Steering nonholonomic systems via nilpotent approximations: The general two-trailer system

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Abstract

Existing methods for nonholonomic motion planning can only be applied to exactly nilpotentizable or flat systems. For nonholonomic systems that do not fall into the above classes, we conjecture that globally defined nilpotent approximations will allow the synthesis of efficient steering and stabilization strategies. In this paper, a car towing two off-hooked trailers is considered as a case study. First, it is shown how to derive a nilpotent approximation that is valid in the neighborhood of both regular and singular points. Then, such approximate model is used to compute steering controls. Simulation results are reported to show the satisfactory performance of the method.

1 Introduction

Nonholonomic motion planning and control are problems of particular interest in mobile robotics [1]. Substantial progresses have been recently achieved: for example, we know how to steer to a desired configuration systems that are exactly nilpotentizable [2, 3] or flat [4]. The same is true for stabilization tasks, e.g., see [5]. However, some nonholonomic systems do not possess the above properties; for them, even the steering problem is still open—let alone stabilization.

Another common weak point of planning methods is that, making use of canonical forms (e.g., chained forms), they ignore the occurrence of singularities, e.g., points where the degree of nonholonomy changes. Such a limitation may be unacceptable in applications.

The synthesis of approximate steering techniques is a first step towards the solution of the steering problem for *general* nonholonomic systems. If the approximation is accurate, this approach may already be a satisfactory practical solution. Furthermore, feedback stabilization to a point can be obtained by iterated application of approximate steering controls [6]; see [7] for an application to underactuated manipulators.

Approximate steering methods can be devised using nilpotent approximations [8–10]. These are high-order local approximations that are useful when linearization does not retain controllability, as in nonholonomic systems. At the same time, closed-form integration of the approximate model (which is polynomial and triangular) under parameterized inputs is possible, thereby allowing the computation of steering controls.

Various techniques exist for computing nilpotent approximations [11–13]. Basically, it is first necessary to express the system in a particular frame defined by the input vector fields and their iterated Lie brackets. The new vector fields are then expanded in Taylor series up to a proper order. As a result, nilpotent approximations are intrinsically local. In particular, not only the values of the approximate system vector fields depend on the given point, but even their structure is altered if the degree of nonholonomy changes.

The main contribution of this paper is to show how to deal with both regular and singular points in a unified framework. A globally valid approximation is obtained and the same algorithm can be used to steer (approximately) the system between any two points. The method is illustrated with reference to a car towing two *off-hooked* trailers, a canonical example of non-nilpotentizable, non-flat nonholonomic system.

The paper is organized as follows. In Sect. 2, we give the model of the general two-trailer system. Section 3 extends the local point of view of approximate nilpotentization techniques to a global one, in order to obtain an approximation which is valid at both regular and singular points. The use of nilpotent approximations for the solution of steering problems is discussed in Sect. 4, and illustrated by simulations in Sect. 5. In the Appendix, we review a classical procedure for deriving nilpotent approximations of driftless systems.

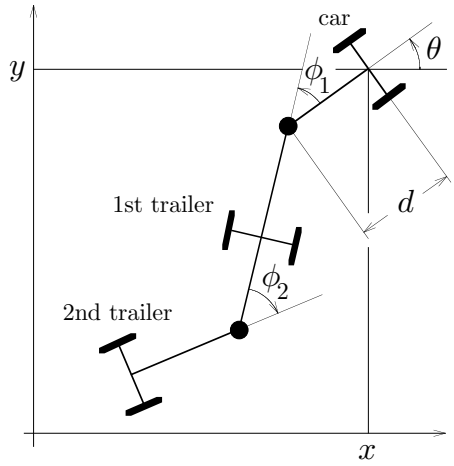

Figure 1: The car with two off-hooked trailers

2 The general two-trailer system

Consider the system shown in Fig. 1, consisting of a car towing two trailers, each hooked at a distance d from the preceding wheel axle (*off-hooking*). The generalized coordinates are $(x, y, \theta, \phi_1, \phi_2)$, where x, y are the cartesian coordinates of the car reference point, θ is the car orientation with respect to the x axis, while ϕ_1, ϕ_2 are the angles formed by the first and the second trailer with the car and the first trailer, respectively. Without loss of generality, we assume $d = 1$.

The kinematic model of the vehicle has the form

$$\dot{\xi} = g_1(\xi)u_1 + g_2(\xi)u_2, \tag{1}$$

where $\xi = (x, y, \theta, \phi_1, \phi_2)$, while u_1 and u_2 are the driving and steering velocities of the car. The system vector fields are expressed as

$$g_1(\xi) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -s_1 \\ s_{12} + s_1 \end{pmatrix}, \quad g_2(\xi) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 - c_1 \\ c_{12} + c_1 \end{pmatrix}.$$

having set $s_i = \sin\phi_i$, $c_i = \cos\phi_i$, $s_{ij} = \sin(\phi_i - \phi_j)$ and $c_{ij} = \cos(\phi_i - \phi_j)$ for $i, j = 1, 2$.

Note that if $\phi_1 = \pi$ or $\phi_2 = \pi$ the system is not controllable. This singularity is however immaterial in practice, in view of joint limits present in actual vehicles. In the following, we consider points of the state space defined as $\mathcal{M} = I\!R^2 \times S^1 \times (S^1 - \{\pi\})^2$.

Consider the first 6 elements of the P. Hall [3] family $g_1, g_2, g_3 = [g_1, g_2]$, $g_4 = [g_1, [g_1, g_2]]$, $g_5 = [g_2, [g_1, g_2]]$, $g_6 = [g_1, [g_1, [g_1, g_2]]]$. Vector fields $g_1, g_2,\ g_3, g_4, g_5$ span the tangent space of $\mathcal{M}$ at points such that $\phi_1 \neq \phi_2$ (*regular points*), while g_1, g_2, g_3, g_4, g_6 span the tangent space everywhere, including points such that $\phi_1 = \phi_2$ (*singular points*). Hence, the system is controllable and the degree of nonholonomy is 3 at regular points and 4 at singular points.

It is well known that system (1) cannot be put in chained form [14], nor does it satisfy the necessary and sufficient condition for flatness given in [4]. Therefore, we shall derive a nilpotent approximation of the system in order to address the steering problem. To this end, one can follow the approach proposed by Bellaïche [10], briefly summarized in the Appendix.

System (3) can be approximated with Bellaïche's method around regular as well as singular points. However, as the degree of nonholonomy increases at singular points, also the basis used to define the coordinate transformation will change, together with the corresponding order of the privileged coordinates. In particular, $g_1, \ldots, g_5$ provide a system of privileged coordinates which is valid only at regular points, while at singular points one should use $g_1, \ldots, g_4, g_6$. Hence, two different nilpotent approximations are obtained.

A simple solution would seem to use the basis $g_1, \ldots, g_4, g_6$ to build the privileged coordinates $z_1, \ldots, z_5$ at regular as well as singular points. However, the orders of the associated privileged coordinates would still depend on the point. In particular, the order of the privileged coordinate z_5 would be 3 at regular and 4 at singular points,[1] a fact to keep into account when computing the Taylor expansion. Thus, this approach does not yield a globally valid approximation, unless a degradation in accuracy is accepted.

In the next section we shall see that it is possible to obtain a globally valid approximation by resorting to a higher-order *nonhomogeneous*, nilpotent form.

3 A global nilpotent approximation

To compute a nilpotent approximation valid at any point $\xi_0 \in \mathcal{M}$ (either regular or singular), we propose the procedure given below.

1. Evaluate vectors g_1, g_2, g_3, g_4, g_6 at ξ_0 and denote the result as $\gamma_1(\xi_0), \ldots, \gamma_5(\xi_0)$.

2. Define $A(\xi_0)$ as the matrix whose columns are $\gamma_1(\xi_0), \ldots, \gamma_5(\xi_0)$. The coordinates $y_1, \ldots, y_n$ of a point $\xi \in \mathcal{M}$, expressed in the frame attached to ξ_0 with basis $\gamma_1(\xi_0), \ldots, \gamma_5(\xi_0)$, can be obtained as
$$y(\xi) = A(\xi_0)^{-1}(\xi - \xi_0).$$
It is easy to prove that such coordinates satisfy condition (4) of the Appendix. The weights of co-

[1] In fact, $g_6(\xi_0)$ can be obtained as a linear combination of $g_1(\xi_0), \ldots, g_5(\xi_0)$ at a regular point ξ_0.

ordinates $y_1, \ldots, y_5$ are respectively $\{1, 1, 2, 3, 4\}$ for singular and $\{1, 1, 2, 3, 3\}$ for regular points.

3. Assign to coordinates $y_1, \ldots, y_5$ the weights $\{1, 1, 2, 3, 4\}$, regardless of the regularity of ξ_0.

4. Define the privileged coordinates using eq. (5) of the Appendix. Coordinates $z_1, \ldots, z_4$ have respectively orders $1, 1, 2, 3$, while z_5 has order 3 at regular and 4 at singular points.

5. Express the system in privileged coordinates and truncate the Taylor series of the i-th component of the vector fields at the order $w_i - 1$ $(i = 1, \ldots, 5)$.

The nilpotent approximation of system (1) obtained with the above procedure is

$$\begin{aligned} \dot{z}_1 &= u_1 \\ \dot{z}_2 &= u_2 \\ \dot{z}_3 &= -z_2 u_1 \\ \dot{z}_4 &= \sum_{j=1}^{2} \hat{g}_{j4}(z_1, \ldots, z_3) u_j \\ \dot{z}_5 &= \sum_{j=1}^{2} \hat{g}_{j5}(z_1, \ldots, z_4) u_j, \end{aligned} \tag{2}$$

having set

$$\begin{aligned} \hat{g}_{j4}(z_1, \ldots, z_3) &= a^2_{j4} z_1^2 + b^2_{j4} z_1 z_2 + c^2_{j4} z_2^2 + d^2_{j4} z_3 \\ \hat{g}_{j5}(z_1, \ldots, z_4) &= a^2_{j5} z_1^2 + b^2_{j5} z_1 z_2 + c^2_{j5} z_2^2 + d^2_{j5} z_3 \\ &\quad + a^3_{j5} z_1^3 + b^3_{j5} z_1 z_3 + c^3_{j5} z_1^2 z_2 \\ &\quad + d^3_{j5} z_2 z_3 + e^3_{j5} z_2^3 + f^3_{j5} z_1 z_2^2 + g^3_{j5} z_4. \end{aligned}$$

Coefficients $a^2_{j4}, \ldots, d^2_{j4}$, $a^2_{j5}, \ldots, d^2_{j5}$ and $a^3_{j5}, \ldots, g^3_{j5}$ $(j = 1, 2)$ are functions of ξ_0; their expression and the mapping from ξ to z are omitted for lack of space.

Note that system (2):

- has the form (6–7) given in the Appendix. However, while $\hat{g}_{jk}(z_1, \ldots, z_{k-1})$ $(j = 1, 2,\ k = 3, 4)$ are polynomials of respective homogeneous order 1 and 2, $\hat{g}_{j5}(z_1, \ldots, z_4)$ are now the sum of two polynomial terms of homogeneous order 2 and 3.
- approximates system (1) both around regular and singular points. In fact, since z_5 has order 3 at regular and 4 at singular points, those coefficients of the monomials of homogeneous degree 2 appearing in the expression of $\dot{z}_5$ will vanish at singular points. As a consequence, the functions $\hat{g}_{j5}(z_1, \ldots, z_4)$ will *naturally* result in a sum of polynomials of degree 2 and 3 at regular points and in a polynomial of homogeneous degree 3 at singular points, and the obtained approximation is at least as accurate as the homogeneous one.
- is nilpotent of order 5. This point can be easily proven by the same arguments used in [10].
- is controllable. Results in [10] apply in proving that $\hat{g}_1, \hat{g}_2$ satisfy the Lie algebra rank condition.

4 Steering via nilpotent approximations

In this section, we offer a preliminary discussion on the use of nilpotent approximations for solving the problem of steering the general two-trailer system from an initial point ξ^0 to a desired point ξ^d.

The first step is to compute the nilpotent approximation (2) at ξ^0. By virtue of the globality of the representation, ξ^0 may be a regular or singular point. The synthesis of a control law that transfers the state of system (2) from $z^0 = 0$ (the image of ξ^0) to z^d (the image of ξ^d) is relatively straightforward. In fact, the model equations are polynomial and triangular, and thus closed-form integrable under appropriate parameterized controls. Two possible choices are sinusoidal controls at related frequencies, as in [2], and polynomial controls. Other choices are possible, each resulting in differently shaped trajectories. In any case, the structure of the chosen control law should be designed so as to obtain an invertible mapping between the control parameters and the final state $z(T)$ at time T.

Once a steering control for the nilpotent approximation (2) has been identified, one may use it in order to steer approximately the original system (1). The induced error will obviously be related to the norm of the displacement vector $\xi^d - \xi^0$ as well as to the trajectory described by the system in $[0, T]$.

The steering error induced by the approximation could be acceptable in practice or not. In principle, there are two ways to reduce the error:

1. Divide a steering task into a series of subtasks through the definition of via points. These should be chosen on the basis of the nonholonomic metric induced by the distance estimate [10].

2. Apply the steering control iteratively. Under suitable assumptions (roughly speaking, continuity of the control parameters w.r.t. the required displacement), this strategy provides robust feedback stabilization to the desired configuration [6].

5 Simulation results

We now present simulation results obtained by the two approaches outlined at the end of the previous section.

In the first simulation, the task is to drive the vehicle from $\xi^0 = (0, 0, 0, 0, 0)$ to $\xi^d = (2, 0.02, 0, 0, 0)$ (distances are expressed in meters, angles in radiants).

Note that both ξ^0 and ξ^d are singular points. To reduce the approximation error, we have added the via point $\xi^i = (1, 0.01, 0, 0, 0)$, which is sufficiently near to ξ^0 (according to the distance estimate provided by the approximation [10]), and closer to ξ^d than ξ^0. The following polynomial steering controls are used:

$$\begin{aligned} u_1(\tau) &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 \\ u_2(\tau) &= b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^7, \end{aligned}$$

with $\tau \in [0, 1]$. For each steering subtask, coefficients $a_0, \ldots, a_3$ and $b_0, \ldots, b_4$ are computed so as to steer the nilpotent approximation at the current point (ξ^0 for the first, $\xi(1)$ for the second subtask) from the origin to the z-image of the final point (ξ^i for the first, ξ^d for the second subtask). The redundancy in control parameters is purposely introduced to provide some degree of freedom in 'shaping' the paths.

Figure 2 shows the xy-path of the reference points (i.e., the midpoints of the car axle and of the trailer axles) during the first and the second subtasks. Figures 3–7 show the evolution of the state variables of the two-trailer system and of the approximate system. The evolution of the generalized coordinates x, y, θ, ϕ_1 are accurately described by the approximate system,[2] while a small error appears in the evolution of ϕ_2. The logarithmic plot of norm of the state error with respect to ξ^d is reported in Fig. 8. The configuration actually reached by the vehicle at the end of the first subtask is $(0.998, 0.01, 0, -0.005, 0.0425)$, while the final configuration is $(1.9988, 0.0199, 0, -0.004, 0.0446)$.

In the second simulation, the two-trailer system should be driven from $\xi^0 = (0, 0, 0, -0.7854, 0.5236)$ to $\xi^d = (1, 0.1, 0.2, -0.7, 0.3)$. Both ξ^0 and ξ^d are now regular points. In this case, we apply the iterative steering approach outlined in the previous section. The following sinusoidal steering controls are used:

$$\begin{aligned} u_1(\tau) &= a_0 + a_1 \cos\omega\tau + a_2 \sin\omega\tau \\ u_2(\tau) &= b_0 + b_1 \sin 3\omega\tau + b_2 \cos 2\omega\tau, \end{aligned}$$

with $\tau \in [0, 1]$. In each iteration, coefficients $a_0, \ldots, a_2$ and $b_0, \ldots, b_2$ are computed so as to steer the approximation at the current point (ξ^0 for the first, $\xi^1 = \xi(1)$ for the second iteration, etc.) from the origin to the z-image of the final point (ξ^d for all iterations). Again, there is a redundancy in the control parameters.

We report the results of two iterations. In Fig. 9, the xy-path of the reference points during the first and the second iteration is shown, while Fig. 10 reports the logarithmic plot of the state error norm. After the first iteration the system has reached the configuration $(0.9589, 0.0997, 0.2000, -0.6831, 0.0934)$ which results in a significant error on the ϕ_2 state variable. After the second iteration, however, the state of the two-trailer system is $(1.002, 0.0999, 0.2, -0.6857, 0.1773)$, with a remarkable reduction of the steering error.

[2] We point out that the evolution of θ is actually the same in the two systems in view of the expression of z_2.

6 Conclusion and future work

For nonholonomic systems that are neither exactly nilpotentizable nor flat, approximate steering methods can be devised using nilpotent approximations. In this paper, we have considered a car towing two off-hooked trailers as a case study. First, we have computed a nilpotent approximation valid in the neighborhood of both regular and singular points. The basic idea consists in building a privileged coordinate system from a basis of the tangent space valid everywhere; the critical point is to assign appropriate weights to the coordinates. This approach can be generalized to other nonholonomic systems provided that it is possible to find a basis covering both regular and singular points.

We have also discussed the use of nilpotent approximations for the synthesis of steering controls, sketching two possible methods for keeping the approximation error within acceptable bounds. Simulation results have been reported to show the performance of the proposed approaches.

It should be emphasized that, at the present stage, this paper must be regarded as a preliminary report of our investigation. A number of issues remain to be addressed. In the presented simulations, the computation of control parameters that guarantee steering of the nilpotent system was performed numerically, whereas a closed-form solution is possible. Moreover, concerning the first steering method of Sect. 4, a detailed study is necessary to devise a selection procedure for the via points, based on the nonholonomic metric induced by the distance estimate. Finally, as already mentioned, the second steering method can be converted into an exponential stabilization technique by meeting certain continuity assumptions.

7 Acknowledgments

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Appendix: Nilpotent approximation

We briefly review here the nilpotent approximation procedure due to Bellaïche [10], to which the reader is referred for details.

Let us consider a driftless system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i, \qquad x \in I\!R^n, \tag{3}$$

satisfying the Lie algebra rank condition (LARC) (i.e., vector fields g_i and their iterated brackets span the tangent space) almost everywhere. This guarantees controllability of the system at every point.

Some definitions are recalled below.

Definition 1 *Let $L^s(x_0)$ be the vector space spanned at x_0 by the brackets of $g_1, \ldots, g_m$ of length $\leq s$. Then, the smallest integer $r = r(x_0)$ such that $L^{r(x_0)}(x_0)$ spans the tangent space of system (3) at x_0 is called the* degree of nonholonomy *at x_0.*

Definition 2 *A function $f \in I\!R$ is of* order $\geq s$ *at x_0 if the function $(g_{i_1} \ldots g_{i_q})f$ vanishes at x_0, for any $i_1, \ldots, i_q$ and $q \leq s-1$.*

Here, $(g_{i_1} \ldots g_{i_{s-1}})f$ is the Lie derivative of order $s-1$ of f along the directions of vector fields $g_{i_1} \ldots g_{i_{s-1}}$.

Definition 3 *A vector field g is of* order $\geq q$ *at x_0 if, for every s and every function f having order s at x_0, the Lie derivative gf has order $\geq q+s$ at x_0.*

Let $\gamma_1(x_0), \ldots, \gamma_n(x_0)$ be a basis[3] of $L^{r(x_0)}(x_0)$. Through a linear change of coordinates, it is always possible to find a system of coordinates $y_1, \ldots, y_n$ centered at x_0 such that

$$(\gamma_i y_j)(x_0) = \delta_{ij}, \tag{4}$$

where δ_{ij} is the Kronecker delta.

Definition 4 *The* weight *w_j of a coordinate y_j is the smallest integer s such that dy_j is not identically zero on $L^s(x_0)$ $(s = 1, \ldots, r)$.*

Definition 5 *Local coordinates $z_1, \ldots, z_n$ centered at x_0 form a system of* privileged coordinates *if the order of z_j at x_0 is equal to w_j $(j = 1, \ldots, n)$.*

Consider now a system of coordinates $y_1, \ldots, y_n$ satisfying condition (4). A system of privileged coordinates $z_1, \ldots, z_n$ around x_0 is obtained by the recursive formula

$$z_q = y_q - \sum_{\{\alpha | w(\alpha) < w(q)\}} \frac{1}{\alpha_1! \ldots \alpha_{q-1}!} (\gamma_1^{\alpha_1} \ldots \gamma_{q-1}^{\alpha_{q-1}} y_q)(x_0) z_1^{\alpha_1} \ldots z_{q-1}^{\alpha_{q-1}}, \tag{5}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $w(\alpha) = \sum_1^n w_i \alpha_i$. Coordinates $z_1, \ldots, z_n$ have order $w_1, \ldots, w_n$ by construction.

With the system in privileged coordinates:

[3] Such a basis can be computed from a P. Hall family [3].

– the order of a smooth function f at x_0 is the least weighted-degree monomial actually appearing in the Taylor expansion of f at x_0:

$$f(z) = \sum_{\alpha} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n};$$

– the order of a vector field g can be computed in the same algebraic way as above, i.e., using the Taylor expansion

$$g(z) = \sum_{\alpha, j} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j},$$

assigning to ∂_{z_j} the weight $-w_j$ and considering terms like $a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ as products;

– each g_i can be expanded in terms of vector fields homogeneous with respect to the weighted degree

$$g_i = g_i^{(-1)} + g_i^{(0)} + g_i^{(1)} + \ldots.$$

The nilpotent approximation of system (3) is derived by replacing the vector fields g_i with their principal component $g_i^{(-1)}$. One obtains the triangular form

$$\dot{z}_i = \sum_{j=1}^{m} \hat{g}_{ji} u_j, \qquad i = 1, \ldots, \nu, \tag{6}$$

$$\dot{z}_k = \sum_{j=1}^{m} \hat{g}_{jk}(z_1, \ldots, z_{k-1}) u_j, \quad k = \nu+1, \ldots, n. \tag{7}$$

Here, ν is the dimension of $\mathrm{span}\{g_1, \ldots, g_m\}$ at x_0, $\hat{g}_{1i}, \ldots, \hat{g}_{mi}$ are constant values, for $i = 1, \ldots, \nu$, and $\hat{g}_{jk}(z_1, \ldots, z_{k-1})$ are polynomial functions of homogeneous degree $w_k - 1$, for $k = \nu+1, \ldots, n$.

References

[1] J.P. Laumond, S. Sekhavat, , and F. Lamiraux, "Guidelines in nonholonomic motion planning for mobile robots", in *Robot Motion Planning and Control*, J.P. Laumond, Ed., vol. 229. Springer-Verlag, 1998.

[2] R. M. Murray and S. Sastry, "Steering nonholonomic systems using sinusoids", *IEEE Int. Conf. on Decision and Control*, pp. 2097–2101, 1990.

[3] G. Laferriere and H. J. Sussmann, "A differential geometric approach to motion planning", in *Nonholonomic Motion Planning*, Z. Li and J. F. Canny, Eds. Kluwer Academic Publishers, 1992.

[4] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: Introductory theory and examples", *Int. J. of Control*, vol. 61, pp. 1327–1361, 1995.

[5] C. Samson, "Control of chained systems. application to path following and time-varying point-stabilization of mobile robots", *IEEE Trans. on Automatic Control*, vol. 40, pp. 64–77, 1995.

[6] P. Lucibello and G. Oriolo, "Stabilization via iterative state steering with application to chained-form systems", *35th IEEE Conf. on Decision and Control*, pp. 1455–1460, 1996.

[7] A. De Luca, R. Mattone, and G. Oriolo, "Stabilization of underactuated robots: Theory and experiments for a planar 2r manipulator", *1997 IEEE Int. Conf. on Robotics and Automation*, pp. 3274–3280, 1997.

[8] H. Hermes, "Nilpotent and high-order approximations of vector field systems", *SIAM Review*, vol. 33, pp. 238–26, 1991.

[9] R. M. Bianchini and G. Stefani, "Controllability along a trajectory: A variational approach", *SIAM J. on Control and Optimization*, vol. 31, pp. 900–927, 1993.

[10] A. Bellaïche, "The tangent space in sub-riemannian geometry", in *Sub-Riemannian Geometry*, A. Bellaïche and J.-J. Risler, Eds. Birkhäuser, 1996.

[11] A. Bellaïche, J.P. Laumond, and M. Chyba, "Canonical nilpotent approximation of control system: Application to nonholonomic motion planning", *32nd IEEE Conf. on Decision and Control*, pp. 2694–2699, 1993.

[12] W. Liu, "An approximation algorithm for nonholonomic systems", *SIAM J. on Control and Optimization*, vol. 35, pp. 1328–1365, 1997.

[13] H. J. Sussmann, "New differential geometric methods in nonholonomic path finding", in *Systems, Models, and Feedback: Theory and Applications*, A. Isidori and T.J. Tarn Eds., Eds. Birkhäuser, 1992.

[14] D. M. Tilbury, *Exterior differential systems and nonholonomic motion planning*, Memo. UCB/ERL M94/90, University of California, Berkeley, 1994.

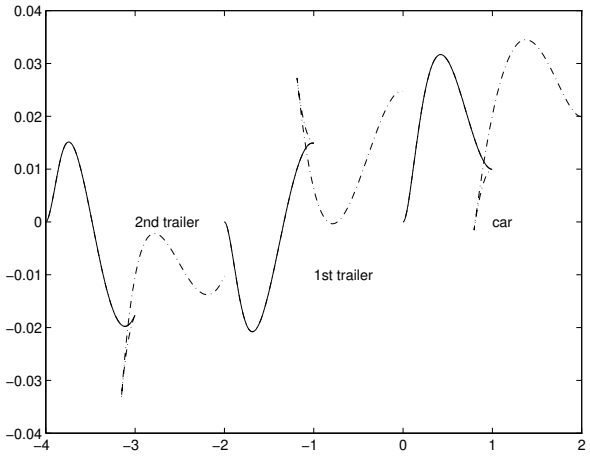

Figure 2: Simulation 1: xy-path of the reference points during the first (—) and the second ($- \cdot -$) subtask

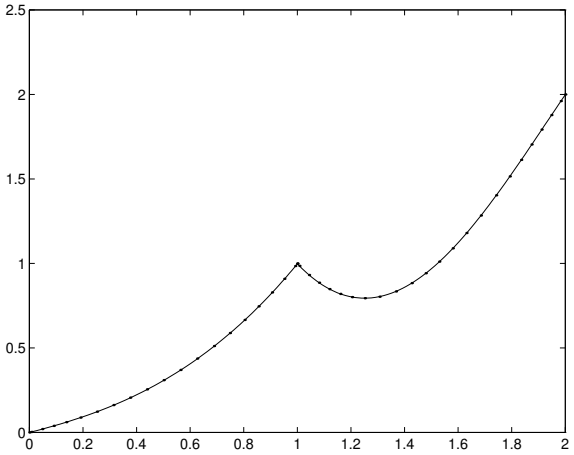

Figure 3: Simulation 1: time evolution of x for the vehicle (—) and the approximated system ($\cdots$)

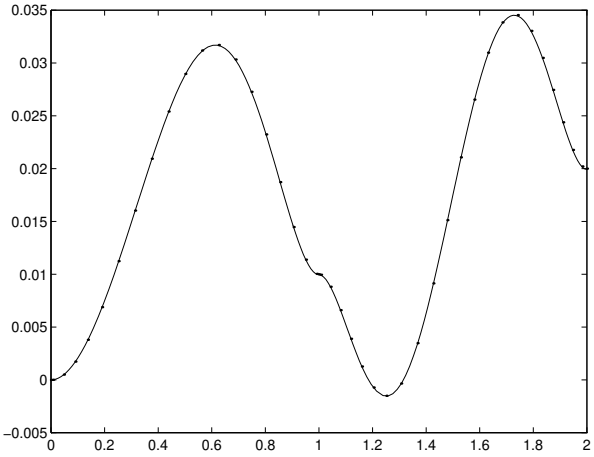

Figure 4: Simulation 1: time evolution of y for the vehicle (—) and the approximated system ($\cdots$)

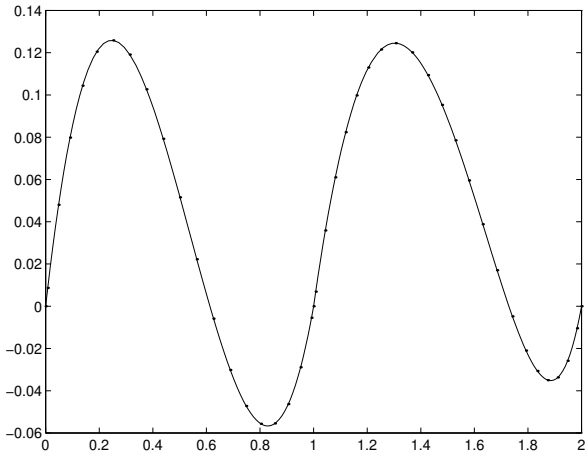

Figure 5: Simulation 1: time evolution of θ for the vehicle (—) and the approximated system $(\cdots)$

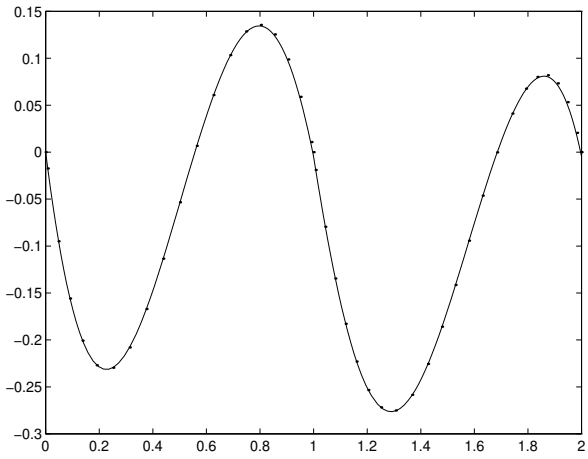

Figure 6: Simulation 1: time evolution of ϕ_1 for the vehicle (—) and the approximated system $(\cdots)$

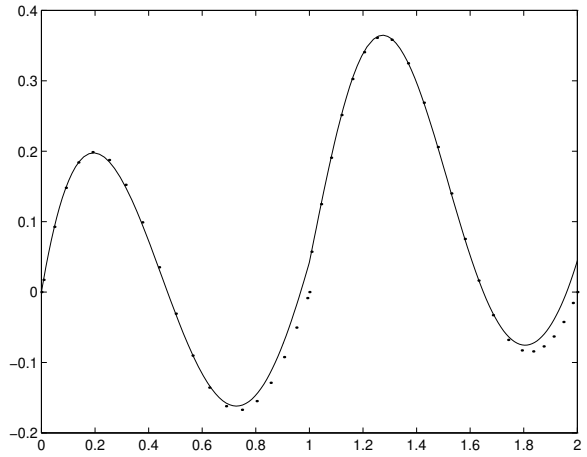

Figure 7: Simulation 1: time evolution of ϕ_2 for the vehicle (—) and the approximated system $(\cdots)$

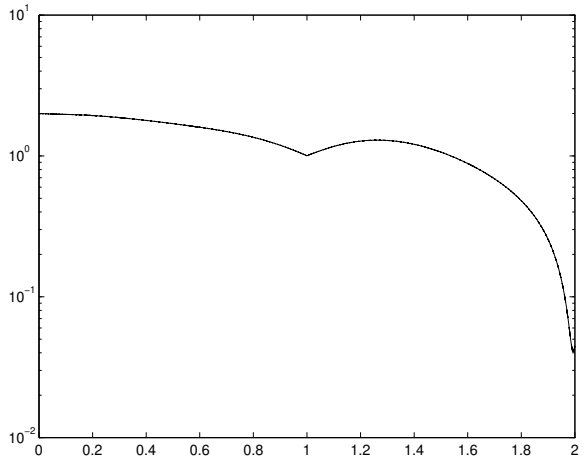

Figure 8: Simulation 1: time evolution of the state error norm

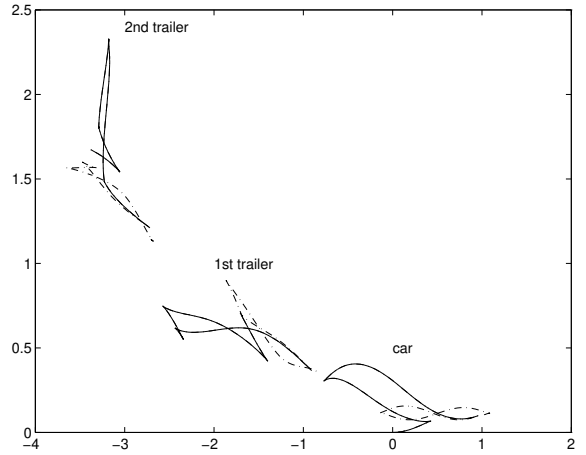

Figure 9: Simulation 2: xy-path of the reference points during the first (—) and the second $(- \cdot -)$ iteration

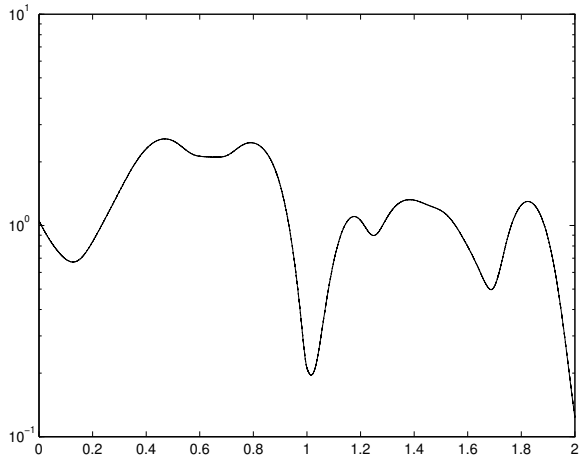

Figure 10: Simulation 2: time evolution of the state error norm