Nonholonomic Behavior in Redundant Robots Under Kinematic Control

Alessandro De Luca and Giuseppe Oriolo

***Abstract*— We analyze the behavior of redundant robots when the joint motion is generated by inverting task velocity commands through a kinematic control scheme. Depending on the chosen inversion scheme, the robot motion is subject to differential constraints that may or may not be integrable. Accordingly, we give a classification in terms of holonomic, partially nonholonomic, and completely nonholonomic behavior, pointing out also the relationship with the so-called cyclicity property. This general classification is illustrated by means of several examples. When the kinematic control scheme is nonholonomic, the whole configuration space of the robot is accessible by a proper choice of the task input commands. Under this assumption, we address the joint reconfiguration problem, namely the design of end-effector velocity commands that drive the robot to a desired joint configuration. To solve this problem, it is possible to borrow existing methods for motion planning of nonholonomic mechanical systems, such as the sinusoidal steering technique for chained-form systems.**

***Index Terms*—Cyclicity, kinematic inversion, nonholonomy, redundant robots.**

I. Introduction

Consider a kinematically redundant robot having n joints and m task coordinates, with $m < n$. Typically, the task coordinates are position and orientation of the robot end-effector. Denoting by $q \in \mathbb{R}^n$ the joint vector and by $p \in \mathbb{R}^m$ the task vector, the differential kinematics is expressed as

$$\begin{aligned}\dot{p} &= \frac{\partial k(q)}{\partial q}\dot{q}\\ &= J(q)\dot{q}\end{aligned}$$

with $k(q)$ the direct kinematic map and $J(q)$ the corresponding analytic Jacobian.

A standard way to generate joint motion at the velocity level is by choosing

$$\dot{q} = G(q)u \tag{1}$$

in which G is any *generalized inverse* [1] of the Jacobian J, and $u \in \mathbb{R}^m$ is the task velocity command. This scheme is often referred to as *kinematic control*. A generalized inverse satisfies $JGJ = J$, implying that any feasible command vector u is exactly realized. A common choice is $G = J^{\#}$, the unique pseudoinverse of J.

In this paper, the following problem is addressed: *When* and *how* is it possible to find a task velocity command $u(t)$ (that is, a trajectory for the task coordinates) such that, under the kinematic control scheme (1), the robot is driven from the initial joint configuration q_0 to a desired joint configuration q_d?

This problem is nontrivial because the kinematic control scheme (1) represents an *underactuated* nonlinear system, i.e., with less inputs than generalized coordinates ($m < n$). The existence of a task input u driving the arm from q_0 to q_d is guaranteed if the whole state space of the driftless control system (1) is *accessible*. This property is strongly connected with the nonholonomic behavior of the robot under the kinematic control scheme. In fact, the motion of the underactuated system (1) is implicitly subject to $n - m$ differential constraints

$$A^T(q)\dot{q} = 0. \tag{2}$$

Constraints (2) are obtained by choosing $A^T(q)$ as any matrix satisfying

$$\mathcal{N}[A^T(q)] = \mathcal{R}[G(q)]$$

or

$$\mathcal{R}[A(q)] = \mathcal{N}[G^T(q)]$$

where $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ are the null and the range space of a matrix, respectively. When the differential constraints (2) are not integrable, system (1) is *nonholonomic* [2].

This viewpoint is particularly convenient for the analysis of kinematic control schemes as well as for the synthesis of reconfiguration methods. If all (or just a subset) of the constraints (2) are integrable, it is not possible to steer the robot from q_0 to an *arbitrary* configuration q_d. Conversely, if the constraint set (2) is not integrable, a solution to our control problem exists and can be found by borrowing techniques from nonholonomic motion planning (see, e.g., [3], [4]). In this framework, an interesting relationship can be established between the integrability of the constraint set (2) and the cyclicity [5] of the corresponding kinematic control scheme.

There are various reasons for addressing the joint reconfiguration problem via task velocity commands.

- Definition of commands directly at the task level is particularly convenient for redundant robots, because it does not require to bypass the built-in inversion scheme G. In this way, the reconfiguration task can be executed with the same software module used for tracking end-effector trajectories.
- Assume that a pick-and place operation must be performed with a redundant robot controlled through the kinematic scheme (1). One may ask whether it is possible to design particular forward and backward *cartesian* motions that drive the robot from the "pick" configuration to the "place" configuration and vice versa. With our approach, this task is split into two separate and symmetric joint reconfiguration problems. In this way, we are able to obtain a closed path in the joint space, even when the scheme (1) is not *cyclic*.
- Redundant robotic systems include also the case of multiple arms carrying an object, like in dextrous grasping with a multifingered hand. With the proper notation, the fundamental grasp constraint for such systems takes the form (1) (see [2, p. 237]). One may be interested in reconfiguring the system for achieving, e.g., maximum manipulability without losing the object grasp. A convenient solution is to plan a suitable cartesian motion of the commonly held object, by choosing the task velocity command $u(t)$ in the kinematic control scheme (1) so as to achieve the desired reconfiguration for the system.

The paper is organized as follows. Basic tools for analyzing nonholonomic systems are briefly recalled in Section II. In Section III, the nonholonomic behavior of various kinematic control schemes is investigated through several examples. The reconfiguration problem is addressed in Section IV, where a method based on sinusoidal steering is applied to a PPR robot. Possible extensions are outlined in the concluding section.

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The authors are with the Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza," 00184 Rome, Italy (e-mail: adeluca@giannutri.caspur.it; oriolo@giannutri.caspur.it).

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II. Nonholonomic Systems Analysis

The configuration of a mechanical system can be described by a vector of *generalized coordinates* $q \in \mathcal{Q}$, where the configuration space $\mathcal{Q}$ is an n-dimensional smooth manifold, locally diffeomorphic to $\mathbb{R}^n$. Suppose that k linearly independent constraints in the *Pfaffian* form

$$a_i^T(q)\dot{q} = 0 \qquad i = 1, \cdots, k < n$$

or

$$A^T(q)\dot{q} = 0 \tag{3}$$

are imposed on the *generalized velocities* $\dot{q}$.

The set of constraints (3) is *holonomic* if it can be integrated to the form

$$h_i(q) = c_i, \qquad i = 1, \cdots, k \tag{4}$$

where the constants $c_1, \cdots, c_k$ depend on the initial condition $q(0) = q_0$. These constraints confine the attainable system configurations to an $(n - k)$-dimensional submanifold of $\mathcal{Q}$, so that k variables can be eliminated from the problem.

When (3) cannot be integrated to the form (4), the constraint set is called *nonholonomic*. This includes the case in which only a *subset* of $p < k$ independent linear combinations of the constraints (3) are integrable, restricting the system configurations to the $(n - p)$-dimensional level surface (*leaf*) where motion is started

$$\{q \in \mathcal{Q}\colon\ h_1(q) = c_1, \cdots, h_p(q) = c_p\}$$

and allowing to eliminate $p < k$ variables. We refer to this situation as *partial* nonholonomy, as opposed to *complete* or *maximal* nonholonomy ($p = 0$).

Although limiting the *instantaneous* mobility of the system, nonholonomic constraints do not imply a loss of global accessibility. If vector fields $\{g_1(q), \cdots, g_{n-k}(q)\}$ are a basis for the $(n - k)$-dimensional null space of matrix $A^T(q)$, all feasible trajectories are obtained as solutions of the nonlinear driftless control system

$$\begin{aligned}\dot{q} &= \sum_{j=1}^{m} g_j(q)u_j \\ &= G(q)u, \qquad m = n - k\end{aligned} \tag{5}$$

with state vector $q \in \mathcal{Q}$ (locally, $q \in \mathbb{R}^n$) and control input $u \in \mathbb{R}^m$. The kinematic control scheme (1) for redundant robots has exactly this form. The complete nonholonomy of the constraint set (3) guarantees that, by a proper choice of u, the *whole* configuration space $\mathcal{Q}$ will be accessible.

To decide about the integrability of a set of kinematic constraints, it is convenient to exploit the tools available from nonlinear control theory. The reader is referred to [6] for the basic concepts of *distribution* and of *Lie bracket* of vector fields. The *accessibility distribution* $\Delta_{\mathcal{A}}$ of system (5) is defined as the closure of the distribution $\Delta = \text{span}\{g_1, \cdots, g_m\}$ under (repeated) Lie bracket operation. Note that $\dim \Delta = m$ by construction. Using Frobenius theorem [6, p. 23], it can be shown that the constraint set (3) is

Holonomic if $\dim \Delta_{\mathcal{A}} = m$.

Nonholonomic if $\dim \Delta_{\mathcal{A}} > m$. In particular, it is

- *Partially nonholonomic* if $\dim \Delta_{\mathcal{A}} = n - p > m$, for $p > 0$. In this case, there exists a subset of p integrable constraints.
- *Completely nonholonomic* if $\dim \Delta_{\mathcal{A}} = n$.

III. Nonholonomy in Redundant Robots

Nonholonomic constraints may arise in redundant robots depending on the particular way in which joint motion is generated, i.e., on the choice of the kinematic control scheme. For example, if the Jacobian pseudoinverse is used in (1), we have

$$\begin{aligned}G(q) &= J^{\#}(q) \\ &= J^T(q)[J(q)J^T(q)]^{-1}\end{aligned}$$

away from kinematic singularities, where the Jacobian loses rank. Hence, it is

$$[I - J^{\#}(q)J(q)]\dot{q} = 0 \tag{6}$$

where $I - J^{\#}J$ is the $n \times n$ orthogonal projection matrix in $\mathcal{N}(J)$, with rank $n - m$. Extracting from the matrix in (6) the maximum number of linearly independent rows, we obtain the Pfaffian constraints (3) as

$$\begin{bmatrix} n_{J,1}^T(q) \\ \vdots \\ n_{J,n-m}^T(q) \end{bmatrix} \dot{q} = 0 \tag{7}$$

where the vectors $\{n_{J,1}, \cdots, n_{J,n-m}\}$ are a basis for the null space of J.

More in general, one can write explicitly the constraints associated to *any* generalized inverse G. For illustration, consider the case $n - m = 1$. All the generalized inverses of a full rank Jacobian J are obtained by varying $r \in \mathbb{R}^{n-1}$ in the expression

$$G(q) = G_1(q) + n_J(q)r^T(q) \tag{8}$$

where G_1 is a particular generalized inverse of J and vector n_J spans $\mathcal{N}(J)$. Using (8), it can be shown that a vector β spanning $\mathcal{N}(G^T)$ is

$$\beta(q) = \{J^T(q)[r(q)n_J^T(q) + G_1^T(q)] - I\}n_J(q),$$

and the underlying Pfaffian constraint is

$$\beta^T(q)\dot{q} = 0. \tag{9}$$

More in general, away from singularities, the robot motion is subject to $k = n - m$ differential constraints (as many as the degree of redundancy), that may or may not be nonholonomic. In the remainder of this section, we present examples of redundant robots under various kinematic control schemes, in which holonomic, partially nonholonomic, or completely nonholonomic behaviors result.

Before, we shall recognize a connection between the holonomy/nonholonomy of the differential constraints (2) and the well-known *cyclicity* property [5], [7], [8]. A given generalized inverse G in (1) yields a *cyclic* inversion scheme if *any* closed task trajectory is mapped to a closed motion in the joint space. As shown in [5], this property is achieved if and only if the *involutivity* condition on the Lie brackets $[g_i, g_j]$ holds

$$[g_i, g_j](q) \in \text{span}\{g_1(q), \cdots, g_m(q)\} \quad \forall i, j, \forall q. \tag{10}$$

The cyclicity condition (10) is equivalent to $\dim \Delta_{\mathcal{A}} = \dim \Delta = m$, i.e., to the kinematic control scheme being holonomic. Conversely, cyclicity is lost whenever the scheme is nonholonomic. Note that, when $n - m = 1$, (10) is violated if and only if $\dim \Delta_{\mathcal{A}} = n$, i.e., in the case of complete nonholonomy. On the other hand, when $n - m > 1$, the violation of (10) becomes only a necessary but not sufficient condition for the accessibility of the whole configuration space.

Cyclicity is a strong property, for it implies that *any* initial joint configuration is recovered after *any* end-effector cycle. However, even if a kinematic inversion scheme is *not* cyclic, there are still two special situations that may occur:

- There may exist a subset $\mathcal{Q}_r \subset \mathcal{Q}$ of *repeatable* joint configurations that repeat themselves after *any* end-effector cycle. This happens when (10) holds for $q \in \mathcal{Q}_r$ and $\mathcal{Q}_r$ is invariant under the kinematic control scheme.
- For some initial joint configurations, it may be possible to find *particular* end-effector cycles leading to their repetition, referred to as *holonomic cycles*. To find a nontrivial holonomic cycle, one can solve two sequential instances of the reconfiguration problem, as will be discussed in Section IV-A.

In conclusion, while the lack of cyclicity is a drawback when tracking a closed end-effector path, it is a prerequisite for the solvability of the reconfiguration problem.

A. Examples of Holonomic Kinematic Control Schemes

We shall now present two examples of kinematic control schemes that are holonomic. For the first robot, we will give the structure of the whole class of generalized inverses yielding a holonomic behavior. The second example is a robot for which the pseudoinverse represents a cyclic inversion technique.

1) Generalized Inversion for a PPR Planar Robot: Consider the PPR planar robot shown in Fig. 1, having one revolute and two prismatic joints. This robot is redundant for the task of positioning the end-effector in the plane with unspecified orientation ($n = 3$, $m = 2$). Denoting by ℓ the length of the third link and letting $s_3 = \sin q_3$ and $c_3 = \cos q_3$, the differential kinematics is

$$\dot{p} = J(q)\dot{q} = \begin{bmatrix} 1 & 0 & -\ell s_3 \\ 0 & 1 & \ell c_3 \end{bmatrix} \dot{q}$$

and the Jacobian is always of full row rank.

Given a positive-definite matrix $W = \operatorname{diag}\{1, 1, w^2\}$, where w is a constant with the same units of ℓ, all the generalized inverses $G = [g_1 \; g_2]$ of the Jacobian can be expressed as in (8), with

$$\begin{aligned} G_1(q) &= J_W^{\#}(q) \\ &= W^{-1}J^T(q)[J(q)W^{-1}J^T(q)]^{-1} \\ &= \frac{1}{1+\left(\frac{\ell}{w}\right)^2} \begin{bmatrix} 1+\left(\frac{\ell}{w}\right)^2 c_3^2 & \left(\frac{\ell}{w}\right)^2 s_3 c_3 \\ \left(\frac{\ell}{w}\right)^2 s_3 c_3 & 1+\left(\frac{\ell}{w}\right)^2 s_3^2 \\ -\frac{\ell}{w^2} s_3 & \frac{\ell}{w^2} c_3 \end{bmatrix} \end{aligned} \tag{11}$$

and

$$n_J(q) = \begin{bmatrix} \ell s_3 \\ -\ell c_3 \\ 1 \end{bmatrix}, \qquad r(q) \in \mathbb{R}^2.$$

The W*-weighted* pseudoinverse $J_W^{\#}$ is used for achieving dimensional homogeneity.

All holonomic generalized inverses are obtained by imposing $\dim \Delta_{\mathcal{A}} = m = 2$, that is

$$\det\,[g_1(r) \quad g_2(r) \quad [g_1(r), g_2(r)]] = 0.$$

Assuming $r = r(q_3)$, we obtain the condition

$$\ell^2 + \ell(c_3 r_2 - s_3 r_1) - (\ell c_3 + r_2)\frac{\partial r_1}{\partial q_3} - (\ell s_3 - r_1)\frac{\partial r_2}{\partial q_3} = 0.$$

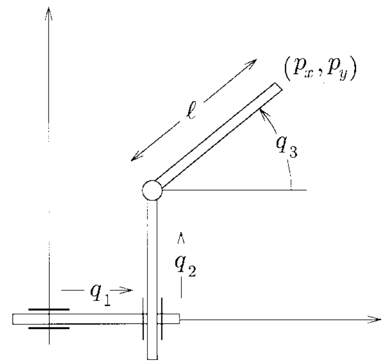

Fig. 1. A PPR planar robot.

For example, the choice

$$G(q) = \begin{bmatrix} 1+\frac{\ell}{w}s_3 & \frac{\ell}{w}s_3 \\ -\frac{\ell}{w}c_3 & 1-\frac{\ell}{w}c_3 \\ \frac{1}{w} & \frac{1}{w} \end{bmatrix} \tag{12}$$

corresponding to $r_1 = (1/w) + \ell/(w^2+\ell^2)\, s_3$, $r_2 = (1/w) - \ell/(w^2+\ell^2)\, c_3$, leads to a differential constraint in the form (9)

$$[1 \quad 1 \quad -w + \ell(c_3 - s_3)]\dot{q} = 0,$$

which can be integrated as

$$h(q) = q_1 + q_2 - wq_3 + \ell(s_3 + c_3) = c$$

where c is a constant.

Therefore, the motion of the PPR robot is constrained to the two-dimensional leaf $\mathcal{H}_0$ of $h(q)$ through q_0, and the corresponding kinematic control (12) is holonomic. The arm cannot be reconfigured from q_0 to an arbitrary q_d, unless $q_d \in \mathcal{H}_0$. On the other hand, if the cartesian command $u(t)$ is cyclic with period T, it will be $q(T) = q_0$. Thus, the generalized inverse (12) gives a cyclic inversion scheme.

2) Pseudoinversion for a 3R Spatial Robot: The 3R spatial robot of Fig. 2 is redundant for the task of positioning the end-effector with respect to the p_x, p_y coordinates, regardless of p_z ($n = 3$, $m = 2$). Denoting by ℓ_2 and ℓ_3 the second and third link lengths, the associated Jacobian is

$$J(q) = \begin{bmatrix} -s_1(\ell_2 c_2 + \ell_3 c_3) & -\ell_2 c_1 s_2 & -\ell_3 c_1 s_3 \\ c_1(\ell_2 c_2 + \ell_3 c_3) & -\ell_2 s_1 s_2 & -\ell_3 s_1 s_3 \end{bmatrix} = \begin{bmatrix} j_1^T \\ j_2^T \end{bmatrix}.$$

Singularities occur for (q_2, q_3) such that $s_2 = s_3 = 0$ or $\ell_2 c_2 + \ell_3 c_3 = 0$.

Assume that pseudoinversion is chosen as kinematic control scheme:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = J^{\#}(q)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = J^T(q)[J(q)J^T(q)]^{-1}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{13}$$

Whenever JJ^T is of full rank, the accessibility of the above system is equivalent to the accessibility of the simpler one

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = J^T(q)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [j_1 \quad j_2]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

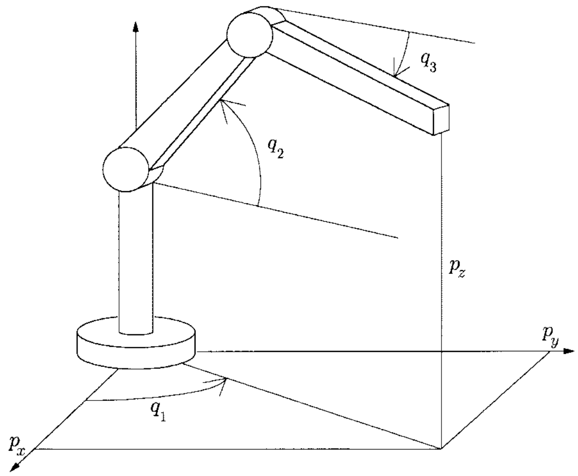

Fig. 2. A 3R spatial robot.

obtained after the input transformation $u = J(q)J^T(q)v$. Simple computations yield

$$[j_1, j_2] = \begin{bmatrix} \ell_2^2 + \ell_3^2 + 2\ell_2\ell_3 c_2 c_3 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\det [j_1 \quad j_2 \quad [j_1, j_2]] = 0.$$

Thus, we have $\dim \Delta_{\mathcal{A}} = m = 2$, and the system is holonomic. Using (7), the differential constraint associated with scheme (13) is

$$[0 \quad \ell_3 \sin q_3 \quad -\ell_2 \sin q_2]\dot{q} = 0.$$

When $\sin q_2(0)$ and $\sin q_3(0)$ are both nonzero, such constraint can be integrated as

$$\tan \frac{q_2}{2} = c\left(\tan \frac{q_3}{2}\right)^{\ell_2/\ell_3}.$$

When $q_2(0) = k\pi$ (with k integer) we obtain the constraint $q_2(t) = k\pi$, $\forall t \geq 0$. Similarly, $q_3(0) = k\pi$ implies $q_3(t) = k\pi$, $\forall t \geq 0$.

As in the previous case, while arbitrary robot reconfigurations are not possible, the use of $J^{\#}$ as an inversion scheme maps cyclic task trajectories to cyclic joint trajectories. Interestingly, one may verify that $\dot{p}_z \neq 0$ during a cyclic task in (p_x, p_y), and thus the end-effector motion is *not* confined to a plane parallel to the xy-plane.

B. Examples of Nonholonomic Kinematic Control Schemes

Four examples of nonholonomic generalized inverses are given below. After studying the weighted pseudoinversion scheme for the PPR robot, we shall consider two possible pseudoinverses for a 3R planar robot, corresponding to different choices of the joint variables. Finally, the pseudoinverse for a 4R planar robot provides an example of a partially nonholonomic scheme.

1) Weighted Pseudoinversion for a PPR Robot: For the PPR robot, consider the kinematic control scheme $\dot{q} = J_W^{\#}(q)u$ directly based on the W-weighted pseudoinverse (11). Denoting by j_{W1} and j_{W2} the columns of $J_W^{\#}$, we have

$$[j_{W1}, j_{W2}] = \rho \begin{bmatrix} \ell s_3 \\ -\ell c_3 \\ \\ \dfrac{1}{w} \end{bmatrix} \qquad \text{with } \rho = \frac{1}{w}\, \frac{\left(\dfrac{\ell}{w}\right)^2}{\left[1 + \left(\dfrac{\ell}{w}\right)^2\right]^2}.$$

Since the above Lie bracket does not belong to the range space of vectors j_{W1} and j_{W2}, it is $\dim \Delta_{\mathcal{A}} = n = 3$, and the system is nonholonomic. The associated differential constraint computed through (9)

$$[\ell s_3 \quad -\ell c_3 \quad w^2]\dot{q} = 0$$

is not integrable. Therefore, the reconfiguration problem is solvable for this mechanism by a proper choice of the task velocities u_1 and u_2.

2) Pseudoinversion for a 3R Planar Robot: Fig. 3 shows two choices of joint variables for a 3R planar robot with equal links of length ℓ, *viz.*, the *absolute* coordinates q and the *relative* coordinates ϑ. This manipulator is redundant for the end-effector positioning task ($n = 3$, $m = 2$).

If absolute coordinates are used, the Jacobian is expressed as

$$J_q(q) = \begin{bmatrix} -\ell s_1 & -\ell s_2 & -\ell s_3 \\ \ell c_1 & \ell c_2 & \ell c_3 \end{bmatrix} = \begin{bmatrix} j_{q1}^T \\ j_{q2}^T \end{bmatrix} \tag{14}$$

while, if relative coordinates are used, we have

$$J_\vartheta(\vartheta) = \begin{bmatrix} -\ell(S_1 + S_{12} + S_{123}) & -\ell(S_{12} + S_{123}) & -\ell S_{123} \\ \ell(C_1 + C_{12} + C_{123}) & \ell(C_{12} + C_{123}) & \ell C_{123} \end{bmatrix} = \begin{bmatrix} j_{\vartheta 1}^T \\ j_{\vartheta 2}^T \end{bmatrix}$$

in which $S_{i\cdots k} = \sin(\vartheta_i + \cdots + \vartheta_k)$ and $C_{i\cdots k} = \cos(\vartheta_i + \cdots + \vartheta_k)$. We shall analyze the two pseudoinversion schemes corresponding to the use of J_q or J_ϑ, respectively, assuming the kinematic singularities $\{q \in \mathbb{R}^3: s_1 = s_2 = s_3\}$ are not encountered.

The behavior of the kinematic control scheme $\dot{q} = J_q^{\#}(q)u$ is characterized by the accessibility distribution generated by j_{q1} and j_{q2}. We obtain

$$[j_{q1}, j_{q2}] = \ell^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

In this case, j_{q1}, j_{q2}, and $[j_{q1}, j_{q2}]$ are not linearly independent for all q, because

$$\det [j_{q1} \quad j_{q2} \quad [j_{q1}, j_{q2}]] = \ell[\sin(q_2 - q_1) + \sin(q_3 - q_2) + \sin(q_1 - q_3)]$$

is zero if one of the following conditions holds

$$\begin{cases} h_q^a(q) = q_1 - q_2 - k\pi = 0 & \text{or} \\ h_q^b(q) = q_2 - q_3 - k\pi = 0 & \text{or} \\ h_q^c(q) = q_1 - q_3 - k\pi = 0, \end{cases} \tag{15}$$

for any integer k. At this point, we need to check whether higher order brackets complete the rank on the two-dimensional planar surfaces (15), which we denote by $\mathcal{M}_q^i$, with $i = a, b, c$. All higher order brackets are easily shown to be linear combinations of j_{q1} and j_{q2}, so that the dimension of $\Delta_{\mathcal{A}}$ is not globally defined, i.e., $\Delta_{\mathcal{A}}$ is *singular*. As a result, the accessibility property holds only in an open and dense submanifold of the configuration space, and it is lost on the surfaces $\mathcal{M}_q^i$.

The same result was obtained in [5] by a geometric reasoning. In fact, being

$$\left.\frac{\partial h_q^i(q)}{\partial q}\right|_{\mathcal{M}_q^i} \cdot J_q^{\#}(q)|_{\mathcal{M}_q^i} u \equiv 0$$

any joint velocity starting from $\mathcal{M}_q^i$ is orthogonal to the surface normal, and the joint evolution is confined to the same surface.

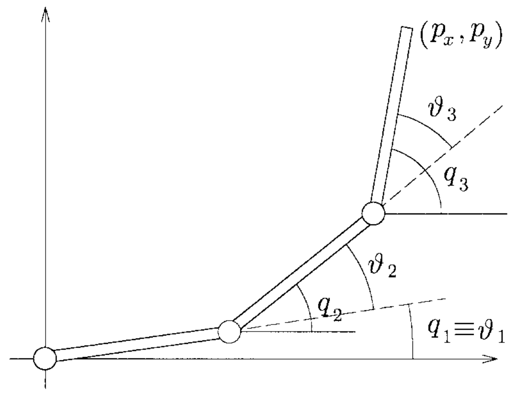

Fig. 3. A 3R planar robot with absolute and relative coordinates.

In summary, pseudoinversion in absolute coordinates is nonholonomic for this robot, provided that the initial joint configuration does not belong to one of the surfaces $\mathcal{M}_q^i$, which consist of *repeatable* configurations.

Consider now the pseudoinversion scheme in relative coordinates $\dot{\vartheta} = J_\vartheta^\#(\vartheta)u$. From

$$[j_{\vartheta 1},\, j_{\vartheta 2}] = \ell^2 \begin{bmatrix} 6 + 4C_3 + 2C_2 + 2C_{23} \\ 5 + 4C_3 + C_2 + C_{23} \\ 3 + 2C_3 + C_{23} \end{bmatrix}$$

we have

$$\det\,[j_{\vartheta 1}\; j_{\vartheta 2}\; [j_{\vartheta 1},\, j_{\vartheta 2}]] = -\frac{\ell^4}{2}[\sin(\vartheta_2 + 2\vartheta_3) - 3\sin\vartheta_2 - \sin(\vartheta_2 - \vartheta_3) + \sin(\vartheta_2 + \vartheta_3)]$$

which is zero on the two-dimensional manifold

$$h_\vartheta(\vartheta) = \tan\vartheta_2 - \frac{\sin\vartheta_3(1 + \cos\vartheta_3)}{1 + \sin\vartheta_3^2} = 0.$$

In this case, one can show that higher order Lie brackets *do* add rank to $\Delta_{\mathcal{A}}$ on $h_\vartheta(\vartheta) = 0$, and conclude that $\dim\Delta_{\mathcal{A}} = n = 3$.

Thus, pseudoinversion in relative coordinates for the 3R planar robot is *always* nonholonomic, allowing arbitrary joint reconfiguration. Equivalently, no repeatable configurations exist for such inversion scheme.

3) Pseudoinversion for a 4R Planar Robot: Consider a 4R planar robot with links of equal length ℓ. For the end-effector positioning task, this manipulator has two degrees of redundancy ($n = 4, m = 2$). Using absolute coordinates, the robot Jacobian is

$$J(q) = \begin{bmatrix} -\ell s_1 & -\ell s_2 & -\ell s_3 & -\ell s_4 \\ \ell c_1 & \ell c_2 & \ell c_3 & \ell c_4 \end{bmatrix} = \begin{bmatrix} j_1^T \\ j_2^T \end{bmatrix}. \tag{16}$$

Assume that pseudoinversion is chosen as a kinematic control scheme. To build the accessibility distribution associated with j_1 and j_2, we compute first

$$[j_1,\, j_2] = \ell^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since $[j_1, [j_1, j_2]] = -\ell^2 j_2$ and $[j_2, [j_1, j_2]] = \ell^2 j_1$, higher order Lie brackets do not increase the rank. Hence, $\dim\Delta_{\mathcal{A}} = 3$ unless

$$q_i - q_j = k\pi, \qquad i, j = 1, \cdots, 4,\; i \neq j,\; k \text{ integer}. \tag{17}$$

Note that this formula generalizes the condition (15). As with the 3R robot, it is easy to verify that the configurations defined by (17) are repeatable. The accessibility distribution $\Delta_{\mathcal{A}}$ is singular and has dimension $3 < n = 4$ in an open and dense subset of the configuration space. Hence, the kinematic control system is only *partially* nonholonomic.

From the discussion in Section II, we infer that the two differential constraints in the form (7) associated to the pseudoinverse of the Jacobian (16)

$$\begin{bmatrix} s_{2-4} & -s_{1-4} & 0 & s_{1-2} \\ s_{2-3} & -s_{1-3} & s_{1-2} & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = 0$$

admit *one* linear combination that is integrable. As a result, the motion of the 4R planar robot under pseudoinversion in absolute coordinates is confined to a three-dimensional manifold defined by this integrable constraint.[1] Reconfiguration is possible only within this submanifold of the four-dimensional configuration space.

IV. The Reconfiguration Problem

Since the problem of reconfiguring a redundant robot via task velocity commands can be solved if and only if the associated kinematic control scheme is completely nonholonomic, we shall make this assumption in the following. Suppose that the configuration of system (1) is initially q_0 and that we wish to transfer it to a desired q_d. It would be desirable to solve this control problem via feedback, i.e., by driving the input u through some function of the configuration error $e = (q_d - q)$. However, for the kinematic system (1) there exists no smooth time-invariant feedback law that can make q_d asymptotically stable. This negative result—which applies to all kinematic nonholonomic systems—is readily established on the basis of Brockett's necessary conditions for smooth stabilizability [9].

Hence, there are two possible approaches to the reconfiguration problem.

1) Use a *feedforward* (open-loop) control law $u = u(t)$. As a byproduct, one obtains a trajectory connecting q_0 to q_d that is feasible, i.e., complies with the nonholonomic constraints.
2) Use a feedback control law that is not ruled out by the above theoretical obstruction. In particular, *nonsmooth* laws $u = u(q, e)$ and/or *time-varying* laws $u = u(q, e, t)$ have been proposed for nonholonomic systems [10], [11].

Since robot reconfiguration is performed here at the kinematic level, we are actually facing a *motion planning* problem. Therefore, the use of a feedforward control is appropriate for our purposes. In the remainder of this section, a representative method of this class is presented with reference to the PPR robot under weighted pseudoinversion. An alternative technique, based on the concept of holonomy angle, has been proposed in [12].

A. Reconfiguration via Sinusoidal Steering

A powerful approach to nonholonomic motion planning relies on the existence of canonical forms for which the problem can be efficiently solved. The most common is the *chained form* [3]. In particular, a two-input system

$$\dot{q} = G(q)u = g_1(q)u_1 + g_2(q)u_2 \tag{18}$$

admits a chained-form representation if there exists a *feedback transformation,* i.e., an invertible input transformation $u = \beta(q)v$

[1] The same conclusion can be drawn for the nR planar robot case.

and a change of coordinates $x = \phi(q)$, such that

$$\begin{aligned}\dot{x}_1 &= v_1\\ \dot{x}_2 &= v_2\\ \dot{x}_i &= x_{i-1}v_1, \qquad i = 3, \cdots, n.\end{aligned}$$

For the PPR robot under the W-weighted pseudoinverse (11), the sufficient conditions given in [3] for the existence of a transformation into chained form are satisfied. Applying the constructive part of the proof of Frobenius theorem, a suitable change of coordinates is obtained as

$$\begin{aligned}x_1 &= q_2 + \ell s_3\\ x_2 &= \frac{\alpha e^{\alpha(q_1+\ell c_3)} \cos q_3}{\eta(q_1, q_3)}\\ x_3 &= 2 \arctan \left[e^{\alpha(q_1+\ell c_3)} \tan \frac{q_3}{2}\right]\end{aligned} \tag{19}$$

where $\alpha = \ell/(w^2 + \ell^2)$ and

$$\eta(q_1, q_3) = \cos^2\left(\frac{q_3}{2}\right) + e^{2\alpha(q_1+\ell c_3)} \sin^2\left(\frac{q_3}{2}\right) \neq 0.$$

The associated input transformation in terms of the original coordinates q is given by

$$\begin{aligned}u_1 &= \frac{1}{2} \sin q_3 \cos q_3 \frac{1 + e^{2\alpha(q_1+\ell c_3)}}{\eta(q_1, q_3)} v_1 + \frac{\eta(q_1, q_3)}{\alpha^2 e^{\alpha(q_1+\ell c_3)}} v_2\\ u_2 &= v_1\end{aligned} \tag{20}$$

with v_1 and v_2 external inputs to be designed. The feedback transformation (19) and (20) is just one possible choice, with the nice feature of being *globally* defined.

With the robot kinematic control system in chained form, the use of sinusoidal steering is particularly advantageous [3]. The initial and the desired joint configurations q_0 and q_d are mapped through (19) into x_0 and x_d, respectively. The reconfiguration task may then be executed in two phases.

1) Steer x_1 and x_2 to their desired values x_{1d} and x_{2d} in a finite time t_1, using nonsmooth feedback laws for v_1 and v_2. For example, one may set $v_i = k_i \operatorname{sign}(x_{id} - x_i)$, with $k_i > 0$, $i = 1, 2$. The variable x_3 will move to $x_3(t_1)$.
2) Use the sinusoidal open-loop commands

$$\begin{aligned}v_1 &= A_1 \sin\left[\frac{2\pi(t - t_1)}{T}\right]\\ v_2 &= A_2 \cos\left[\frac{2\pi(t - t_1)}{T}\right]\end{aligned}$$

where $t \in [t_1, t_1 + T]$, and $T > 0$ is arbitrary. In this way, x_1 and x_2 will cycle returning to their values x_{1d} and x_{2d}. In order to bring x_3 from $x_3(t_1)$ to its desired value x_{3d} at $t = t_1 + T$, the amplitudes are chosen as

$$\begin{aligned}A_1 &= \frac{2}{T}\sqrt{\pi|x_{3d} - x_3(t_1)|}\\ A_2 &= A_1 \operatorname{sign}[x_{3d} - x_3(t_1)].\end{aligned}$$

The above method was simulated for the case $\ell = 1$, $w = 1$, with reconfiguration task $q_0 = (1, 1, \pi/3)$, $q_d = (2, 1, 2\pi/3)$. Note that the initial and final end-effector positions coincide, although this is not required in general. The duration of the two phases is $t_1 = 0.6$ s and $T = 1$ s, respectively. In order to obtain a smoother behavior, we have used a linear feedback within the first phase. The arm is practically at rest at t_1, which is about 30 times the time constant corresponding to the chosen proportional gains. The resulting motion is shown in Figs. 4 and 5.

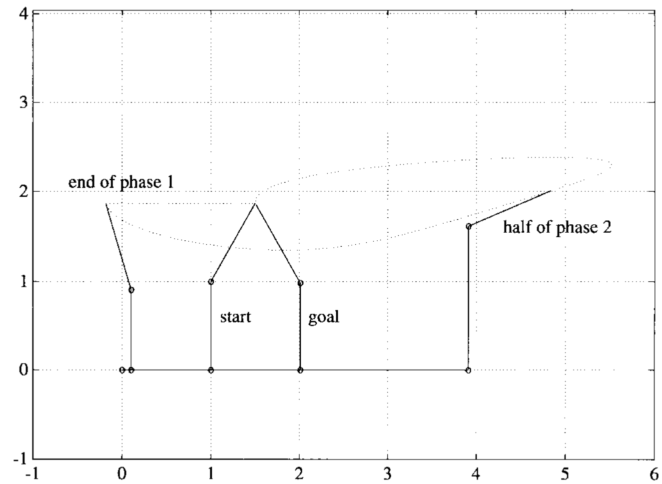

Fig. 4. Reconfiguration of the PPR robot with sinusoidal steering.

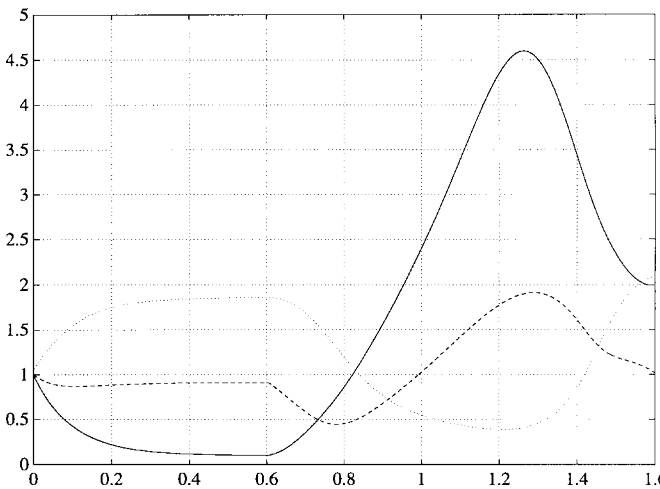

Fig. 5. Joint evolutions of the PPR robot with sinusoidal steering: q_1 (—), q_2 (– –), q_3 (· · ·).

We remark that:

- The first phase is performed in feedback, while the second specifies $v(t)$ in a feedforward mode. In any case, the state-dependent transformation (20) must be applied in both phases in order to obtain the actual task command $u(t)$.
- Sometimes it is convenient to use multiple sinusoidal phases, each achieving part of the reconfiguration for x_3. In this way, it is possible to limit the end-effector displacement along the cycle.
- The same technique can be used to plan a nontrivial (i.e., enclosing a nonzero area) cyclic cartesian path that maps to a closed joint trajectory. Such a holonomic path is built by applying the two-phase steering algorithm twice: the robot is first steered from the initial q_0 to an (arbitrary) intermediate configuration q_i and then driven back[2] from q_i to q_0.

Concerning the general applicability of this method, it has been proven in [13] that nonholonomic systems of the form (18) with $n =$

[2] Due to the two-phase structure of the algorithm, different forward and backward paths are obtained. Instead, by reversing the commands of the forward motion, the robot would trace backward the same path from q_i to q_0 (with zero area enclosed).

3 or 4 can be always put in chained form. Hence, reconfiguration via sinusoidal steering can certainly be achieved for redundant robots with up to $n = 4$ rotary and/or prismatic joints and $m = 2$ task inputs, provided that the kinematic control scheme is nonholonomic. However, we emphasize that deriving the chained form transformation is a difficult task. For example, the 3R planar robot with equal links under pseudoinversion in absolute coordinates is transformed into chained form via the following change of coordinates:

$$\begin{aligned} x_1 &= q_1 \\ x_2 &= 8\,\frac{\sin\left(\frac{q_2-q_1}{2}\right)\sin\left(\frac{q_3-q_1}{2}\right)\sin\left(\frac{q_2-q_3}{2}\right)}{2-\cos(q_2-q_3)-\cos(q_2+q_3-2q_1)} \\ &=: \varphi(q) \\ x_3 &= 2\,\arctan\left[\frac{\tan\left(\frac{q_3-q_1}{2}\right)}{\tan\left(\frac{q_2-q_1}{2}\right)}\right] \end{aligned}$$

and the associated input transformation

$$u = \frac{1}{\ell} J_q J_q^T \begin{bmatrix} -\sin q_1 & \cos q_1 \\ -\sin q_2 & \cos q_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ L_{g_1}\varphi(q) & L_{g_2}\varphi(q) \end{bmatrix}^{-1} v$$

with the Jacobian matrix J_q given by (14). Here, $L_g\varphi$ denotes the Lie derivative of function φ w.r.t. vector field g, and we have set

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ \dfrac{\sin(q_2-q_3)}{\sin(q_2-q_1)} \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 1 \\ \dfrac{\sin(q_3-q_1)}{\sin(q_2-q_1)} \end{bmatrix}.$$

Both the change of coordinates and the input transformation are only locally defined.

V. Conclusions

Redundant robots under a given kinematic control scheme are an example of mechanical system where nonholonomic constraints may arise due to the particular command strategy. Using tools from nonlinear control theory, and in particular analyzing the accessibility distribution of the system, we have shown instances of redundant robots and kinematic control schemes for which holonomic, partially nonholonomic or completely nonholonomic behaviors are enforced. In this framework, the properties of cyclicity and holonomy are recognized to be equivalent.

In the case of nonholonomic behavior, it is possible to achieve arbitrary joint configurations by designing suitable end-effector velocity commands. For this reconfiguration problem, techniques can be borrowed from the area of nonholonomic motion planning and control. In particular, we have applied the sinusoidal steering method for reconfiguring a PPR robot under weighted pseudoinversion.

Although an explicit solution was presented for a robot with a single degree of redundancy, the case of a higher degree of redundancy can be tackled in a similar way. In fact, the sinusoidal steering method can be successfully extended using multiple sinusoids at integrally related frequencies [3].

We are currently working on a dynamical version of the reconfiguration problem. The objective is to drive the redundant arm to a given joint configuration by applying forces and torques to the end-effector. As robot dynamics must be taken into account, a nonlinear control system with drift is obtained, requiring more complex controllability analysis and control design. Preliminary results can be found in [14].

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