulated model.

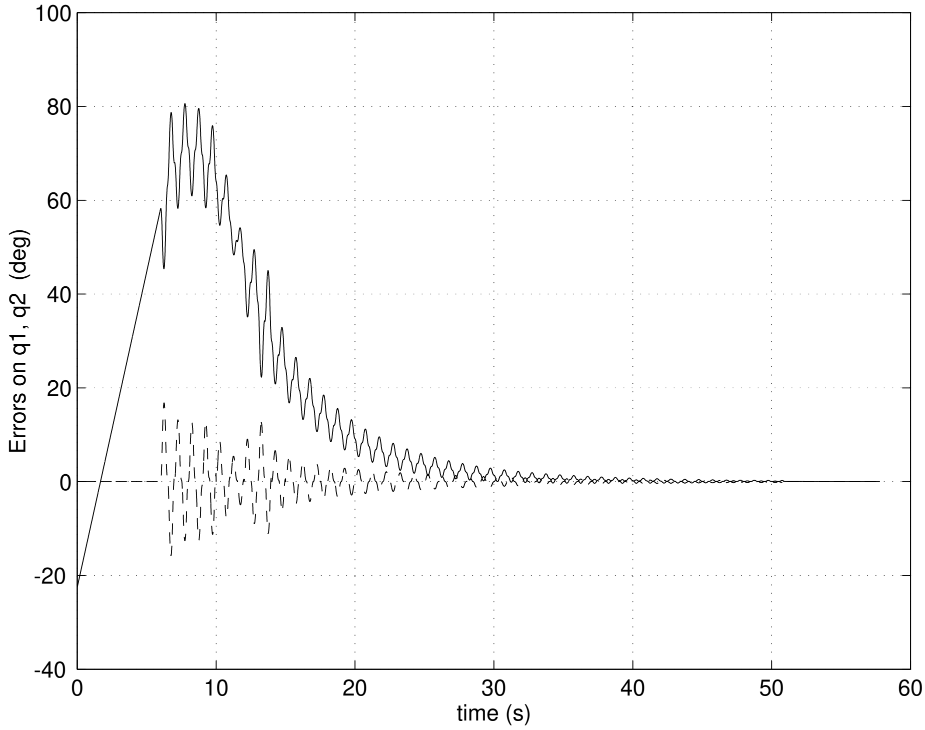

Figure 5: Simulation results: Errors on q_1 (dashed) and q_2 (solid)

3.3 Experimental results

We have applied the proposed stabilization method to the FLEXARM, a lightweight 2R planar manipulator available in our laboratory (see [26] for a description of the robot). The second link, which is flexible, has been stiffened for our purposes by appropriately bonding the forearm, and its driving motor has been switched off. As a nominal model, we have used eq. (8) with $a_1 = 0.867$, $a_2 = 0.195$, and $a_3 = 0.42$ (all in kg·m^2). Using the linearizing feedback (9), a model in the form (10) is obtained, with $K = 0.4643$.

The accuracy of the nominal model is quite poor, due to unmodeled dynamics such as dry and viscous friction on both joints, the second link residual elasticity, and the presence of a bound on the first joint torque (to avoid saturation of the actuator). Besides, no direct measure is available for the joint velocity, which is reconstructed by numeric filtering.

Before proceeding with the experiment, we have simulated the control of the nominal model. The arm is required to move from $q_1^0 = 74°$, $q_2^0 = 91°$ to $q_1^d = 0°$, $q_2^d = 45°$. The result is shown in Fig. 6. The alignment phase (performed with a simple PD control law on the first joint position, see the remark below) lasts approximately 2.5 s, after which the contraction phase is started. Note that no transition phase is needed in this case and that a constant $T_2 = 1$ has been used for contraction (as in the simulation of Sect. 3.2).

When implementing the method on our experimental setup, we introduced some modifications to the basic strategy:

- To avoid chattering, the alignment phase was performed by a PD control law on the first joint position (with gains K_P^I and K_D^I). Although the convergence is only asymptotic, any desired error tolerance can be met in finite time.

- During the contraction phase, in view of the model inaccuracy, the first link was controlled via high-gain PD feedback on the second joint position (with gains K_P^{II} and K_D^{II}) in place of the partially linearizing feedback (9). The position reference signal is obtained by integrating twice the acceleration profile (15).

- Due to the system perturbations, the first joint may not perform exactly a cyclic motion during the iterations of the contraction phase—a small displacement may occur. To prevent the first link to drift away from its desired position, each iteration was actually implemented as a re-alignment phase followed by a contraction phase.

Figures 7–8 show the results of an experiment with the same initial and final desired conditions of Fig. 6. During each alignment phase, the PD control law on the first joint position used the gains $K_P^I = 20$ and $K_D^I = 0.3$. Instead, we have set $K_P^{II} = 70$ and $K_D^{II} = 2$ for the contraction phase, whose period is again chosen as $T_2 = 1$ s.

Joint errors and the first joint torque τ_1 are reported respectively in Fig. 7 and Fig. 8. For the sake of clarity, each contraction phase is marked in bold on the time axis. A comparison with Fig. 6 shows that, due to the presence of friction, stabilization of the robot is obtained in a smaller time. For the same reason, the amplitude of the oscillations in the contraction phase is reduced. Note also that τ_1 saturates during the first alignment phase. In spite of all the unmodeled effects, the results support the claim of a satisfactory robustness of the proposed control strategy, a byproduct of the iterative steering approach.

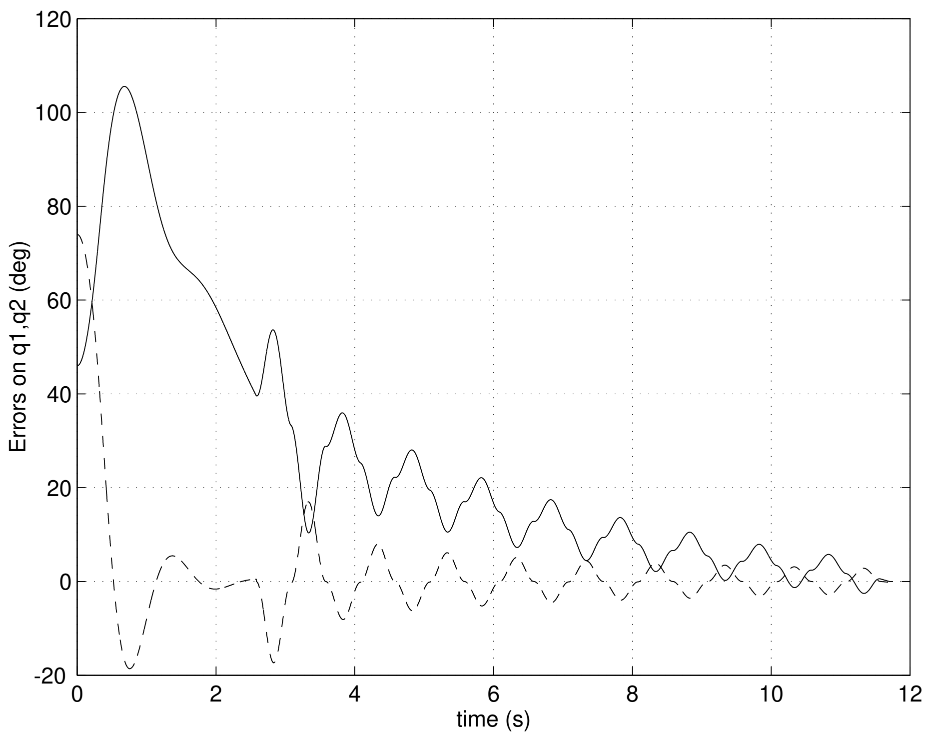

Figure 6: Simulation results on the nominal model of the FLEXARM: Errors on q_1 (dashed) and q_2 (solid)

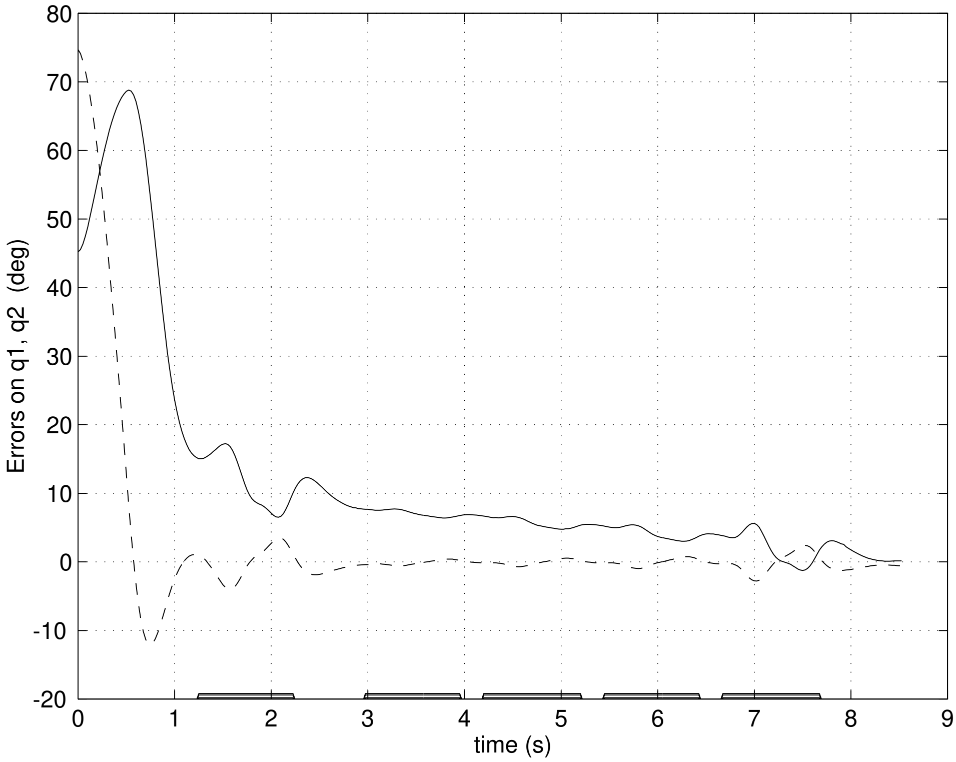

Figure 7: Experimental results on the FLEXARM: Errors on q_1 (dashed) and q_2 (solid)

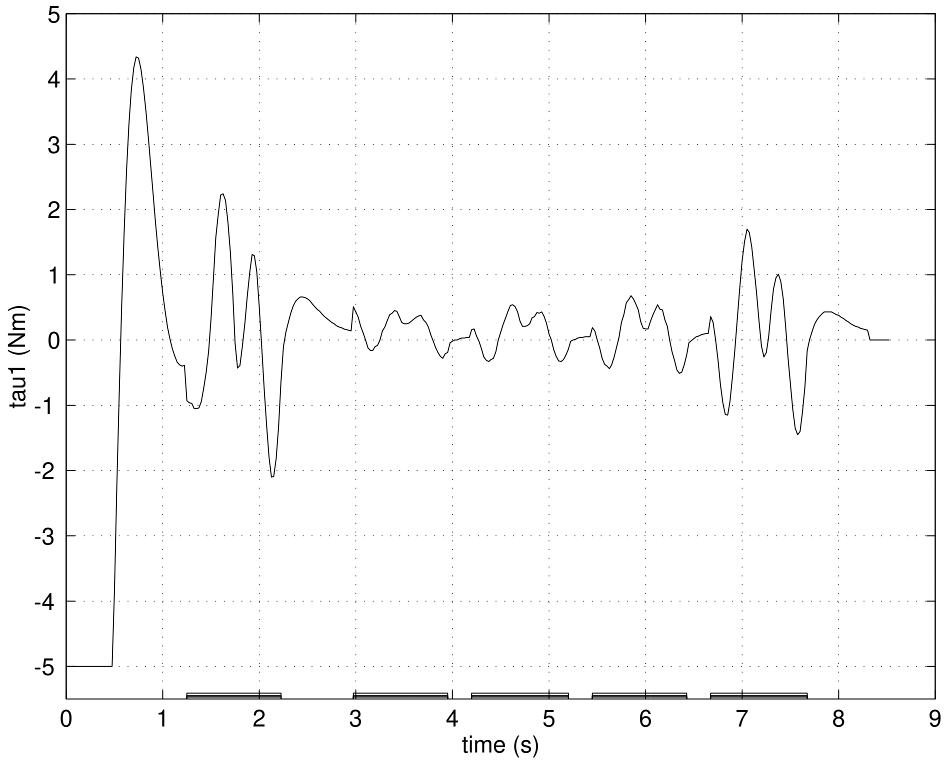

Figure 8: Experimental results on the FLEXARM: Joint torque τ_1

4 Conclusions

We have presented a solution method for the stabilization of underactuated manipulators. Such systems are not smoothly stabilizable in the absence of gravity. Moreover, the presence of a drift term in the dynamic equations complicates remarkably the control synthesis. The stabilization strategy consists of three phases, namely *(i)* alignment, in which the active joints are brought to their desired position, *(ii)* transition, where simple maneuvers are executed to obtain the correct initial condition for *(iii)* contraction, based on the iterative application of a suitable open-loop control designed on a nilpotent approximation of the system.

The proposed approach has been illustrated with reference to a planar 2R robot with a single actuator at the base. The presented simulation and experimental results show the satisfactory performance of the method. In principle, the underlying general approach can be applied to most underactuated mechanisms of interest in robotic applications. However, the application to specific systems or classes of systems must be worked out on a case-by-case basis, and may prove difficult or even impossible for higher degrees of underactuation. The critical point is the closed-form evaluation of parameter values in the chosen parameterized class of open-loop controls that guarantee the contraction conditions for the state error.

Indeed, we have successfully applied the same iterative steering approach for the stabilization of the Acrobot [27], as well as for the robust stabilization of a rigid spacecraft with two control torques [28]. Moreover, the use of nilpotent approximations in conjunction with iterative steering has given encouraging preliminary results also for the control of non-nilpotentizable driftless systems, such as the car with off-hooked trailers [29].

Acknowledgments

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Appendix 1: Nilpotent approximation procedure

We briefly review here the nilpotent approximation procedure of [19], to which the reader is referred for details. The extension to systems with drift is straightforward.

Consider a driftless system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i, \qquad x \in I\!R^n, \tag{26}$$

satisfying the Lie algebra rank condition almost everywhere. Denote by $(g_{i_1} \dots g_{i_{s-1}})f$ the Lie derivative of order $s-1$ of f along $g_{i_1} \dots g_{i_{s-1}}$, and by $L^s(x^0)$ the space spanned at x^0 by the brackets of $g_1, \dots, g_m$ of length $\leq s$. The smallest integer $r(x^0)$ such that $L^{r(x^0)}(x^0)$ spans the tangent space of system (26) at x^0 is called the *degree of nonholonomy* at x^0.

Definition 1 *A function $f \in I\!R$ is of* order $\geq s$ *at x^0 if the function $(g_{i_1} \dots g_{i_q})f$ vanishes at x^0, for any $i_1, \dots, i_q$ and $q \leq s-1$.*

Definition 2 *A vector field g is of* order $\geq q$ *at x^0 if, for every s and every function f having order s at x^0, the Lie derivative gf has order $\geq q+s$ at x^0.*

Definition 3 *The* weight w_j *of a coordinate y_j is the smallest integer s such that dy_j is not identically zero on $L^s(x^0)$ ($s = 1, \ldots, r$).*

Definition 4 *Local coordinates $z_1, \ldots, z_n$ centered at x^0 form a system of* privileged coordinates *if the order of z_j at x^0 is equal to w_j ($j = 1, \ldots, n$).*

Let $\gamma_1(x^0), \ldots, \gamma_n(x^0)$ be a basis of $L^{r(x^0)}(x^0)$. Through a linear change of coordinates, it is always possible to find a system of coordinates $y_1, \ldots, y_n$ centered at x^0 such that $(\gamma_i y_j)(x^0) = \delta_{ij}$, with δ_{ij} the Kronecker delta. From this, a system of privileged coordinates $z_1, \ldots, z_n$ around x^0 is obtained by the recursive formula

$$z_q = y_q - \sum_{\{\alpha : w(\alpha) < w(q)\}} \frac{1}{\alpha_1! \ldots \alpha_{q-1}!} (\gamma_1^{\alpha_1} \ldots \gamma_{q-1}^{\alpha_{q-1}} y_q)(x^0) z_1^{\alpha_1} \ldots z_{q-1}^{\alpha_{q-1}},$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $w(\alpha) = \sum_1^n w_i \alpha_i$. Coordinates $z_1, \ldots, z_n$ have order $w_1, \ldots, w_n$ by construction. With the system in privileged coordinates, the order of a smooth function f at x^0 is the least weighted-degree monomial actually appearing in the Taylor expansion $f(z) = \sum_\alpha a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ of f at x^0. The order of a vector field g can be computed in the same algebraic way as above, i.e., using the Taylor expansion $g(z) = \sum_{\alpha,j} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$, assigning to ∂_{z_j} the weight $-w_j$ and considering terms like $a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ as products;

In privileged coordinates, each g_i can be expanded in terms of vector fields homogeneous with respect to the weighted degree as $g_i = g_i^{(-1)} + g_i^{(0)} + g_i^{(1)} + \ldots$. The nilpotent approximation of system (26) is derived by replacing the vector fields g_i with their principal component $g_i^{(-1)}$. One obtains the triangular form

$$\dot{z}_i = \sum_{j=1}^m \hat{g}_{ji} u_j, \qquad i = 1, \ldots, \nu, \tag{27}$$

$$\dot{z}_k = \sum_{j=1}^m \hat{g}_{jk}(z_1, \ldots, z_{k-1}) u_j, \qquad k = \nu+1, \ldots, n. \tag{28}$$

being ν the dimension of $\mathrm{span}\{g_1, \ldots, g_m\}$ at x^0, $\hat{g}_{1i}, \ldots, \hat{g}_{mi}$ constants for $i = 1, \ldots, \nu$, and $\hat{g}_{jk}(z_1, \ldots, z_{k-1})$ polynomial functions of homogeneous degree $w_k - 1$ for $k = \nu+1, \ldots, n$.

Appendix 2: Stabilization via iterative steering

In this section, the stabilization technique based on iterative state steering is summarized (see [17] for details and proofs).

Consider the control system

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x) u_i, \qquad x \in I\!R^n. \tag{29}$$

Without loss of generality, we assume that $f(0,0) = 0$, i.e., the origin is an equilibrium.

Consider a sequence of time instants $\{t_k\}$ $(k = 0, 1, 2, \ldots)$ with $t_{k+1} = t_k + T_{k+1}$, and $0 < T_m \leq T_{k+1} \leq T_M < \infty$. For compactness, let $x(t_k) = x_k$. On each time interval $I_{k+1} = [t_k, t_{k+1}]$, apply the steering control law

$$u(t) = u_{k+1}(t) = \alpha(x, x_k, t), \qquad t \in I_{k+1}. \tag{30}$$

Let

$$\dot{x} = f(x, \alpha(x, x_k, t)) = \tilde{f}(x, x_k, t), \qquad t \in I_{k+1}, \tag{31}$$

be the closed-loop dynamics of system (29) within the $(k+1)$-th interval I_{k+1}.

Assumption A *The steering control function α is such that:*

a) $\alpha(x, 0, t) = 0$, for any $(x, t) \in I\!\!R^n \times I_{k+1}$;

b) $\tilde{f}$ is locally lipschitz in x, continuous in x_k and piecewise-continuous in t, for $t \in I_{k+1}$;

c) $|x_{k+1}| \leq \eta\, |x_k|, \quad \eta < 1, \quad \forall x_k$ (contraction).

The requirement that $\tilde{f}$ (and hence, the steering control α) is continuous in x_k is essential for the proposed stabilization strategy.

Theorem 1 *Under Assumption A, for the controlled system (31):*

1. *The origin is a uniformly asymptotically stable equilibrium point.*
2. *If the additional condition holds*

$$|\tilde{f}(0, x_k, t)| \leq \mu |x_k|^r, \qquad t \in I_{k+1},\ \mu \geq 0,\ r > 0, \tag{32}$$

then the rate of convergence is exponential. In particular, if $r \geq 1$, then the origin is an exponentially stable equilibrium point.

In particular, the convergence rate is $r|\log\beta|/T_M$ if $r < 1$ or $|\log\beta|/T_M$ if $r \geq 1$. Condition (32) is referred to as *Hölder-continuity* of order r at the origin.

For a characterization of the robustness of the iterative steering approach, see [17].

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